XI – MATHS

Name :

Class : Sec :

School :

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$EXERCISE : 1.1$

$Cartesian\ Product$

$Eg\ 1.1 : Find\ the\ number\ of\ subset\ of\ A\ if\ A = \{x : x = 4n + 1, 2 \leq n \geq 5, n \in \mathbb{N}\}$

$$\mathrm{A} = \{x : x = 4n + 1, n = 2,3,4,5\}$$

$$x = 4n + 1$$

$$n = 2; x = 4(2) + 1 = 8 + 1 \implies x = 9$$

$$n = 3; x = 4(3) + 1 = 12 + 1 \implies x = 13$$

$$n = 4; x = 4(4) + 1 = 16 + 1 \implies x = 17$$

$$n = 5; x = 4(5) + 1 = 20 + 1 \implies x = 21$$

$$\mathrm{A} = \{9,13,17,21\} \implies n(A) = 4$$

$$n[\mathcal{P}(A)] = 2^n \implies where\ n = 4$$

$$n[\mathcal{P}(A)] = 2^4 = 2 \times 2 \times 2 \times 2 = 16$$

Example 1.2 : In a survey of 5000 person in a town, it was found 45% of the persons know language A, 25% know language B, 10% know language C, 5% know languages A and B, 4% know language B and C, and 4% know languages A and C. If 3% of the persons know all the three languages, find the number of persons who knows only language A.

$Given\ that : Number\ of\ person\ in\ a\ town = 5000$

$$Language\ A = 45\%\ of\ the\ person \implies n(A) = \frac{45}{\cancel{100}} \times 50\cancel{00} = 45 \times 50$$

$$n(A) = 2250 \implies Language\ B = 25\%\ of\ the\ person$$

$$n(B) = \frac{25}{\cancel{100}} \times 50\cancel{00} = 25 \times 50 \implies n(B) = 1250$$

$$Language\ C = 10\%\ of\ the\ person$$

$$n(C) = \frac{10}{\cancel{100}} \times 50\cancel{00} = 10 \times 50 \implies n(C) = 500$$

$$Language\ A\ and\ B = 5\%\ of\ the\ person$$

$$n(A \cap B) = \frac{5}{\cancel{100}} \times 50\cancel{00} = 5 \times 50 \implies n(A \cap B) = 250$$

$$Language\ B\ and\ C = 4\%\ of\ the\ person$$

$$n(B \cap C) = \frac{4}{\cancel{100}} \times 50\cancel{00} = 4 \times 50 \implies n(B \cap C) = 200$$

$$Language\ C\ and\ A = 4\%\ of\ the\ person$$

$$n(C \cap A) = \frac{4}{\cancel{100}} \times 50\cancel{00} = 4 \times 50 \implies n(C \cap A) = 200$$

$$All\ the\ three\ Language\ = 3\%\ of\ the\ person$$

$$n(A \cap B \cap C) = \frac{3}{\cancel{100}} \times 50\cancel{00} = 3 \times 50$$

$$n(A \cap B \cap C) = 150$$

$$A \cap B' = A - B$$

$$A \cap (B \cap C)' = A - A \cap (B \cap C)$$

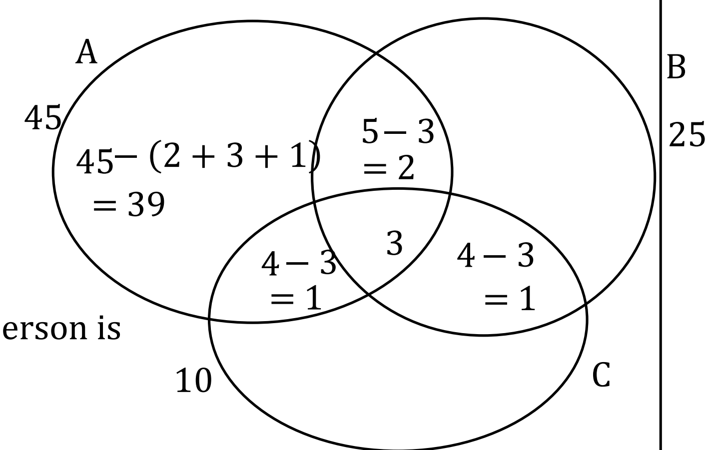
$The\ number\ of\ persons\ who\ knows\ only\ A\ is$

$$\begin{aligned} n(A \cap B' \cap C') &= n\{A \cap (B \cap C)'\} \\ &= n(A) - n\{A \cap (B \cap C)\} \\ &= n(A) - n\{(A \cap B) \cup (A \cap C)\} \\ &= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C) \\ &= 2250 - 250 - 200 + 150 = 1950 \end{aligned}$$

Thus the required number of person is 1950

Example 1.2 : In a survey of 5000 person in a town, it was found 45% of the persons know language A, 25% know language B, 10% know language C, 5% know languages A and B, 4% know language B and C, and 4% know languages A and C. If 3% of the persons know all the three languages, find the number of persons who knows only language A.

From figure, the percentage of personwho know only language A is 39.



Therefore, the required number of person is

$$\frac{39}{\cancel{100}} \times 50\cancel{00} = 1950$$

Example 1.3

Prove that $\left((A \cup B' \cup C) \cap (A \cap B \cap C')\right) \cup \left((A \cup B \cup C') \cap (B' \cap C')\right) = B' \cap C'$

we have $A \cap B' \cap C' \subseteq A \subseteq A \cup B' \cup C$

and hence $(A \cup B' \cup C) \cap (A \cap B' \cap C') = A \cap B' \cap C'$

$Also.$ $B' \cap C' \subseteq C' \subseteq A \cup B \cup C'$

and hence $(A \cup B \cup C') \cap B' \cap C' = B' \cap C'.$

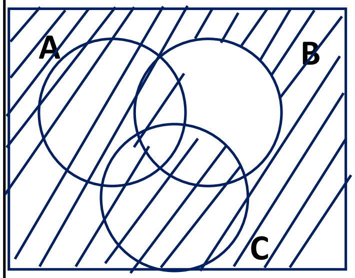
$Now\ as\ A \cup B' \cap C' \subseteq B' \cap C'.$

we have $\left((A \cup B' \cup C) \cap (A \cap B \cap C')\right) \cup \left((A \cup B \cup C') \cap (B' \cap C')\right) = B' \cap C'$

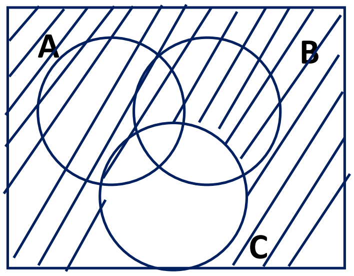
Example 1.3

Prove that $\left((A \cup B' \cup C) \cap (A \cap B \cap C')\right) \cup \left((A \cup B \cup C') \cap (B' \cap C')\right) = B' \cap C'$

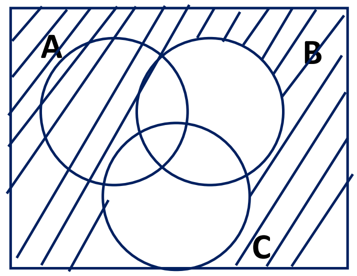
$(A \cup B' \cup C)$



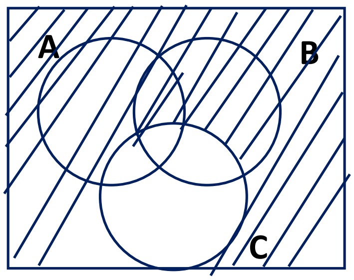
$(A \cap B \cap C')$



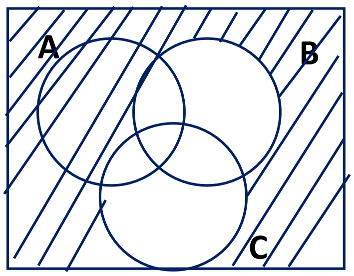
$(A \cup B' \cup C) \cap (A \cap B \cap C')$



$(A \cup B \cup C')$



$(B' \cap C')$



Example 1.4

If $X = \{1, 2, 3 \ldots 10\}$ and $A = \{1, 2, 3, 4, 5\}$, find the number of sets B $\subseteq$ X such that A $-$ B $= \{4\}$

For every subset C of $\{6,7,8,9,10\}$,

let $B = C \cup \{1,2,3,5\}$. Then $A - B = \{4\}$.

In other words, for every subset C of $\{6,7,8,9,10\}$, we are a unique set B so that $A - B = \{4\}$. so number of set $B \subseteq X$ such that $A - B = \{4\}$ and the number of substes of $\{6,7,8,9,10\}$ are the same. So the number of sets $B \subseteq X$ Such that $A - B = \{4\}$ *is* $2^5 = 32$.

Example 1.5

if A and B are two sets so that $n(B - A) = 2n(A - B) = 4n(A \cap B)$ *and if* $n(A \cup B) = 14$, *then find* $n[p(A)]$

To find $n[p(A)]$. *We need* $n(A)$.

Let $n(A \cap B) = k$.

$n(B - A) = 2n(A - B) = 4n(A \cap B)$

$n(B - A) = 2n(A - B) = 4k$

$n(B - A) = 4k$

$2n(A - B) = 4k \Longrightarrow n(A - B) = 2k$

$A = (A - B) \cup (A \cap B)$

$n(A) = n(A - B) + n(A \cap B)$

$n(B) = n(B - A) + n(A \cap B)$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A \cup B) = n(A - B) + \cancel{n(A \cap B)} + n(B - A) + n(A \cap B) - \cancel{n(A \cap B)}$

$n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$

$n(A \cup B) = 2k + 4k + k$

where $n(A \cup B) = 14$

$14 = 7k \Longrightarrow \quad k = 2$

$n\,(A \cap B) = k \quad \Longrightarrow n\,(A \cap B) = 2$

$n(B - A) = 4k \;\Longrightarrow\; n(B - A) = 4(2) \;\Longrightarrow\; n(B - A) = 8$

$n(A - B) = 2k \;\Longrightarrow\; n(A - B) = 2(2) \;\Longrightarrow\; n(A - B) = 4$

$n(A) = n(A - B) + n(A \cap B) \Longrightarrow n(A) = 4 + 2$

$n(A) = 6$ and hence $n[p(A)] = 2^n \Longrightarrow n[p(A)] = 2^6 \Longrightarrow \quad n[p(A)] = 64.$

Example 1.6

Two sets have m and k elements. If the total number of subsets of the first set is 112 more than that of the second set, find the value of m and k.

Let A and B be the two sets with $n\,(A) = m$ *and* $n(B) = k$.

$$n[P(A)] = 2^m \text{ and } n[P(B)] = 2^k$$

Since A contains more elements than B, we have $m > k$.

Difference between number of subset of A and B $= 112$

$$n[P(A)] - n[P(B)] = 112$$

$$\begin{array}{r|l} 2 & 112 \\ \hline 2 & 56 \\ \hline 2 & 28 \\ \hline 2 & 14 \\ \hline & 7 \end{array}$$

$$2^m - 2^k = 112 \;\Longrightarrow\; 2^k\left(\frac{2^m}{2^k} - \frac{2^k}{2^k}\right) = 112$$

$$2^k\left(2^m \times 2^{-k} - 1\right) = 112 \Longrightarrow 2^k(2^{m-k} - 1) \;= 2^4 \times 7$$

$$2^k = 2^4 \text{ and } 2^{m-k} - 1 = 7 \Longrightarrow \; k = 4$$

$$2^{m-k} = 7 + 1 \Longrightarrow 2^{m-k} = 8 \Longrightarrow \; 2^{m-k} = 2^3 \Longrightarrow m - k = 3$$

sub $k = 4$

$$m - 4 = 3 \;\Longrightarrow\; m = 7$$

Example 1.7

If $\boldsymbol{n\,(A) = 10}$ ***and*** $\boldsymbol{n(A \cap B) = 3}$***, find*** $\boldsymbol{n[(A \cap B)' \cap A]}$.

$\boxed{A \cap B' = A - B}$

$\boxed{A' \cap A = \emptyset}$

$\boxed{A - B = A - A \cap B}$

$(A \cap B)' \cap A = (A' \cup B') \cap A \;= (A' \cap A) \cup (B' \cap A)$

$= \emptyset \cup (B' \cap A) \;= B' \cap A \;\; = A - B$

$n[(A \cap B)' \cap A] = n(A - B) = n\big(A - (A \cap B)\big) = n(A) - n(A \cap B) = 10 - 3 = 7$

Example 1.8

If $A = \{1,2,3,4\}$ *and* $B = \{3,4,5,6\}$ *find* $n[(A \cup B) \times (A \cap B) \times (A \,\Delta\, B)]$

$A \cup B = \{1,2,3,4\} \cup \{3,4,5,6\} \Rightarrow A \cup B = \{1,2,3,4,5,6\}$

$A \cap B = \{1,2,3,4\} \cap \{3,4,5,6\} \Rightarrow A \cap B = \{3,4\}$

$A \,\Delta\, B = (A - B) \cup (B - A)$

$A - B = \{1,2,3,4\} - \{3,4,5,6\} \Rightarrow A - B = \{1,2\} \Rightarrow B - A = \{3,4,5,6\} - \{1,2,3,4\}$

$B - A = \{5,6\} \Rightarrow A \,\Delta\, B = \{1,2\} \cup \{5,6\} \Rightarrow A \,\Delta\, B = \{1,2,5,6\}$

$n(A \cup B) = 6, n(A \cap B) = 2 \text{ and } n(A \,\Delta\, B) = 4$

$n[(A \cup B) \times (A \cap B) \times (A \,\Delta\, B)] = n(A \cup B) \times n(A \cap B) \times n(A \,\Delta\, B)$

$= 6 \times 2 \times 4 \quad = 48$

Example 1.9

If $p(A)$ denotes the power set of A, then find $n\{p[p(p(\emptyset))]\}$

$n[P(A)] = 2^n \Longrightarrow A = \emptyset = \{\ \} \Longrightarrow n = 0 \Longrightarrow n[P(\emptyset)] = 2^0 = 1$

Since p $(\emptyset)$contains 1 element.$\Longrightarrow n = 1 \quad For\ P[P(\emptyset)]$

$n\{P[P(\emptyset)]\} = 2^1 = 2 \Longrightarrow$ $p(p(\emptyset))$contains 2 element$\Longrightarrow n = 2 \quad For$ $p[p(p(\emptyset))]$

$n\{p[p(p(\emptyset))]\} = 2^2$ contains 4 elements.

1. Write the following in roster form.

$(i)\{x \in N : x^2 < 121 \text{ and } x \text{ is a prime}\}$.

Let us check x is a prime and $x^2 < 121$

x^2

$x = 2 \Longrightarrow 2^2 = 4$

$x = 3 \Longrightarrow 3^2 = 9$

$x = 5 \Longrightarrow 5^2 = 25$

$x = 7 \Longrightarrow 7^2 = 49$

$x = 11 \Longrightarrow 11^2 = 121$

$A = \{2,3,5,7\}$

(ii) *The set of positive roots of the equation* $(x-1)(x+1)(x^2-1) = 0$

$(x-1)(x+1)(x^2-1) = 0$

$x - 1 = 0, x + 1 = 0, x^2 - 1 = 0$

$x = 1, x = -1, x^2 - 1^2 = 0$

$(x-1)(x+1) = 0$

$x - 1 = 0, x + 1 = 0 \Longrightarrow x = 1, x = -1$

$A = \{1\}$

$(iii)\ \{x \in N : 4x + 9 < 52\}$.

$4x + 9$

$x = 1 \Longrightarrow 4(1) + 9 = 4 + 9 = 13$

$x = 2 \Longrightarrow 4(2) + 9 = 8 + 9 = 17$

$x = 3 \Longrightarrow 4(3) + 9 = 12 + 9 = 21$

$Similiarly : 4x + 9$

$x = 4 \Longrightarrow 25$, $x = 8 \Longrightarrow 41$

$x = 5 \Longrightarrow 29$, $x = 9 \Longrightarrow 45$

$x = 6 \Longrightarrow 33$, $x = 10 \Longrightarrow 49$

$x = 7 \Longrightarrow 37$, $x = 11 \Longrightarrow 53 > 52$

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$\boldsymbol{(iv)\left\{\frac{x-4}{x+2} = 3, x \in R - \{-2\}\right\}}$

$\frac{x-4}{x+2} = 3 \Longrightarrow x - 4 = 3(x + 2)$

$x - 4 = 3x + 6 \Longrightarrow x - 3x = 6 + 4$

$-2x = 10 \Longrightarrow -x = 5$

$x = -5 \Longrightarrow \{-5\}$

2. Write the set $\{-1, 1\}$ in set builder form.

$x = -1, x = 1$

$x + 1 = 0, x - 1 = 0 \Longrightarrow (x + 1)(x - 1) = 0$

$x^2 - 1^2 = 0 \Longrightarrow x^2 - 1 = 0$

$A = \{x \vdots x \; is \; a \; root \; of \; x^2 - 1 = 0\}$

3. State whether the following sets are finite or infinite

$\boldsymbol{(i)\ \{x \in N \vdots x \; is \; an \; even \; prime \; number\}}$

$\boxed{Finite}$

$\boldsymbol{(ii)\ \{x \in N \vdots x \; is \; an \; odd \; prime \; number\}}$.

$\boxed{Infinite.}$

$\boldsymbol{(iii)\ \{x \in Z \vdots x \; is \; even \; and \; less \; than \; 10\}}$

$\boxed{Infinite}$

$\boldsymbol{(iv)\{x \in R \vdots x \; is \; a \; rational \; number\}}$.

$\boxed{Infinite}$

$\boldsymbol{(v)\{x \in N \vdots x \; is \; a \; rational \; number\}}$.

$\boxed{Infinite}$

4. By taking suitable sets A, B, C, verify the follwoing results.

$\boldsymbol{(i) A \times (B \cap C) = (A \times B) \cap (A \times C)}$.

To prove, $\mathrm{A} \times (\mathrm{B} \cap \mathrm{C}) = (\mathrm{A} \times \mathrm{B}) \cap (\mathrm{A} \times \mathrm{C})$

Let $A = \{a\}$, $\mathrm{B} = \{a, b\}$, $\mathrm{C} = \{a, b, c\}$

$L.H.S = \mathrm{A} \times (\mathrm{B} \cap \mathrm{C})$

$B \cap C = \{a, b\} \Rightarrow \therefore \mathrm{A} \times (\mathrm{B} \cap \mathrm{C}) = \{a\} \times \{a, b\}$

$\mathrm{A} \times (\mathrm{B} \cap \mathrm{C}) = \{(a, a), (a, b)\} \dots (1)$

$R.H.S = (\mathrm{A} \times \mathrm{B}) \cap (\mathrm{A} \times \mathrm{C})$

$A \times B = \{a\} \times \{a, b\} \Rightarrow A \times B = \{(a, a), (a, b)\}$

$A \times C = \{a\} \times \{a, b, c\} \Rightarrow A \times C = \{(a, a), (a, b), (a, c)\}$

$(A \times B) \cap (A \times C) = \{(a, a), (a, b)\} \cap \{(a, a), (a, b), (a, c)\}$

$(A \times B) \cap (A \times C) = \{(a, a), (a, b)\} \dots (2)$

From (1)and (2) $\therefore$ L.H.S = R.H.S

(ii) $\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$

To Prove : $\mathrm{A} \times (\mathrm{B} \cup \mathrm{C}) = (\mathrm{A} \times \mathrm{B}) \cup (\mathrm{A} \times \mathrm{C})$

Let $A = \{a\}, \mathrm{B} = \{a, b\}, \mathrm{C} = \{a, b, c\}$

$L.H.S = \mathrm{A} \times (\mathrm{B} \cup \mathrm{C})$

$\mathrm{B} \cup \mathrm{C} = \{a, b\} \cup \{a, b, c\}$

$\mathrm{B} \cup \mathrm{C} = \{a, b, c\}$

$\mathrm{A} \times (\mathrm{B} \cup \mathrm{C}) = \{a\} \times \{a, b, c\} \Rightarrow \mathrm{A} \times (\mathrm{B} \cup \mathrm{C}) = \{(a, a), (a, b), (a, c)\} \dots (1)$

$R.H.S = (\mathrm{A} \times \mathrm{B}) \cup (\mathrm{A} \times \mathrm{C})$

$A \times B = \{a\} \times \{a, b\} \Rightarrow A \times B = \{(a, a), (a, b)\}$

$A \times C = \{a\} \times \{a, b, c\} \Rightarrow A \times C = \{(a, a), (a, b), (a, c)\}$

$(A \times B) \cup (A \times C) = \{(a, a), (a, b)\} \cup \{(a, a), (a, b), (a, c)\}$

$(\mathrm{A} \times \mathrm{B}) \cup (\mathrm{A} \times \mathrm{C}) = \{(a, a), (a, b), (a, c)\} \dots (2)$

$From\ (1) and\ (2)$

$\mathrm{A} \times (\mathrm{B} \cup \mathrm{C}) = (\mathrm{A} \times \mathrm{B}) \cup (\mathrm{A} \times \mathrm{C})$

(iii) $(\mathbf{A} \times \mathbf{B}) \cap (\mathbf{B} \times \mathbf{A}) = (\mathbf{A} \cap \mathbf{B}) \times (\mathbf{B} \cap \mathbf{A})$

To prove : $(\mathrm{A} \times \mathrm{B}) \cap (\mathrm{B} \times \mathrm{A}) = (\mathrm{A} \cap \mathrm{B}) \times (\mathrm{B} \cap \mathrm{A})$

Let $A = \{a, b\}, \mathrm{B} = \{a, b, c\}$

$\mathrm{A} \times \mathrm{B} = \{a, b\} \times \{a, b, c\}$

$A \times B = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c)\}$

$\mathrm{B} \times \mathrm{A} = \{a, b, c\} \times \{a, b\}$

$\mathrm{B} \times \mathrm{A} = \{(a, a), (a, b), (b, a), (b, b), (c, a), (c, b)\}$

$(\mathrm{A} \times \mathrm{B}) \cap (\mathrm{B} \times \mathrm{A}) = \{(a, a), (a, b), (b, a), (b, b)\} \dots (1)$

$A \cap B = \{a, b\} \cap \{a, b, c\} \Rightarrow A \cap B = \{a, b\}$

$B \cap A = \{a, b, c\} \cap \{a, b\} \Rightarrow B \cap A = \{a, b\}$

$(A \cap B) \times (B \cap A) = \{a, b\} \times \{a, b\}$

$(A \cap B) \times (B \cap A) = \{(a, a), (a, b), (b, a), (b, b)\} \ldots (2)$

From (1)and (2) $\therefore$ L. H. S = R. H. S

(iv) $\mathbf{C - (B - A) = (C \cap A) \cup (C \cap B')}$

To prove: $C - (B - A) = (C \cap A) \cup (C \cap B')$

$U = \{4, 8, 12, 16, 20, 24, 28\}, A = \{8, 16, 24\}, B = \{4, 16, 20, 28\}$ and $C = \{12, 24, 28\}$

$L.H.S = C - (B - A)$

$B - A = \{4, 16, 20, 28\} - \{8, 16, 24\} \Rightarrow B - A = \{4, 20, 28\}$

$C - (B - A) = \{12, 24, 28\} - \{4, 20, 28\}$

$C - (B - A) = \{12, 24\} \ldots (1)$

$R.H.S = (C \cap A) \cup (C \cap B')$

$C \cap A = \{12, 24, 28\} \cap \{8, 16, 24\} \Rightarrow C \cap A = \{24\}$

$B' = U - B = \{\cancel{4}, 8, 12, \cancel{16}, \cancel{20}, 24, \cancel{28}\} - \{4, 16, 20, 28\}$

$B' = \{8, 12, 24\} \Rightarrow (C \cap B') = \{12, 24, 28\} \cap \{8, 12, 24\}$

$C \cap B' = \{12, 24\} \Rightarrow (C \cap A) \cup (C \cap B') = \{24\} \cup \{12, 24\}$

$(C \cap A) \cup (C \cap B') = \{12, 24\} \ldots (2)$

From (1)and (2) L. H. S = R. H. S

(v) $\mathbf{(B - A) \cap C = (B \cap C) - A = B \cap (C - A)}$

To prove: $(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$

$\mathbf{(B - A) \cap C}$

$B - A = \{4, 16, 20, 28\} - \{8, 16, 24\} \Rightarrow B - A = \{4, 20, 28\}$

$(B - A) \cap C = \{4, 20, 28\} \cap \{12, 24, 28\} \Rightarrow (B - A) \cap C = \{28\} \ldots (1)$

$\boldsymbol{(B \cap C) - A}$

$B \cap C = \{4, 16, 20, 28\} \cap \{12, 24, 28\} = \{28\}$

$(B \cap C) - A = \{28\} - \{8, 16, 24\}$

$(B \cap C) - A = \{28\} \ldots (2)$

$\boldsymbol{B \cap (C - A)}$

$C - A = \{12, 24, 28\} - \{8, 16, 24\} \Rightarrow C - A = \{12, 28\}$

$B \cap (C - A) = \{4, 16, 20, 28\} \cap \{12, 24, 28\}$

$B \cap (C - A) = \{28\} \ldots (3)$

$From\ (1), (2)\ and\ (3)$

$(\mathrm{B}-\mathrm{A}) \cap \mathrm{C} = (\mathrm{B} \cap \mathrm{C}) - \mathrm{A} = \mathrm{B} \cap (\mathrm{C}-\mathrm{A})$

$\boldsymbol{(vi)\ (B-A) \cup C = (B \cup C) - (A-C)\ by\ taking\ suitable\ A, B, C.}$

To prove: $(\mathbf{B}-\boldsymbol{A}) \cup \mathbf{C} = (\boldsymbol{B} \cup \boldsymbol{C}) - (\boldsymbol{A}-\boldsymbol{C})$

$\boldsymbol{L.H.S} = (\mathbf{B}-\boldsymbol{A}) \cup \mathbf{C}$

$B-A = \{4,16,20,28\} - \{8,16,24\} \Longrightarrow B-A = \{4,20,28\}$

$(B-A) \cup C = \{4,20,28\} \cup \{12,24,28\}$

$(B-A) \cup C = \{4,12,20,24,28\} \ldots (1)$

$R.H.S = (B \cup C) - (A-C)$

$\mathrm{B} \cup \mathrm{C} = \{4,16,20,28\} \cup \{12,24,28\} \Longrightarrow \mathrm{B} \cup \mathrm{C} = \{4,12,16,20,24,28\}$

$A - \mathrm{C} = \{8,16,\cancel{24}\} - \{12,24,28\} \Longrightarrow A - \mathrm{C} = \{8,16\}$

$(B \cup C) - (A-C) = \{4,12,\cancel{16},20,24,28\} - \{8,16\}$

$(B \cup C) - (A-C) = \{4,12,20,24,28\}$

From (1)and (2)

$(B-A) \cup C = (B \cup C) - (A-C)$

5. ***Justify the trueness of the statement***:

"An element of a set can never be a subset of itself".

Let $\mathrm{S} = \{a, b, c\}$

P (S) = { ∅ ,{a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}}

For example if we take any elements from the given set

if we take $a \in \mathrm{S}$ *will not be sub set of itself. it means a is an element of S.*

But a is subset of S i.e $\{a\} \subset \mathrm{S}$. *Every set is a subset of itself.*

6. If $\mathbf{n}(\mathbf{P}(\mathbf{A})) = \mathbf{1024}, \mathbf{n}(\mathbf{A} \cup \mathbf{B}) = \mathbf{15}$ **and** $\mathbf{n}(\mathbf{P}(\boldsymbol{B})) = \mathbf{32}$**, Find** $\mathbf{n}(\mathbf{A} \cap \boldsymbol{B})$**.**

$n[P(A)] = 1024$

$n[P(A)] = 2^{10} \Longrightarrow \boxed{n(A) = 10}$

$n[P(B)] = 32 \Longrightarrow n[P(B)] = 2^5 \Longrightarrow \boxed{n(B) = 5}$

$\mathbf{n}(\mathbf{A} \cup \boldsymbol{B}) = \mathbf{n}(\boldsymbol{A}) + \mathbf{n}(\boldsymbol{B}) - \mathbf{n}(\mathbf{A} \cap \mathbf{B})$

$15 = 10 + 5 - \mathrm{n}(\mathrm{A} \cap \mathrm{B}) \Longrightarrow \mathrm{n}(\mathrm{A} \cap \mathrm{B}) = 0$

7. if $\mathbf{n}(\mathbf{A} \cap \mathbf{B}) = \mathbf{3}$ **and** $\mathbf{n}(\mathbf{A} \cup \mathbf{B}) = \mathbf{10}$**, then find** $\mathbf{n}(\mathbf{P}(\mathbf{A}\, \Delta\, \mathbf{B}))$**.**

Given $\mathrm{n}(\mathrm{A} \cap \mathrm{B}) = 3$ and $\mathrm{n}(\mathrm{A} \cup \mathrm{B}) = 10$

$A\, \Delta\, B = (A \cup B) - (A \cap B) \Longrightarrow n(A\, \Delta\, B) = \mathrm{n}(\mathrm{A} \cup \mathrm{B}) - \mathrm{n}(\mathrm{A} \cap \mathrm{B})$

$n(A\, \Delta\, B) = 10 - 3 \Longrightarrow n(A\, \Delta\, B) = 7 \Longrightarrow n[p(A\, \Delta\, B)] = 2^7$

$n[p(A\, \Delta\, B)] = 128$

8. $\boldsymbol{For\ a\ set\ A, A \times A\ contains\ 16\ elements\ and\ two\ of\ its\ elements\ are\ (1,3)\ and\ (0,2). Find\ the\ elements\ of\ A.}$

$$n(A \times A) = 16 \implies n(A) \times n(A) = 4 \times 4 \implies n(A) = 4$$

$$A \times A \in \{(1,3),(0,2)\}$$

$$A = \{0,1,2,3\}$$

9. $\boldsymbol{Let\ A\ and\ B\ be\ two\ sets\ such\ that\ n(A) = 3\ and\ n(B) = 2. If (x,1)(y,2)(z,1) are\ in\ A \times B, find\ A\ and\ B, where\ x, y, z\ are\ distinct\ elements.}$

$$n(A) = 3, n(B) = 2\ and\ \{(x,1)(y,2)(z,1)\} \in A \times B$$

$$\therefore A = \{x, y, z\}, B = \{1,2\}$$

10. If $\mathbf{A \times A}$ has 16 elements, $\mathbf{S = \{(a,b) \in A \times A : a < b\}; (-1,2)}$ and $\mathbf{(0,1)}$ are two elements of S, then write the remaining elements of S.

Let $\mathrm{A} = \{-1,0,1,2\}$

$A \times \mathrm{A} = \{-1,0,1,2\} \times \{-1,0,1,2\}$

$A \times A = \{(-1,-1)(-1,0), (-1,1), (-1,2), (0,-1), (0,0), (0,1), (0,2), (1,-1), (1,0), (1,1), (1,2), (2,-1), (2,0), (2,1), (2,2)\}$

$\mathrm{S} = \{(\mathrm{a},\mathrm{b}) \in \mathrm{A} \times \mathrm{A}; (\mathrm{a} < \mathrm{b})\}$

$= \{(-1,0), (-1,1), (-1,2), (0,1), (0,2), (1,2)\}$

Give that $(-1,2), (0,1)$ are in S.

Other elements are $(-1,0), (-1,1), (0,2), (1,2)$

Constant and variable:

A quantity, which retains the same value throughout a mathematical process, is called a constant

A variable is a quantity which can have different values in a particular mathematical process.

It is customary to represent constants by the letters *a*, *b*, *c*, … and variable by *x*, *y*, *z*.

Independent / dependent variables

$$\boldsymbol{V = \frac{1}{3}\pi r^2 h \; cubic\ units}$$

$$\boldsymbol{A = \pi r^2 \; sq.units}$$

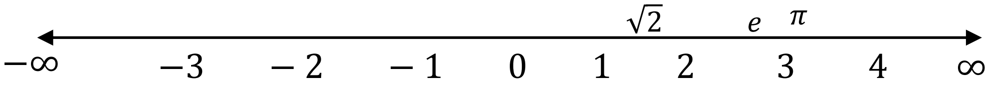
$$\boldsymbol{S = 4\pi r^2 h \; cubic\ units}$$

V, A and S are dependent variables the quantities,

r and *h* are independent variables

INTERVALS AND NEIGHBOURHOODS

The real numbers can be represented geometrically as points on a number line called the real line



Any real number can be identified as a point on the line, We call the line as the real line.

The value increase as we go right and decreases as we go left.

There are infinitely many Real number between any two real numbers.

Definition 1.1

A subset I of R is said to be an interval if

(i) I contains at least two elements and

(ii) $a, b \in I$ and $a < c < b$ then $c \in I$

Geomerically, intervals correspond to rays and line segments on the real line.

Note that the set of all natural numbers, the set of all non − negative integers, Set of all odd integers, set of all even integers, set of all prime numbers are not intervals.

Consider the following sets

(*i*) *The set of all real numbers greater than* 0

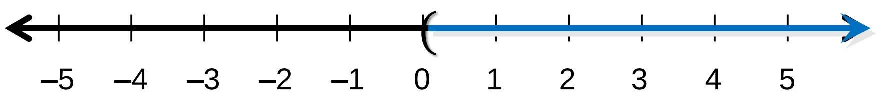
(*ii*) *The set of all real numbers greater than* 5 *and less than* 7

(*iii*) *The set of all real numbers* x *such than* $1 \le x \le 3$.

(*iv*) *The set of all real numbers* x *such that* $1 < x \le 2$.

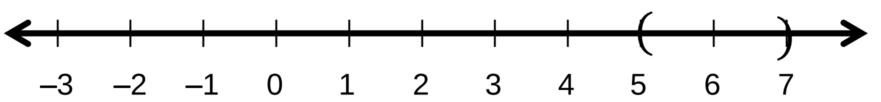
The above four sets are intervals.

(*i*)

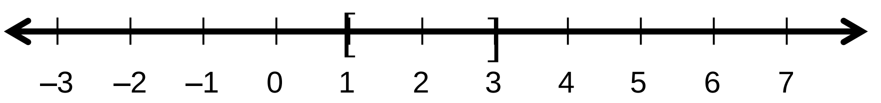


(*i*) *is an infinite interval*

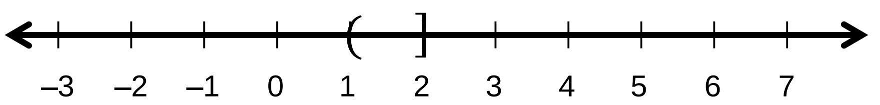
(*ii*) *The set of all real numbers greater than* 5 *and less than* 7



(*iii*) *The set of all real numbers* x *such than* $1 \le x \le 3$.



(*iv*) *The set of all real numbers* x *such that* $1 < x \le 2$.



(*ii*), (*iii*) *and* (*iv*) *are finite intervals*

The term “finite interval” does not mean that the interval contains Only finitely many real numbers, however both ends are finite numbers.

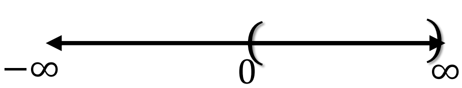
Both finite and Infinite intervals are infinite sets.

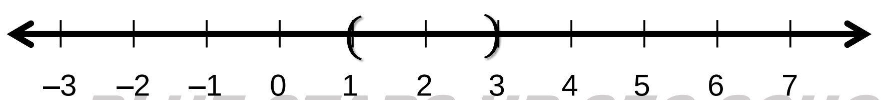
The intervals correspond to line segments are finite Intervals whereas that correspond to rays and the entire real line are infinite intervals.

A finite interval is said to be closed if it contains both of its end points and open if it contains neither of its end points.

( ) *parentheses indicate open interval*

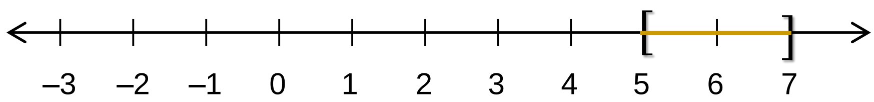
$(0, \infty)$ *and* $(1, 3)$





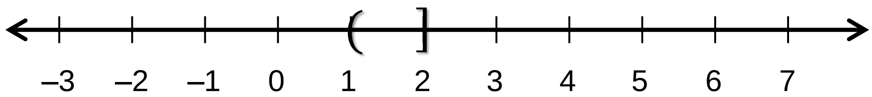
[ ] *square brackets indicate closed interval.*

$[5, 7]$



$(\quad]$ *one end open and other end closed*

$(1, 2]$



In paricular $[5, 7]$ *contains both* 5 *and* 7 *and in between real numbers.*

The interval $(1, 3)$. *Does not contain* 1 *and* 3 *but contains all in between the number.*

The interval $(1, 2]$ *does not contain* 1 *but contains* 2 *and all in between numbers.*

Note that ∞ *is not a number*

The symbols $-\infty$ *and* ∞ *are used to indicate the ends of Real line*

Further, the intervals (a, b) *and* $[a, b]$ *are subsets of* R

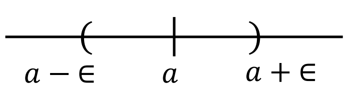
Neighbourhood of a point $'a'$ *is any open interval containing* $'a'$.

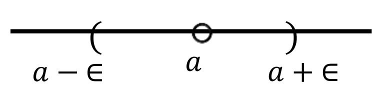
If $\in$ *is a positive number, usually very small,*
then the $\in -$ *neighbourhood of* $'a'$

Is the open interval $(a- \in, a + \in)$.

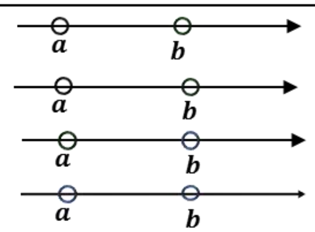
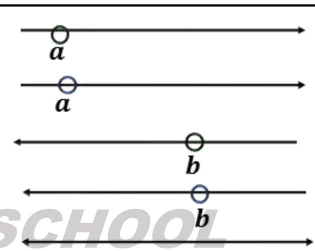
The set $(a - \in, a + \in) - \{a\}$ is called deleted Neighborhood of ‘a’

and it is denoted as $0 < |x - a| < \in$





Type of intervals

	Notation	Set	Graph
Finite	(a, b)	$\{x \,/\, a < x < b\}$	
	$[a, b)$	$\{x \,/\, a \le x < b\}$	
	$(a, b]$	$\{x \,/\, a < x \le b\}$	
	$[a, b]$	$\{x \,/\, a \le x \le b\}$	
Infinite	(a, ∞)	$\{x \,/\, x > a\}$	
	$[a, \infty)$	$\{x \,/\, x \ge a\}$	
	$(-\infty, b)$	$\{x \,/\, x < b\}$	
	$(-\infty, b]$	$\{x \,/\, x \le b\}$	
	$(-\infty, \infty)$	$\{x \,/\, -\infty < x < \infty\}$	

write the following intervals in symbolic form

$(\boldsymbol{i})\{\boldsymbol{x} : \boldsymbol{x} \in \boldsymbol{R}, -\mathbf{2} \leq \boldsymbol{x} \leq \mathbf{0}\}$ $(\boldsymbol{ii})\{\boldsymbol{x} : \boldsymbol{x} \in \boldsymbol{R}, \mathbf{0} < \boldsymbol{x} < \mathbf{8}\}$

$(\boldsymbol{iii})\{\mathbf{x}: \boldsymbol{x} \in \boldsymbol{R}, -\mathbf{8} < \boldsymbol{x} \leq -\mathbf{2}\}$ **(iv)** $\{\boldsymbol{x} : \boldsymbol{x} \ni \boldsymbol{R}, -\mathbf{5} \leq \boldsymbol{x} \leq \mathbf{9}\}$.

(i) $[-2, 0]$ (ii) $(0, 8)$ (iii) $(-8, -2]$ (iv) $[-5, 9]$

RELATION

EXERCISE : 1.2

Example* 1.10 *Check the relation $\boldsymbol{R = \{(1,1),(2,2),(3,3),\ldots,(n,n)\}}$ ***defined on the set*** $\boldsymbol{S = \{1,2,3,\ldots n\}}$ ***for the three basic relations.***

As $(a,a) \in R$ *for all* $a \in S$, R *is reflexive.*

For every pair $(a,b) \in R$ (b,a) *is also in* R. *Thus* R *is symmetric.*

We cannot, find two pairs (a,b) *and* (b,c) *in* R, *such that* $(a,c) \notin R$.
hence R *is Transitive. Since* R *is reflexive, symmetric and transitive*
This relation is an equivalence relation.

Example* 1.11 *Let $\boldsymbol{S = \{1,2,3\}}$ ***and*** $\boldsymbol{\rho = \{(1,1),(1,2),(2,2),(1,3),(3,1)\}}$

(i) Is ρ reflexive ? If not, state the reason and write the minimum set of ordered Pairs to be included to ρ so as to make it reflexive.

(ii) Is ρ symmetric? If not, state the reason, write minimum number of orderedPairs to be included to ρ so as to make it symmetric and write minimum number of Ordered pairs to be deleted from ρ so as to make it symmetric.

(iii) Is ρ transwitive? If not, state the reason, write minimum number of ordered Pairs to be included to ρ so as to make it transitive and write minimum number of Ordered pairs to be deleted from ρ so as to make it transitive.

(iv) Is ρ an equivalence relation? If not, write the mininmum ordered pairs to be Included to ρso as to make it an equivalence relation

(i) Is ρ reflexive ? If not, state the reason and write the minimum set of ordered Pairs to be included to ρ so as to make it reflexive.

$\rho = \{(1,1),(1,2),(2,2),(1,3),(3,1)\}$

ρ *is not reflexive because* $(3,3)$ *is not in* ρ. *As* $(1,1)$ *and* $(2,2)$ *are in* ρ, *it is enough to include the pair* $(3,3)$ *to* ρ *so as to make it reflexive.*

$\rho = \{(1,1),(1,2),(2,2),(1,3),(3,1),(3,3)\}$ *is Reflexive.*

(ii) Is ρ symmetric? If not, state the reason, write minimum number of ordered Pairs to be included to ρ so as to make it symmetric and write minimum number of Ordered pairs to be deleted from ρ so as to make it symmetric.

(iii) ρ *is not symmetric because* $(1,2)$ *is in* ρ, *but* $(2,1)$ *is not in* ρ.

It is enough to included the pair $(2,1)$ *to* ρ *so as to make it symmetric.*

$\rho = \{(1,1),(1,2),(2,2),(1,3),(3,1),(2,1)\}$ *It is enough to remove the pair* $(1,2)$ *to* ρ *so as to make it symmetric.* $\rho = \{(1,1),(2,2),(1,3),(3,1)\}$

(iii) Is ρ transitive? If not, state the reason, write minimum number of ordered Pairs to be included to ρ so as to make it transitive and write minimum number of Ordered pairs to be deleted from ρ so as to make it transitive.

$\rho = \{(1,1),(1,2),(2,2),(1,3),(3,1)\}$

$(\underset{a}{1},\underset{b}{1}) \in R \; and \; (\underset{b}{1},\underset{c}{2}) \in R \Rightarrow (\underset{a}{1},\underset{c}{2}) \in R$

$(\underset{a}{1},\underset{b}{1}) \in R \; and \; (\underset{b}{1},\underset{c}{3}) \in R \Rightarrow (\underset{a}{1},\underset{c}{3}) \in R$

$(\underset{a}{1},\underset{b}{2}) \in R \; and \; (\underset{b}{2},\underset{c}{2}) \in R \Rightarrow (\underset{a}{1},\underset{c}{2}) \in R$

$(\underset{a}{1},\underset{b}{3}) \in R \; and \; (\underset{b}{3},\underset{c}{1}) \in R \Rightarrow (\underset{a}{1},\underset{c}{1}) \in R$

$(\underset{a}{3},\underset{b}{1}) \in R \; and \; (\underset{b}{1},\underset{c}{3}) \in R \Rightarrow (\underset{a}{3},\underset{c}{3}) \notin R$

$(\underset{a}{3},\underset{b}{1}) \in R \; and \; (\underset{b}{1},\underset{c}{2}) \in R \Rightarrow (\underset{a}{3},\underset{c}{2}) \notin R$

So $(3,3)$ *and* $(3,2)$ *are to be included into so as to make* ρ *transitive.*

$\rho = \{(1,1),(1,2),(2,2),(1,3),(3,1),(3,3),(3,2)\}$

$\rho = \{(1,1),(1,2),(2,2),(1,3),(3,1)\}$

But if we remove $(3,1)$ *from* ρ, *then it becomes transitive.*

(iv) *Is* ρ *an equivalence relation? If not, write the mininmum ordered pairs to be Included to* ρ *so as to make it an equivalence relation*

$\rho = \{(1,1),(1,2),(2,2),(1,3),(3,1)\}$

(iv)
- *To make* ρ *reflexive, we have to include* $(3,3)$
- To make ρ symmetric, we have to include (2,1)
- *And to make* ρ *transitive, we have to include* $(3,2)$.

To make ρ *as an equivalence relation we have to include all these pairs.*

It becomes $\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(3,2)\}$

But this relation is not symmetric because $(3,2)$ *is in the relation and* $(2,3)$ *is not in the relation. So we have to include* $(2,3)$ *also.*

Now the new relation becomes

$\{(1,1),(2,2),(3,3),(1,2),(2,1)(1,3),(3,1),(3,2),(2,3)\}$

Now this relation is reflexive, symmetric and transitive, and hence its an equivalence relation.

Example 1.12

on A of the following tyLet $A = \{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}\}$. ***Construct relationspes;***

(i) ***Not reflexive, not symmetric, not transitive,***

(ii) ***Not reflexive, not symmertric, transitive.***

(iii) ***Not reflexive, symmetric, not transitive.***

(iv) ***Not reflexive, symmetric, transitive.***

(v) ***reflexive, not symmetric, not transitive.***

(vi) ***reflexive, not symmetric, transitive.***

(vii) ***reflexive, symmetric, not transitive.***

$(viii)$ ***reflexive, symmetric, transitive.***

$A = \{0, 1, 2, 3\}$

(i) *Not reflexive, not symmetric, not transitive,*

(i) *Let us use the pair* $(1, 2)$ *to make the relation* "not symmetric"

Consider the Relation $\{(1, 2)\}$. *It is transitive.*

If we include $(2, 3)$ *then the relation is not Transitive.*

So the relation $\{(1, 2),(2, 3)\}$ *is not reflexive, not symmetric and not transitive.*

(ii) *Not reflexive, not symmertric, transitive.* $\{(1,2)\}$ *is transitive, not reflexive and not Symmetric.*

(iii) *Not reflexive, symmetric, not transitive.*

Let us start with the pair $(1, 2)$. *to make the relation* "symmetric" *we have to include . The pair* $(2, 1)$

$\{(1, 2)\ , (2, 1)\}$ *is not reflexive; it is symmetric; and it is not transitive.*

$A = \{0, 1, 2, 3\}$

(iv) *Not reflexive, symmetric, transitive.*

Let us take the relation $\{(1, 2), (2, 1)\}$ *is symmetric*

To make transitive

$(\underset{a}{1}, \underset{b}{2}) \in R$ *and* $(\underset{b}{2}, \underset{c}{1}) \in R \Longrightarrow (\underset{a}{1}, \underset{c}{1}) \notin R$

Relation $\{(1, 2), (2, 1), (1, 1)\}$ *is transitive.*

$(\underset{a}{2}, \underset{b}{1}) \in R$ *and* $(\underset{b}{1}, \underset{c}{1}) \in R \Longrightarrow (\underset{a}{2}, \underset{c}{1}) \in R$

$(\underset{a}{2}, \underset{b}{1}) \in R$ *and* $(\underset{b}{1}, \underset{c}{2}) \in R \Longrightarrow (\underset{a}{2}, \underset{c}{2}) \notin R$

Relation $\{(1, 2), (2, 1), (1, 1), (2, 2)\}$ *is transitive.*

Thus $\{(1, 2), (2, 1), (1, 1), (2, 2)\}$ *is not reflexive; it is symmetric and it is transitive.*

(v) *reflexive, not symmetric, not transitive.*

For a relation on $\{0,1,2,3\}$ to be reflexive, it must have the pairs $(0, 0), (1, 1)(2, 2), (3, 3)$

Fortunately, it becomes symmetric and transitive if we insert $(1, 2)$

Relation $\{(0, 0), (1, 1)(2, 2), (3, 3), (1, 2)\}$ *it becomes not symmetric*

$(\underset{a}{1}, \underset{b}{2}) \in R$ *and* $(\underset{b}{2}, \underset{c}{2}) \in R \Longrightarrow (\underset{a}{1}, \underset{c}{2}) \in R$ *Fortunately, it becomes transitive*

Now we insert $(2, 3)$ *Thus* $\{(0, 0), (1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$

$(\underset{a}{1}, \underset{b}{2}) \in R$ *and* $(\underset{b}{2}, \underset{c}{3}) \in R \Longrightarrow (\underset{a}{1}, \underset{c}{3}) \notin R$ *is not transitive.*

Hence it is reflexive, it is not symmetric and it is not transitive.

(vi) *reflexive, not symmetric, transitive.*

The relation $\{(0,0), (1,1), (2,2), (3,3), (1, 2)\}$

That is reflexive, transitive and not symmetric.

(vii) *reflexive, symmetric, not transitive.*

Thus $\{(0,0),(1,1),(2,2),(3,3),(1,2),(2,3)\}$

$(\underset{a}{1},\underset{b}{2}) \in R$ *and* $(\underset{b}{2},\underset{c}{3}) \in R \Longrightarrow (\underset{a}{1},\underset{c}{3}) \notin R$ *is not transitive.*

Fortunately, it becomes not symmetric

To make it symmetric. It is enough to included the pair $(2,1)$ *and* $(3,2)$

The relation $\{(0,0),(1,1),(2,2),(3,3),(1,2),(2,3),(2,1),(3,2)\}$ *that*
Is reflexive, symmetric and not transitive.

$(viii)$ *reflexive, symmetric, transitive.*

The relation $\{(0,0),(1,1),(2,2),(3,3)\}$ *which is reflexive, symmetric and transitive.*

Example 1.13 : In the set Z of integers, define m R n if m − n is a multiple of 12. Prove That R is an equivalence relation.

$$m\,R\,n \Longrightarrow m-n = 12k$$
$$when\ ,k = 0$$
$$m - n = 0\ \ i.e\ n = m$$

$m\,R\,n \Longrightarrow m\,R\,m$ *hence m R m proving that R is reflexive*

$$m\,R\,n \Longrightarrow m-n = 12k$$
$$-(n-m) = 12k$$
$$n - m = -12k$$
$$n - m = 12(-k) \Longrightarrow n\,R\,m.$$

$m\,R\,n \Longrightarrow n\,R\,m$ *. This shows that R is symmetric.*

Let m R n and n R p implies m R p

$$Let\ m\,R\,n \Longrightarrow m - n = 12k \ldots (1)$$
$$and$$
$$Let\ n\,R\,p \Longrightarrow n - p = 12l \ldots (2)$$

Adding (1) *and* (2)

$$m - n = 12k$$
$$n - p = 12l$$
$$m - p = 12k + 12l$$
$$m - p = 12(k + l)$$

So $m - p = 12(k+l)$ *and hence m R p. This shows that R is transitive*

$$m\,R\,n \text{ and } n\,R\,p \Longrightarrow m\,R\,p$$

Thus R is an equivalence relation.

Theorem 1.1

The number of relations from a set containing m elements to a set Containing n elements is 2^{mn}. In particular the number of relations on a set containing N elements is $2n^2$.

Proof. Let A and B be sets containing m an n elements respectively.

$n(A) = m$ and $n(B) = n$. Then $A \times B$

$n(A \times B) = n(A) \times n(B) \Rightarrow n(A \times B) = mn$

$A \times B$ Contains mn elements and $A \times B$ has 2^{mn} subsets.

Since every subset of $A \times B$ is a relation from A to B,

The number of relations from a set containing m elements.
to a set containing n elements $= 2^{mn}$

Taking $A = B$

The number of relations on a set containing n elements $= 2^{n^2}$

Note:

(i) The number of reflexive relations on a set containing n elements is 2^{n^2-n}

(ii) The number of symmetric relations on a set containing n elements is $2^{\frac{n^2+n}{2}}$

Definition 1.5

If R is a relation from A to B, then the relation R^{-1} defined from B and A by $R^{-1} = \{(b, a)\colon a, b \in R\}$ is called the inverse of the relation R. For example, if $R = \{(1, a), (2, b), (3, a)\}$, then $R^{-1} = \{(a, 1), (b, 2), (c, 2), (a, 3)\}$

It is easy to see that the domain of R becomes the range of R^{-1} and the range of R becomes the domain of R^{-1}

An equivalence relation on a set decomposes it into a disjoint union of its subsets ("equivalence classes). Such a decomposition is called a partition. This is explained in the following example.

For $a, b \in Z$, aRb if and only if $a - b = 3k, k \in Z$ is an equivalence relation on Z.

$Z_0 = \{x \in Z\colon xR0\} = \{\dots, -6, -3, 0, 3, 6, \dots\}$

$Z_1 = \{x \in Z\colon xR1\} = \{\dots, -5, -2, 1, 4, 7, \dots\}$

$Z_2 = \{x \in Z\colon xR2\} = \{\dots, -4, -1, 2, 5, 8, \dots\}$

Thus $Z = Z_0 \cup Z_1 \cup Z_2$ and all are disjoint subsets.

For a given partition $S_1 \cup S_2 \cup \dots \cup S_n$ of as set S into disjoint subsets, *one* Can construct an equivalence relation R on S by xRy if x,y ∈ S_i for some i.

1. $Discuss\ the\ following\ relation\ for\ reflexivity, symmetricity\ and\ transitivity$

2. $(i)\ The\ relation\ R\ defined\ on\ the\ set\ of\ all\ positive\ integers\ by\ mRn\ if\ m\ divides\ n.$

$(ii)\ Let\ P\ denote\ the\ set\ of\ all\ straight\ lines\ in\ a\ plane. The\ relation\ R\ defined\ by\ lRm\ if\ l\ is\ perpendicular\ to\ m".$

$(iii)\ Let\ A\ be\ the\ set\ consisting\ of\ all\ the\ members\ of\ a\ family.$ $The\ relation\ R\ defined\ by\ aRb\ if\ a\ is\ not\ a\ sister\ of\ b.$

$(iv)\ Let\ A\ be\ the\ set\ consisting\ of\ all\ the\ female\ members\ of\ a\ family.$ $The\ relation\ R\ defined\ by\ a\ R\ b\ if\ a\ is\ not\ a\ sister\ of\ b.$

$(v)\ On\ the\ set\ of\ natural\ numbers\ the\ relation\ R\ defined\ by\ x\ R\ y\ if\ x+2y=1.$

$(i)\ The\ relation\ R\ defined\ on\ the\ set\ of\ all\ positive\ integers\ by\ m\ R\ n\ if\ m\ divides\ n.$

$If\ m\ R\ m \Rightarrow m\ divides\ m$

$which\ is\ true\ every\ integer\ divides\ itself \Rightarrow Reflexive.$

$For\ example:\ a\ divides\ a,\ b\ divides\ b\ etc.$

$(b)\ if\ m\ divides\ n\ then\ it\ is\ not\ true\ for\ n\ divides\ m$

$$m\ R\ n \neq n\ R\ m \Rightarrow not\ symmetric$$

$For\ example:\ 3\ divides\ 9\ \Rightarrow\ 9\ does\ not\ divide\ 3.$

$(c)\ If\ m\ R\ n\ and\ n\ R\ p \Rightarrow m\ R\ p$

$if\ m\ divides\ n\ and\ n\ divides\ p \Rightarrow m\ divides\ p$

$\boldsymbol{For\ example: 3\ divides\ 15 and\ 15\ divides\ 45 \Rightarrow 3\ divides\ 45}$

$(ii)\ Let\ P\ denote\ the\ set\ of\ all\ straight\ lines\ in\ a\ plane. The\ relation\ R\ defined\ by\ lRm\ if\ l\ is\ perpendicular\ to\ m".$

$R\ is\ defined\ by\ l\ R\ m\ if\ l\ is\ perpendicular\ to\ m.$

$(a)\ A\ line\ cannot\ be\ perpendicular\ to\ itself$

$l\ R\ l\ is\ not\ true \Rightarrow not\ reflexive.$

$(b)\ If\ a\ line\ l\ is\ perpendicular\ to\ a\ line\ m\ then\ m\ is\ perpendicular\ to\ l.$

$if\ l\ R\ m \Rightarrow m\ R\ l \Rightarrow symmetric.$

$(c)\ If\ l\ is\ perpendicular\ to\ m\ and\ m\ is\ perpendicular\ to\ n.$

$It\ is\ not\ true\ that\ l\ is\ perpendicular\ to\ n.$

$$l\ R\ m\ and\ m\ R\ n \Rightarrow l \cancel{R} n\ \ is\ not\ transitive.$$

(iii) *Let A be the set consisting of all the members of a family.*
The relation R defined by a R b if a is not a sister of b.

A is the set of all members of a family.

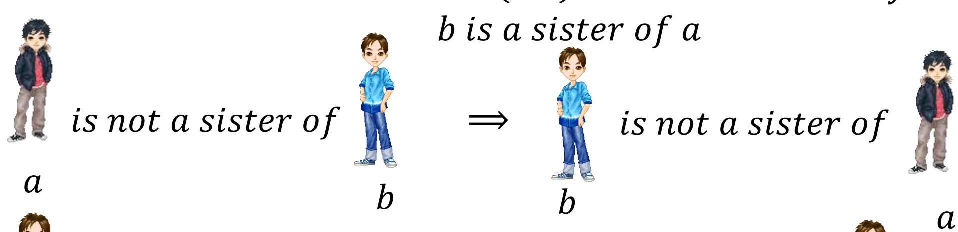
R is defined by a R b if a is not a sister of b

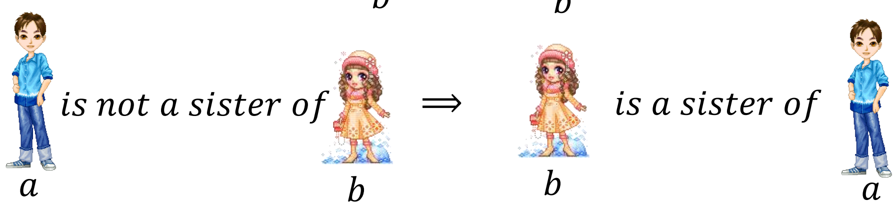
(a) *A member of a family cannot be sister of herself a R a is true*
is Reflexive.

(b) *If a is not a sister of b,* $\Rightarrow$ *b is not a sister of a*

(OR) *is not symmetric.*

b is a sister of a



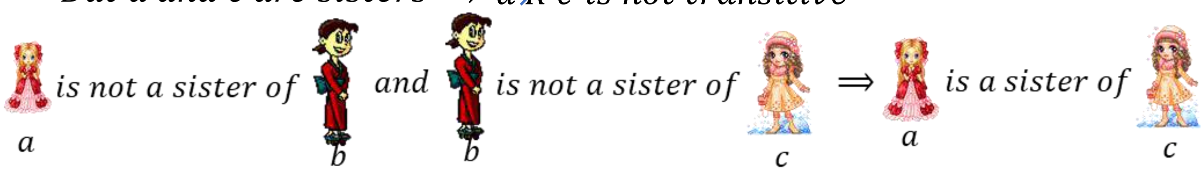


(c) *Suppose in a family if we take mother b and two daughters a and c*

it is true that a is not a sister of b $\Rightarrow a\,R\,b$

b is not a sister of c $\Rightarrow b\,R\,c.$

But a and c are sisters $\Rightarrow a\,\cancel{R}\,c$ *is not transitive*



(iv) *Let A be the set consisting of all the female members of a family.*
The relation R defined by a R b if a is not a sister of b.
A is the set of all female members of a family.
$a\,R\,b \Rightarrow$ *a is not a sister of b*

(a) *A female member of the family cannot be sister of herself a R a*
is Reflexive.

(b) *if a is not a sister of b then b is not a sister of a*
$a\,R\,b \Rightarrow b\,R\,a$ *is Symmetric.*

(c) *Take for example mother b and two daughters a and c it is true that*
a is not a sister of b and b is not a sister of c

But a and c are sisters $\Rightarrow a\,\cancel{R}\,c$ *is not transitive*

$a\,R\,b$ and $b\,R\,c \Rightarrow a\,\cancel{R}\,c$

(v) $On\ the\ set\ of\ natural\ numbers\ the\ relation\ R\ defined\ by\ x\ R\ y$ $if\ x + 2y = 1.$

(a) $R\ is\ defined\ by\ x\ R\ y\ if\ x + 2y = 1$

$$x\ R\ y \Rightarrow x + 2y = 1$$

$$x\ R\ x \Rightarrow x + 2x = 1$$

$$3x = 1 \Rightarrow x = \frac{1}{3} \notin N\ \therefore Not\ reflexive.$$

(b) $x\ R\ y \Rightarrow x + 2y = 1\ is\ an\ empty\ set$

$Empty\ set\ is\ symmetric\ and\ transitive$

$Let\ X = \{a, b, c, d\}\ and\ R = \{(a, a), (b, b), (a, c)\}. Write\ down\ the\ minimum\ number\ of\ ordered\ pairs\ to\ be\ included\ to\ R\ to\ make\ it$
$(i) reflexive \quad (ii) symmetric \quad (iii)\ transitive \quad (iv)\ equivalence.$

$R = \{(a, a), (b, b), (a, c)\}$

$(i)\ reflexive$

$we\ have\ to\ include\ (c, c), (d, d)\ for\ reflexive.$

$R = \{(a, a), (b, b), (a, c), (c, c), (d, d)\}\ is\ reflexive.$

$(ii)\ symmetric$

$we\ have\ to\ include\ (c, a) for\ symmetric.$

$R = \{(a, a), (b, b), (a, c), (c, a)\}\quad is\ symmetric$

$(iii)\ transitive$

$(iii)\ we\ have\ to\ include\ (c, a) for\ transitive$

$R = \{(a, a), (b, b), (a, c), (c, a)\}\quad is\ transitive$

$R = \{(a, a), (b, b), (a, c)\}$

$(iv)\ equivalence.$

$we\ have\ to\ include\ (c, c), (d, d), (c, a)\ for\ equivalence\ relation.$

$R = \{(a, a), (b, b), (a, c), (c, c), (d, d), (c, a)\}\quad is\ equivalance$

3. $\boldsymbol{Let\ A = \{a, b, c\}\ and\ R = \{(a, a), (b, b), (a, c)\}. Write\ down\ the\ minimum\ number\ of\ ordered\ pairs\ to\ be\ included\ to\ R\ to\ make\ it}$
$\boldsymbol{(i) reflexive \quad (ii) symmetric \quad (iii) transitive \quad (iv)\ equivalence.}$

$Let\ A = \{a, b, c\}\ and\ R = \{(a, a), (b, b)(a, c)\}$

$(i)\ reflexive$

$we\ have\ to\ include\ (c, c) for\ reflexive.$

$R = \{(a, a), (b, b), (a, c), (c, c)\}\ is\ reflexive.$

(ii) *symmetric*

we have to include (c,a) *for symmetric.*

$R = \{(a,a),(b,b),(a,c),(c,a)\}$ *is symmetric*

(iii) *transitive*

(iii) *we have to include* $(c,c),(c,a)$ *for transitive*

$R = \{(a,a),(b,b),(a,c),(c,a),(c,c)\}$ *is transitive*

$R = \{(a,a),(b,b),(a,c)\}$

(iv) *equivalence.*

we have to include $(c,c),(c,a)$ *for equivalence relation.*

$R = \{(a,a),(b,b),(c,c),(c,a),(a,c)\}$ *is equivalance*

4. Let P be the set of all triangles in a plane and R be the relation defined on P as $a\,R\,b$ if a similar to b. Prove that R is an equivalence relation.

$P =$ *Set of all triangles in a plane*

$a\,R\,b$ *if* a *is similar to* b.

(i) *Every triangle is similar to itself*

$a\,R\,a \Longrightarrow$ *Reflexive.*

if a triangle 'a' *is similar to triangle* 'b' *then* 'b' *is similar to* 'a'

If $a\,R\,b$ *then it implies* $b\,R\,a$

$aRb \Longrightarrow bRa$ *is Symmetric*

(iii) *if* $a\,R\,b$ *and* $b\,R\,c \Longrightarrow a\,R\,c$ *is transitive*

(iv) *Since* R *is reflexive, symmetric and transitive it is an equivalence relation.*

5. On the set of natural number let R be the relation defined by aRb if $2a + 3b = 30$. Write down the relation by listing all the pairs. Check Whether it is (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence.

$a\,R\,b \Longrightarrow 2a + 3b = 30$ *where* $a, b \in N$

$N = \{1,2,3,\ldots.\}$

$2a + 3b = 30 \Longrightarrow 3b = 30 - 2a$

$$b = \frac{30 - 2a}{3}$$

$$a = 3;\ b = \frac{30 - 2(3)}{3} = \frac{30 - 6}{3} = \frac{24}{3}$$

$$b = 8$$

$$a = 6;\ b = \frac{30 - 2(6)}{3} = \frac{30 - 12}{3} = \frac{18}{3}$$

$$b = 6$$

$a = 9; b = \dfrac{30 - 2(9)}{3} = \dfrac{30 - 18}{3} = \dfrac{12}{3}$

$b = 4$

$a = 12; b = \dfrac{30 - 2(12)}{3} = \dfrac{30 - 24}{3} = \dfrac{6}{3}$

$b = 2$

$R = \{ (3,8), (6,6), (9,4), (12,2)\}$

(i) *Not reflexive*

(ii) *Not symmetric.*

(iii) *transitive.*

(iv) *Not equivalence relation.*

6. *Prove that the relation friendship is not an equivalence relation on the set of all people in Chennai.*

Every people in chennai is not a friend of itself

"a is a friend of a" $\Longrightarrow$ $a \cancel{R} a$ *is not reflexive*

If a is a friend of b and b is a friend of a

 *is a friend of*  $\Longrightarrow$  *is a friend of* 

$a\ R\ b \Longrightarrow b\ R\ a$ *is symmetric.*

If a is a friend of b and b is a friend of c then a need not be a friend of c

 *is a friend of*  *and*  *is a friend of*  $\Longrightarrow$  *is not a friend of* 

$a\ R\ b$ *and* $b\ R\ c$ *does not imply* $a\ R\ c$ *is not transitive*

Hence it is not equivalent.

7. *On the set of natural number let R be the relation defined by aRb if $a + b \le 6$. Write down the relation by listing all the pairs. Check whether it is (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence.*

R is defined by a R b if $a + b \le 6.$

$R = \{(1,5), (1,4), (1,3), (1,2), (1,1), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2)$
$(3,3), (4,1), (4,2), (5,1)\}$

(i) *Not reflexive as* $(4,4)$ *and* $(5,5) \notin R.$

(ii) *Symmetric.*

(iii) *Not transitive.*

(iv) *Not equivalence because it is not reflexive, transitive.*

8. Let $A = \{a, b, c\}$. What is the equivalence relation of smallest cardinality on A? What is the equivalence relation of larger cardinality on A?

$A = \{a, b, c\}$

$A \times A = \{a, b, c\} \times \{a, b, c\}$

$A \times A = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$

Let $R_1 = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$

R_1 *is equivalence relation* . $\therefore n(R_1) = 9$. *This is the largest cardinality.*

Let $R_2 = \{(a,a),(b,b),(c,c)\}$ R_2 *is equivalence relation.* $\therefore n(R_2) = 3$

This is the smallest cardinality.

9. In the set Z of integers, define mRn if $m - n$ is divisible by 7. Prove that R is an equivalence relation.

$mRn \Longrightarrow (m-n)$ *is divisible by* 7

$m\,R\,n \Longrightarrow m - n = 7k$

when, $k = 0$

$m - n = 0$

$n = m$

$m\,R\,n \Longrightarrow m\,R\,m$ *hence* $m\,R\,m$ *proving that* R *is reflexive*

$m\,R\,n \Longrightarrow m - n = 7k$

$-(n-m) = 7k$

$n - m = -7k$

$n - m = 7(-k) \Longrightarrow n\,R\,m.$

$m\,R\,n \Longrightarrow n\,R\,m$. *This shows that* R *is symmetric.*

Let $m\,R\,n$ *and* $n\,R\,p$ *implies* $m\,R\,p$

Let $m\,R\,n \Longrightarrow m - n = 7k \ldots (1)$

and

Let $n\,R\,p \Longrightarrow n - p = 7l \ldots (2)$

Adding (1) *and* (2)

$$m - \cancel{n} = 7k$$
$$\cancel{n} - p = 7l$$

$$m - p = 7k + 7l$$

$$m - p = 7(k+l)$$

So $m - p = 7\,(k+l)$ *and hence* $m\,R\,p$. *This shows that* R *is transitive*

$m\,R\,n$ *and* $n\,R\,p \Longrightarrow m\,R\,p$

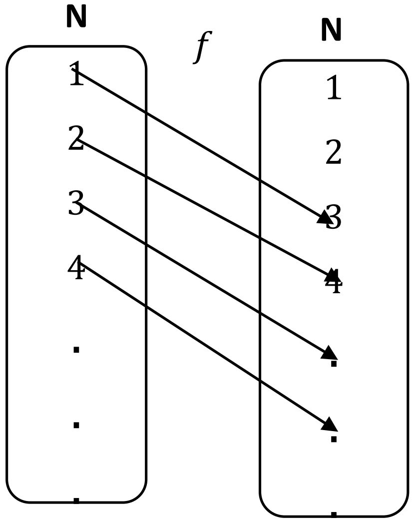
Thus R *is an equivalence relation.*

$EXERCISE : 1.3$

Example 1.14

Check whether the following functions are one-to-one and onto.
(i) $f:\mathbb{N}\to\mathbb{N}$ defined by $f(n) = n+2$. (ii) $f:\mathbb{N}\cup\{-1,0\}\to\mathbb{N}$ defined by $f(n) = n+2$.

(i) If $f(n) = f(m)$, then $n+2 = m+2$ and hence $m = n$.



$f(n) = n+2.$

$n = 1: f(1) = 1 + 2 = 3$

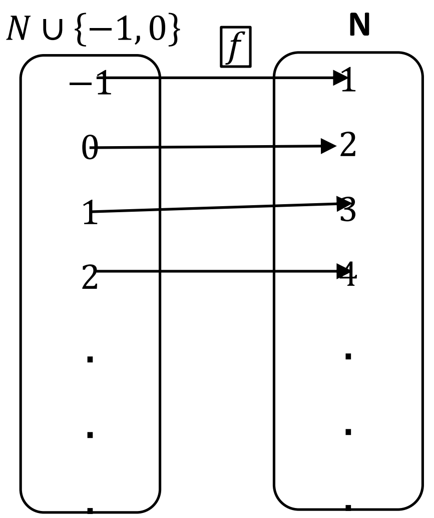
$n = 2: f(2) = 2 + 2 = 4$

$n = 3: f(3) = 3 + 2 = 5$

$\vdots$

(i) Thus f is one$-$to$-$one. As 1 & 2 has no pre$-$image, this function is not onto.

(ii) $f:\mathbb{N}\cup\{-1,0\}\to\mathbb{N}$ defined by $f(n) = n+2$.



$f(n) = n+2$

$n = -1: f(-1) = -1 + 2 = 1$

$n = 0: f(0) = 0 + 2 = 2$

$n = 1: f(1) = 1 + 2 = 3$

$n = 2: f(2) = 2 + 2 = 4$

$\vdots$

(ii) As above, this function is one-to-one.

$f(n) = m$

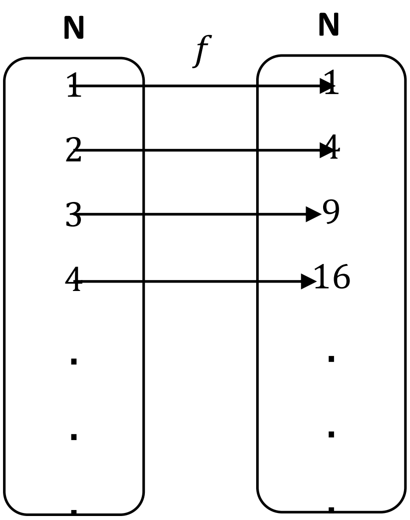
$m = n + 2 \Rightarrow n = m - 2$

$f(m - 2) = m - 2 + 2$

$f(m - 2) = m \quad thus\ m\ has\ a\ pre-image\ and\ hence\ this\ function\ is\ onto$

Example 1.15

Check the following functions for one-to-one and ontoness.
(i) f: ℕ→ℕ defined by $f(n) = n^2$ (ii) f: ℝ→ ℝ defined by $f(n) = n^2$



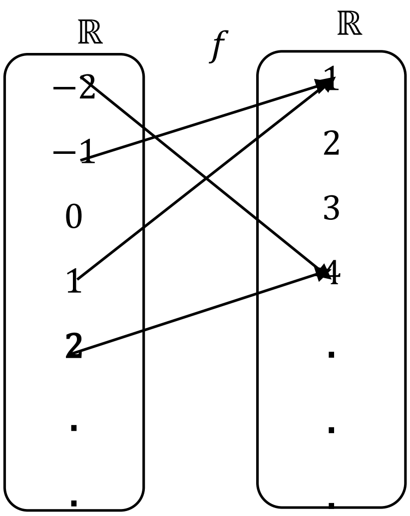
(i) $f(n) = f(m)$

$n^2 = m^2 \implies n = m$

since $m, n \in ℕ$. Thus f is one−to−one

But, non-perfect square elements in the co-domain do not have pre-images and hence not onto.

(ii) f: ℝ→ ℝ defined by $f(n) = n^2$



(ii) Two different elements in the domain have same images and hence f is not one-to-one. Clearly the range of f is a proper subset of ℝ . Thus it is not onto.

Example 1.16

Check whether the following for one − to − oneness and ontones

$(i) f\colon \mathbb{R} \to \mathbb{R}$ ***defined by*** $f(x) = \dfrac{1}{x}$ $(ii) f\colon \mathbb{R} - \{0\} \to \mathbb{R}$ ***defined by*** $f(x) = \dfrac{1}{x}$

(i) *This is not at all a function because* $f(x)$ *is not defined for* $x = 0$.

(ii) *This function is one − to one (verify) but not onto because* 0 *has no pre − image.*

Example 1.17

If $f: \mathbb{R} - \{-1,1\} \to \mathbb{R}$ defined by $f(x) = \dfrac{x}{x^2-1}$, verify whether f is one−to−one or not.

We start with the assumption $f(x) = f(y)$. *Then,*

$$\frac{x}{x^2-1} = \frac{y}{y^2-1} \Rightarrow x(y^2-1) = y(x^2-1)$$

$$xy^2 - x = yx^2 - y$$

$$xy^2 - x - yx^2 + y = 0 \Rightarrow y - x + xy^2 - yx^2 = 0$$

$$y - x + xy\,(y-x) = 0 \Rightarrow (y-x)(1+xy) = 0$$

$$y - x = 0 \text{ and } 1 + xy = 0 \Rightarrow xy = -1$$

$$x = y$$

So, if we select two numbers x and y so that $xy = -1$, then $f(x) = f(y)$.

$$y = -\frac{1}{x}$$

$$\text{when } x = 2 \Rightarrow y = -\frac{1}{2} \quad \therefore \left(2, -\frac{1}{2}\right)$$

$$\text{when } x = 7 \Rightarrow y = -\frac{1}{7} \quad \therefore \left(7, -\frac{1}{7}\right)$$

$$\text{when } x = -2 \Rightarrow y = \frac{1}{2} \quad \therefore \left(-2, \frac{1}{2}\right)$$

$\left(2, -\frac{1}{2}\right), \left(7, -\frac{1}{7}\right), \left(-2, \frac{1}{2}\right)$ are some among the infinitely many possible pairs

$$f(x) = \frac{x}{x^2-1}$$

$$f(2) = \frac{2}{2^2-1} = \frac{2}{3}$$

$$f(y) = \frac{y}{y^2-1}$$

$$f\left(-\frac{1}{2}\right) = \frac{-\frac{1}{2}}{\left(-\frac{1}{2}\right)^2 - 1} = \frac{-\frac{1}{2}}{\frac{1}{4} - 1} = \frac{-\frac{1}{2}}{-\frac{3}{4}} = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}$$

$$f(2) = f\left(-\frac{1}{2}\right) = \frac{2}{3}$$

i.e. $f(x) = f(y)$ *does not imply* $x \quad y$. *Hence it is not one* − *to* − *one.*

Example 1.18

If f: $\mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = 2x^2 - 1$, find pre−image of 17, 4 and –2.

To find the pre−image of 17 $i.e\ f(x) = 17$

$$2x^2 - 1 = 17 \implies 2x^2 = 17 + 1$$

$$2x^2 = 18 \implies x^2 = 9$$

$$x = \sqrt{9}$$

$$x = \pm 3$$

3 and –3 are the pre−images of 17 under f.

To find the pre−image of 4 $i.e\ f(x) = 4$

$$2x^2 - 1 = 4 \implies 2x^2 = 4 + 1$$

$$2x^2 = 5 \implies x^2 = \frac{5}{2}$$

$$x = \pm\sqrt{\frac{5}{2}}$$

$\sqrt{\frac{5}{2}}$ and $-\sqrt{\frac{5}{2}}$ are the pre−images of 4 under f.

To find the pre−image of -2 $i.e\ f(x) = -2$

$$2x^2 - 1 = -2 \implies 2x^2 = -2 + 1$$

$$2x^2 = -1 \implies x^2 = -\frac{1}{2}$$

x^2 *negative and* x *will be imaginary* which has no solution in $\mathbb{R}$ and hence – 2 has no pre−image under f.

Example 1.19

If f: [–2, 2]→B is given by $f(x) = 2x^3$, then find B so that f is onto. $\boldsymbol{f(x) = 2x^3}$

$$x = -2 \implies f(-2) = 2(-2)^3$$

$$f(-2) = 2(-8)$$

$$f(-2) = -16$$

$$x = 2 \implies f(2) = 2(2)^3$$

$$f(2) = 2(8)$$

$$f(2) = 16$$

when $x = -2$, *minimum value is* $f(-2) = -16$

when $x = 2$ *maximum value is* $f(2) = 16$

So B, is [–16, 16].

Example 1.19

If f: [–2, 2]→B is given by $f(x) = 2x^3$, then find B so that f is onto. $f(x) = 2x^3$

$x = -2 \Rightarrow f(-2) = 2(-2)^3$

$f(-2) = 2(-8)$

$f(-2) = -16$

$x = 2 \Rightarrow f(2) = 2(2)^3$

$f(2) = 2(8)$

$f(2) = 16$

when $x = -2$, *minimum value is* $f(-2) = -16$

when $x = 2$ *maximum value is* $f(2) = 16$

So B, is [–16, 16].

Example 1.20

Check whether the function $f(x) = x\,|x|$, defined on [–2, 2] is one–to–one or not. If it is one–to–one, find a suitable co–domain so that the function becomes a bijection.

Let $x, y \in [-2,2]$ *such that* $f(x) = f(y)$. *If* $y = 0$, *then* $x = 0$

let $y \neq 0$ *and hence* $x \neq 0$

since $f(x) = f(y)$

$x|x| = y|y| \Rightarrow \frac{x}{y} = \frac{|y|}{|x|} \Rightarrow \frac{x}{y} = \frac{y}{x}$ *since* $\frac{|y|}{|x|} > 0, \frac{x}{y} > 0$

x and y are either both positive or both negative and hence $x^2 = y^2$

so if $f(x) = f(y)$ *we must have* $x^2 = y^2$

$\sqrt{x^2} = \sqrt{y^2} \Rightarrow \pm x = \pm y$

. Also x and y are either both negative or both positive

. This is possible only if $x = y$

Thus f is one − to − one.

$f(x) = x\,|x| \Rightarrow f(x) = x\,(\pm x)$

When $x < 0, f(x) = x(-x) \Rightarrow x < 0, f(x) = -x^2$

When $x \geq 0, f(x) = x(x) \Rightarrow f(x) = x^2$

$x = -2: f(x) = -x^2$

$f(-2) = -(-2)^2 \Rightarrow f(-2) = -4$

$x = 2: f(x) = x^2$

$f(2) = (2)^2 \Rightarrow f(2) = 4$

So the range is $[-4, 4]$. *So f becomes a bijection from* $[-2, 2]$ *to* $[-4, 4]$

Example 1.21

Find the largest possible domain for the real valued function f defined by

$$f(x) = \sqrt{x^2 - 5x + 6}$$

$$x^2 - 5x + 6 \geq 0 \; for\; all\; x\; in\; the\; Domain$$

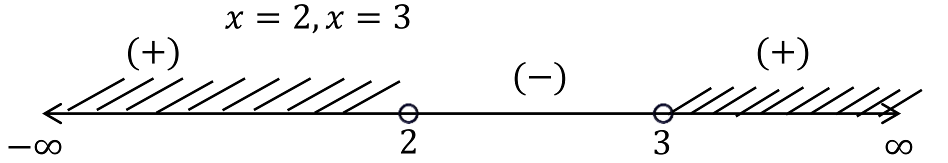
$$(x-2)(x-3) \geq 0$$

Equating the linear factors to zero.

$$(x-2)(x-3) = 0$$

$$x - 2 = 0, x - 3 = 0$$

$$x = 2, x = 3$$



$We\ have\ three\ intervals\ (-\infty, 2), (2,3)\ and (3, \infty).$

$Since\ the\ sign\ of (x-2)(x-3) is\ positive$

$select\ the\ intervals\ in\ which (x-2)(x-3) is\ positive$

$$\therefore x \in (-\infty, 2] \cup [3, \infty)$$

$(i)\ Take\ any\ point\ in\ (-\infty, 2), say\ x = 1. Clearly\ x^2 - 5x + 6\ is\ positive$

$(ii)\ Take\ any\ point\ in\ (2,3),\ say\ x = 2.5\ \ Clearly\ x^2 - 5x + 6\ is\ negative$

$(iii)\ Take\ any\ point\ in (3, \infty), say\ x = 4\ \ Clearly\ x^2 - 5x + 6\ is\ positive$

$For\ all, x\ in\ the\ intervals (-\infty, 2)\ and (3, \infty), x^2 - 5x + 6\ is\ positive$

$At\ x = 2, 3\ the\ value\ of x^2 - 5x + 6\ is\ zero\ .$

$$\sqrt{x^2 - 5x + 6}\ is\ defined\ for\ x \in (-\infty, 2] \cup [3, \infty)$$

$\boldsymbol{Example}: \mathbf{1.25}$

$\boldsymbol{Let\ f = \{(1,2), (3,4), (2,2)\}\ g = \{(2,1), (3,1), (4,2)\} Find\ g \circ f\ and\ f \circ g.}$

$f = \{(1,2), (3,4), (2,2)\}$

$$Domain\ of\ f = \{1,2,3\}$$

$$Range\ of\ f = \{2,4\}$$

$$g = \{(2,1), (3,1), (4,2)\}$$

$$Domain\ of\ g = \{2,3,4\}$$

$$Range\ of\ g = \{1,2\}$$

$Range\ of\ f\ is\ contained\ in\ the\ domain\ of\ g\ \ To\ find\ g \circ f$

$$g \circ f(1) = g[f(1)] = g(2) = 1$$

$$g \circ f(2) = g[f(2)] = g(2) = 1$$

$$g \circ f(3) = g[f(3)] = g(4) = 2$$

$$g \circ f = \{(1,1), (2,1), (3,2)\}$$

$To\ find\ f \circ g$

$$f \circ g(2) = f[g(2)]$$
$$= f(1) = 2$$
$$f \circ g(3) = f[g(3)]$$
$$= f(1) = 2$$
$$f \circ g(4) = f[g(4)]$$
$$= f(2) = 2$$
$$f \circ g = \{(2,2),(3,2),(4,2)\}$$

$\boldsymbol{Example: 1.26:}$ $\boldsymbol{Let\ f = \{(1,4),(2,5),(3,5)\}}$ $\boldsymbol{and\ g = \{(4,1),(5,2),(6,4)\}\ Find\ g \circ f can\ you\ find\ f \circ g.}$

$$f = \{(1,4),(2,5),(3,5)\}$$
$$Domain\ of\ f = \{1,2,3\}$$
$$Range\ of\ f = \{4,5\}$$
$$g = \{(4,1),(5,2),(6,4)\}$$
$$Domain\ of\ g = \{4,5,6\}$$
$$Range\ of\ g = \{1,2,4\}$$

$Range\ of\ f\ contained\ in\ the\ domain\ of\ g$

$To\ find\ g \circ f$

$$g \circ f(1) = g[f(1)] = g(4) = 1$$
$$g \circ f(2) = g[f(2)] = g(5) = 2$$
$$g \circ f(3) = g[f(3)] = g(5) = 2$$
$$g \circ f = \{(1,1),(2,2),(3,2)\}$$

$But\ f \circ g\ is\ not\ defined\ because\ the\ Range\ of\ g\ is\ not\ contained\ in\ Domain\ of\ f.$

$\boldsymbol{Example: 1.27}$

$\boldsymbol{Let\ f\ and\ g\ be\ the\ two\ function\ from\ \mathbb{R}\ to\ \mathbb{R}\ defined\ by}$ $\boldsymbol{f(x) = 3x - 4\ and\ g(x) = x^2 + 3. Find\ g \circ f and\ f \circ g}$

$To\ find\ g \circ f$

$$g \circ f(x) = g[f(x)]$$
$$= g(3x-4) = (3x-4)^2 + 3$$

$$= (3x)^2 - 2(3x)(4) + 4^2 + 3$$
$$= 9x^2 - 24x + 16 + 3$$
$$= 9x^2 - 24x + 19$$

To find $f \circ g$

$$f \circ g(x) = f[g(x)]$$
$$= f(x^2+3) = 3(x^2+3) - 4$$
$$= 3x^2 + 9 - 4 = 3x^2 + 5$$
$$\therefore f \circ g \neq g \circ f$$

In general composition of function is not commutative

Theorem 1.2:

$\boldsymbol{f : A \longrightarrow B}$ **and** $\boldsymbol{g : B \longrightarrow C}$ **be two functions. If** $\boldsymbol{f}$ **and** $\boldsymbol{g}$ **are one − to − one then** $\boldsymbol{g \circ f}$

proof:

Let $x \neq y$ *in* A *since* f *is one to one,* $f(x) \neq f(y)$

Since g *is one to one* $g[f(x)] \neq g[f(y)]$

That if $x \neq y \implies g \circ f(x) \neq f \circ g(y)$

Hence $g \circ f$ *is one to one*

Example: 1.29

Let $\boldsymbol{f, g : \mathbb{R} \longrightarrow \mathbb{R}}$ **be defined as** $\boldsymbol{f(x) = 2x - |x|}$ **and** $\boldsymbol{g(x) = 2x + |x|}$ **Find** $\boldsymbol{f \circ g}$.

$$|x| = \begin{cases} -x \text{ if } x \leq 0 \\ x \text{ if } x > 0 \end{cases}$$

$$f(x) = 2x - |x|$$

$$f(x) = \begin{cases} 2x - (-x) \text{ if } x \leq 0 \\ 2x - (x) \text{ if } x > 0 \end{cases} \implies f(x) = \begin{cases} 3x \text{ if } x \leq 0 \\ x \text{ if } x > 0 \end{cases}$$

$$g(x) = 2x + |x|$$

$$g(x) = \begin{cases} 2x + (-x) \text{ if } x \leq 0 \\ 2x + (x) \text{ if } x > 0 \end{cases} \implies g(x) = \begin{cases} x \text{ if } x \leq 0 \\ 3x \text{ if } x > 0 \end{cases}$$

Let $x \leq 0$ $\quad g \circ f(x) = g[f(x)] = g(3x) = 3x$

Let $x > 0$ $\quad g \circ f(x) = g[f(x)] = g(x)$

$$= 3x$$

$\therefore g \circ f(x) = 3x$ *for all* x

Example: 1.30

If $f:\mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = 2x - 3$ prove that f is a bijection and find its inverse.

Method 1: Let $f(x) = f(y)$

$$2x - 3 = 2y - 3$$

$$2x = 2y \implies x = y$$

$$\therefore f(x) = f(y) \implies x = y \text{ Thus } f \text{ is one} - \text{one}$$

onto: Let $y \in \mathbb{R}$ $y = 2x - 3$

$$y + 3 = 2x \implies x = \frac{y+3}{2}$$

$$f(x) = 2x - 3$$

$$= 2\left(\frac{y+3}{2}\right) - 3 = y + 3 - 3$$

$$= y \quad \text{Thus } f \text{ is onto.}$$

To find inverse: let $y = 2x - 3$

$$y + 3 = 2x \implies x = \frac{y+3}{2}$$

$$\boxed{\begin{array}{l} f(x) = y \\ x = f^{-1}(y) \end{array}}$$

$$f^{-1}(y) = \frac{y+3}{2}$$

By replasing y as x $\quad f^{-1}(x) = \dfrac{x+3}{2}$

Method 2:

Let $y = 2x - 3$

$$y + 3 = 2x \implies x = \frac{y+3}{2}$$

$$\text{Let } g(y) = \frac{y+3}{2}$$

$$g \circ f(x) = g[f(x)] = g(2x - 3)$$

$$= \frac{2x - 3 + 3}{2} = \frac{2x}{2} = x$$

$$(f \circ g)(y) = f[g(y)] = f\left(\frac{y+3}{2}\right)$$

$$= 2\left(\frac{y+3}{2}\right) - 3 = y + 3 - 3 = y$$

$f(x) = 2x - 3$

$g(y) = \dfrac{y+3}{2}$

$$g \circ f = I_x \text{ and } f \circ g = I_y$$

$f^{-1}(y) = \dfrac{y+3}{2}$ Replacing y by x $f^{-1}(x) = \dfrac{x+3}{2}$

Example: 1.22 : **Find the domain of** $f(x) = \dfrac{1}{1 - 2\cos x}$.

The function is defined for all $x \in \mathbb{R}$

Except: $1 - 2\cos x = 0$. $\qquad \cos\theta = \cos\alpha$

$\theta = 2n\pi \pm \alpha\ , n \in \mathbb{Z}$

$$-2\cos x = -1 \Rightarrow\ 2\cos x = 1$$

$$\cos x = \frac{1}{2}$$

$$\cos x = \cos 60^\circ \Rightarrow\ \cos x = \cos\frac{\pi}{3}$$

$$i.e\ Except\ x = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$$

Hence the domain is $\mathbb{R} - \left\{2n\pi \pm \dfrac{\pi}{3}\right\}\ , n \in \mathbb{Z}$

Example: 1.23 : **Find the range of the fucntion** $\dfrac{1}{1 - 3\cos x}$

Range of cosine is $-1 \le \cos x \le 1$

$$-1 \le \cos x \le 1$$

Multipying throughout by 3

$$-3 \le 3\cos x \le 3$$

$$3 \ge -3\cos x \ge -3$$

$$3 + 1 \ge 1 - 3\cos x \ge -3 + 1$$

$$4 \ge 1 - 3\cos x \ge -2$$

Taking reciprocal

$$\frac{1}{4} \le \frac{1}{1 - 3\cos x} \le -\frac{1}{2}$$

$$\text{Range is } \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{4}, \infty\right)$$

7. Find the largest possible domain of the real valued function

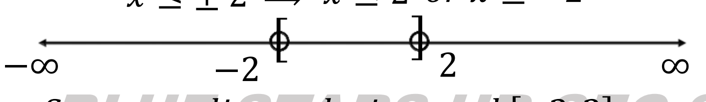
$$f(x) = \frac{\sqrt{4 - x^2}}{\sqrt{x^2 - 9}}$$

Given : $f(x) = \dfrac{\sqrt{4 - x^2}}{\sqrt{x^2 - 9}}$

$$4 - x^2 \ge 0 \Rightarrow -x^2 \ge -4$$

$$x^2 \le 4\ \Rightarrow x \le \sqrt{4}$$

$$x \le \pm 2 \Rightarrow\ x \le 2\ \text{ or } x \ge -2$$

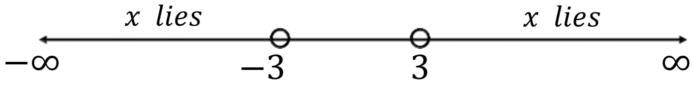


So x must lie on the interval $[-2, 2]$.

$x^2-9>0 \Longrightarrow \ x^2>9$

$x>\sqrt{9} \ \ \Longrightarrow x>\pm 3$

$x>3 \ \ or\ x<-3$



$That\ is, x\ must\ lie\ on\ (-\infty,-3)\cup(3,\infty).$

$[-2,2]\cap\{(-\infty,-3)\cup(3,\infty)\}=\{\ \}$

$For\ no\ real\ value\ of\ x, to\ define\ f(x).$

$\boldsymbol{Example: 1.24:\ Find\ the\ largest\ possible\ domain\ for\ the\ real\ valued}$

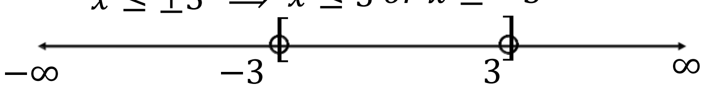
$\boldsymbol{function\ \ given\ by\ f(x)=\frac{\sqrt{9-x^2}}{\sqrt{x^2-1}}.}$

$Given: f(x)=\frac{\sqrt{9-x^2}}{\sqrt{x^2-1}}$

$9-x^2\geq 0 \Longrightarrow -x^2\geq -9$

$x^2\leq 9 \ \Longrightarrow \ x\leq\sqrt{9}$

$x\leq\pm 3 \ \Longrightarrow \ x\leq 3\ or\ x\geq -3$

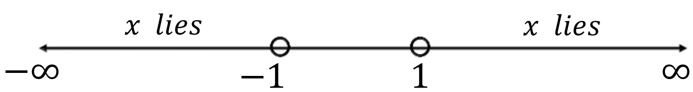


$So\ x\ must\ lie\ on\ the\ interval\ [-3,3].$

$x^2-1>0 \Longrightarrow \ x^2>1$

$x>\sqrt{1} \ \ \Longrightarrow x>\pm 1$

$x>1 \ \ or\ x<-1$



$That\ is, x\ must\ lie\ on\ (-\infty,-1)\cup(1,\infty).$

$Combining\ these\ two\ conditions, the\ largest\ possible\ domain\ for\ f\ is$

$[-3,3]\cap\{(-\infty,-1)\cup(1,\infty)\}$

$That\ is,\ [-3,-1)\cup(1,3]$

Suppose that 120 students are studying in 4 sections of eleventh standard in a school. Let A denote the set of students and B denote the set of the sections. Define a relation from A to B as " x related to y'' if the student x belongs to the section"y. Is this relation a function? What can you say about the inverse relation? Explain your answer.

$A\ Let\ A=Set\ of\ all\ 120\ students: \{1,2,3,\ldots 120\}$

$B\ =Set\ of\ sections: \ \{A,B,C,D\}$

$R: x\ R\ y \Longrightarrow x$ *is any one of the* 120 *students belongs to any one of the sections* y.

(i) *Since every student will belong to any one of the* 4 *classes, all the elements of* A *will have a unique image in* B

[*A student will not belong to more than* 1 *section*], *this relation is a function.*

(ii) *All the* 4 *sections will have students.*

In other words all the elements of B *will have per image in* A. *therefore it is onto function.*

(iii) *It is not one − one because the number of elements in* A *is not equal to number of elements in* B.

(iv) *The inverse relation from* B *to* A *is not a fucntion, since element in* B *will not have a unique image in* A.

2. *Write the values of f at* −4, 1, −2, 7, 0

$$\boldsymbol{if\ f(x) = \begin{cases} -x+4 \ if \ -\infty < x \le -3 \\ x+4 \ if \ -3 < x < -2 \\ x^2 - x \ if \ -2 \le x < 1 \\ x - x^2 \ if \quad 1 \le x < 7 \\ 0 \qquad\qquad otherwise \end{cases}}$$

$$x = -4 \in (-\infty, -3] \ \therefore f(x) = -x + 4$$

$$f(-4) = 4 + 4 = 8$$

$$x = 1 \in [1, 7) \therefore f(x) = x - x^2$$

$$f(1) = 1 - 1^2 = 0$$

$$x = -2 \in [-2, 1) \therefore f(x) = x^2 - x$$

$$f(-2) = (-2)^2 - (-2)$$
$$= 4 + 2 = 6$$

$x = 7$ *cannot belongs to any intervals* $\therefore f(7) = 0$

$$x = 0 \in [-2, 1) \therefore f(x) = x^2 - x$$

$$f(0) = 0^2 - 0 = 0$$

3. *Write the values of f at* −3, 5, 2, −1, 0

$$\boldsymbol{if\ f(x) = \begin{cases} x^2 + x - 5 \quad if\ x \in (-\infty, 0) \\ x^2 + 3x - 2 \quad if\ x \in (3, \infty) \\ x^2 \qquad\quad if\ x \in (0, 2) \\ x^2 - 3 \quad otherwise \end{cases}}$$

$x = -3 \in (-\infty, 0) \therefore f(x) = x^2 + x - 5$

$$f(-3) = (-3)^2 - 3 - 5$$
$$= 9 - 3 - 5 = 9 - 8 = 1$$

$x = 5 \in (3, \infty) \therefore f(x) = x^2 + 3x - 2$

$$f(5) = 5^2 + 3(5) - 2 = 25 + 15 - 2 = 38$$

$x = 2 \ in \ \ f(x) = x^2 - 3$

$$f(2) = (2)^2 - 3 = 4 - 3 = 1$$

$x = -1 \in (-\infty, 0) \therefore f(x) = x^2 + x - 5$

$$f(-1) = (-1)^2 + (-1) - 5 = 1 - 1 - 5 = -5$$

$x = 0 \ in \ \ f(x) = x^2 - 3$

$$f(0) = 0^2 - 3 = -3$$

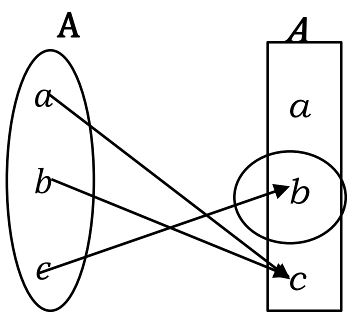
4. *State whether the following relations are functions or not. If it is a function check for one − to − oneness and ontoness. If it is not a function, state why?*

(*i*) *If* $A = \{a, b, c\}$ *and* $f = \{(a, c), (b, c), (c, b)\}; (f: A \to A)$.

(*ii*) *If* $X = \{x, y, z\}$ *and* $f = \{(x, y), (x, z), (z, x)\}; (f: X \to X)$.

(i) Let $A = \{a, b, c\}$

$f: A \to A \ defined \ by \ f: \{(a, c), (b, c), (c, b)\}$

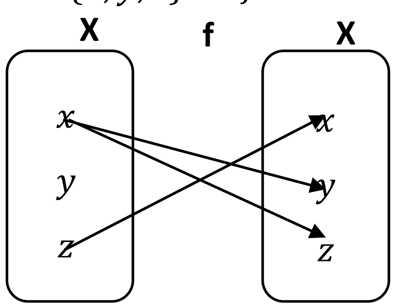


**Every elements in A has a image in A .
Hence it is a function**

(i) Different elements of A does not have different image in A .Thus, f is not one-one function

It is not onto fuction because $a \in A$ does not have pre − image in A.

$(ii) Let \ X = \{x, y, z\}$ $\quad f : X \to X \ defined \ by \ f : \{(x, y)(x, z)(z, x)\}$



It is not a fuction since x has more than one image, and y has no image in B

5. $\boldsymbol{Let\ A = \{1, 2, 3, 4\}\ and\ B = \{a, b, c, d\}. Give\ a\ function\ from\ A \to B\ for\ each\ of\ the\ followung:}$

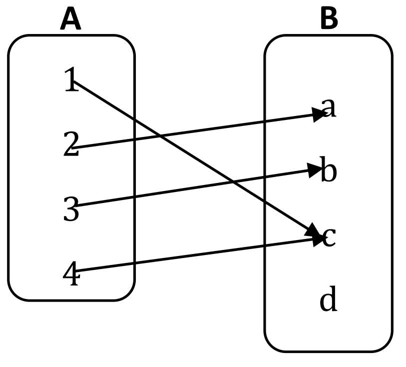
$\boldsymbol{(i)\ neither\ one - to - one\ and\ nor\ onto,}$

$\boldsymbol{(ii)\ not\ one - to - one\ but\ onto}$

$\boldsymbol{(iii)\ one - to - one\ but\ not\ onto, (iv) one - to - one\ and\ onto.}$

$Let\ A = \{1, 2, 3, 4\}\ \ B = \{a, b, c, d\}$

$(i)\ f_1 : A \to B\ defined\ by\ f_1 : \{(1, c)(2, a)(3, b)(4, c)\}$



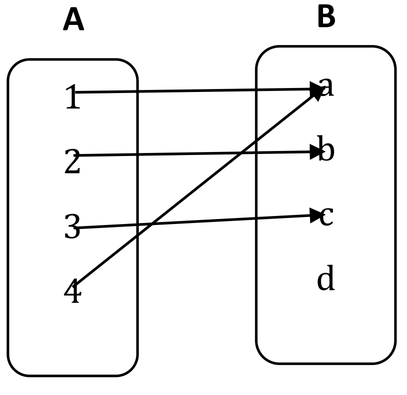
Every element in A has only One image in B . Hence it is a function.

The element d in B has no pre-image in A. Thus, it is not onto Function.

The elements 1 and 4 in A have same image c in B. Hence it is not one-one.

$f\ is\ neither\ one - to - one\ and\ nor\ onto$

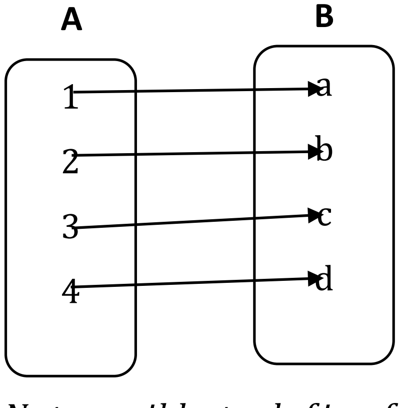
$(ii)\ not\ one - to - one\ but\ onto$



$(ii)\ Not\ possible, to\ define\ function\ which\ is\ not\ one - to - one\ but\ onto.$

$since\ n(A) = n(B)$

$(iii)\ one - to - one\ but\ not\ onto$



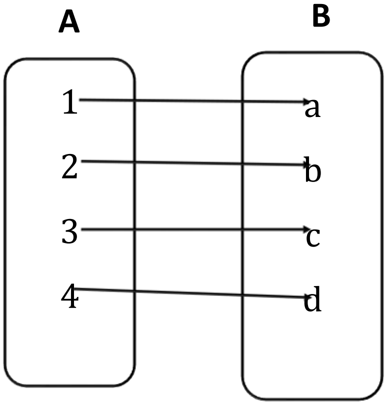
$(ii)\ Not\ possible, to\ define\ function\ which\ is\ one - to - one\ but\ not\ onto.$

$since\ n(A) = n(B)$

(iv) $one - to - one\ and\ onto.$

Different elements of A into different elements of B.

Hence, it is one-one function



Each element in B pre-image in A. Thus, it is onto. Hence, it is a bijective function.

$(iv)\ f_2 : A \rightarrow B\ defined\ by\ (1,a)(2,b)(3,c)(4,d). It\ is\ one - to - one\ and\ onto.$

6. *Find the domain of* $\mathbf{\dfrac{1}{1-2\sin x}}$

$$Let\ f(x) = \frac{1}{1-2\sin x}$$

$The\ function\ is\ defined\ for\ all\ x \in \mathbb{R}$

$Except: 1 - 2\sin x = 0 \Rightarrow 1 = 2\sin x$

$$\sin x = \frac{1}{2} \Rightarrow \sin x = \sin 30°$$

$$\sin x = \sin\frac{\pi}{6}$$

$$Except:\ x = n\pi + (-1)^n \frac{\pi}{6}$$

$$\sin\theta = \sin\alpha$$

$$\theta = n\pi + (-1)^n\alpha, n \in Z$$

$$Domain\ is\ R - \left\{n\pi + (-1)^n \frac{\pi}{6}\right\}$$

8. *Find the range of the fucntion* $\mathbf{\dfrac{1}{2\cos x - 1}}$

$Range\ of\ cosine\ is\ -1 \le \cos x \le 1$

$$-1 \le \cos x \le 1$$

$Multipying\ throughout\ by\ 2$

$$-2 \le 2\cos x \le 2$$

$$-2 - 1 \le 2\cos x - 1 \le 2 - 1$$

$$-3 \le 2\cos x - 1 \le 1$$

$$Taking\ reciprocal\quad -\frac{1}{3} \ge \frac{1}{2\cos x - 1} \ge 1$$

Range is $\left(-\infty, -\frac{1}{3}\right] \cup [1, \infty)$

9. *Show that the relation $xy = -2$ is a function for a suitable domain. Find the domain and the range of the function.*

Given : *Relation is* $xy = -2$

$$y = -\frac{2}{x} \Longrightarrow f(x) = -\frac{2}{x}$$

$$f(x) = f(y) \quad \Longrightarrow \quad -\frac{2}{y} = -\frac{2}{x}$$

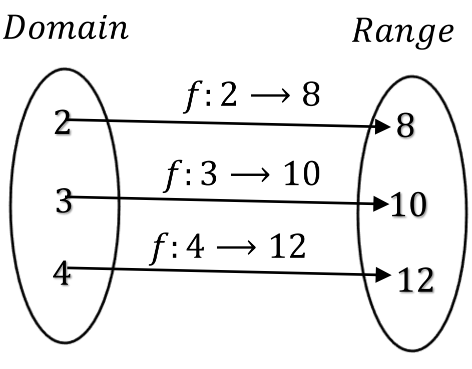
$$\frac{1}{y} = \frac{1}{x} \quad \Longrightarrow \quad x = y$$

f *is a one* $-$ *one function*

$f(x)$ *is not defined for* $x = 0$

Domain $= R - \{0\}$ *and Range* $= R - \{0\}$

$y = f(x) = 2x + 4$



$x = 2$: $f(2) = 2(2) + 4 = 4 + 4$

$f(2) = 8$

$x = 3$: $f(3) = 2(3) + 4 = 6 + 4$

$f(3) = 10$

$x = 4$: $f(4) = 2(4) + 4 = 8 + 4$

$f(4) = 12$

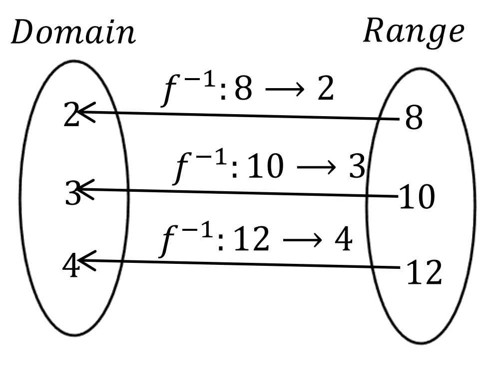
$y = 2x + 4$

$y - 4 = 2x$

$$\frac{y-4}{2} = x$$

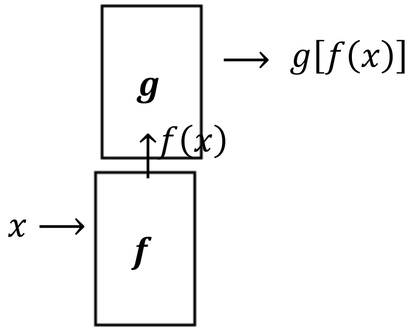
$$x = \frac{y-4}{2}$$

$$f^{-1}(y) = \frac{y-4}{2}$$



$$f^{-1}(8) = \frac{8-4}{2} = \frac{4}{2} = 2 \Longrightarrow \quad f^{-1}(10) = \frac{10-4}{2} = \frac{6}{2} = 3$$

$$f^{-1}(12) = \frac{12-4}{2} = \frac{8}{2} = 4$$



10. $\boldsymbol{If\ f, g: R \to R\ are\ defined\ by\ f(x) = |x| + x\ and\ g(x) = |x| - x,}$
$\boldsymbol{find\ g \circ f\ and\ f \circ g.}$

$f(x) = |x| + x$ *and* $g(x) = |x| - x$

$$|x| = \begin{cases} -x\ if\ x \le 0 \\ x\ if\ x > 0 \end{cases}$$

$$f(x) = |x| + x$$

$$f(x) = \begin{cases} -x + x\ if\ x \le 0 \\ x + x \quad if\ x > 0 \end{cases} \implies f(x) = \begin{cases} 0 \quad if\ x \le 0 \\ 2x \quad if\ x > 0 \end{cases}$$

$$g(x) = |x| - x$$

$$g(x) = \begin{cases} -x - x \quad if\ x \le 0 \\ x - x \quad if\ x > 0 \end{cases} \implies g(x) = \begin{cases} -2x\ if\ x \le 0 \\ 0\ if\ x > 0 \end{cases}$$

$Let\ x \le 0,\ then\ (g \circ f)x = g\big(f(x)\big) = g(0) = 0$

$Next, Let\ x \le 0, then\ (f \circ g)x = f\big(g(x)\big) = f(-2x) = 0$

$Let\ x > 0, then\ (g \circ f)x = g\big(f(x)\big) = g(2x) = 0$

$x > 0, then\ (f \circ g)x = f\big(g(x)\big) = f(0) = 0$

11. $\boldsymbol{If\ f, g, h\ are\ real\ valued\ functions\ defined\ on\ R, then\ prove\ that}$
$\boldsymbol{(f + g) \circ h = f \circ h + g \circ h\ \ What\ can\ you\ say\ about\ f \circ (g + h)?}$
$\boldsymbol{Justify\ your\ answer.}$

To prove $(f + g) \circ h = f \circ h + g \circ h$

$$[(f + g) \circ h](x) = (f + g)[h(x)]$$
$$= f[h(x)] + g[h(x)]$$
$$= (f \circ h)x + (g \circ h)x$$
$$(f + g) \circ h = f \circ h + g \circ h$$

$$f\circ(g+h)(x)=f[(g+h)(x)]$$
$$=f[g(x)+h(x)]=f[g(x)]+f[h(x)]$$
$$=(f\circ g)x+(f\circ h)(x)$$
$$f\circ(g+h)=(f\circ g)+(f\circ h)$$
$$(f+g)\circ h\neq f\circ(g+h)$$

12. $\boldsymbol{If\ f: R\to R\ is\ defined\ by\ f(x)=3x-5, prove\ that\ f\ is\ a\ bijection\ and\ find\ its\ inverse.}$

$f: R\to R\ be\ defined\ as\ f(x)=3x-5$

$Let\ y=3x-5 .Then\ x=\dfrac{y+5}{3}$

$$y+5=3x$$
$$\frac{y+5}{3}=x$$

$Let\ g(y)=\dfrac{y+5}{3}$

$$(f\circ g)(y)=f[g(y)]$$
$$=f\left(\frac{y+5}{3}\right)=3\left(\frac{y+5}{3}\right)-5=y+5-5$$
$$=y$$

$$Now\ (g\circ f)(x)=\ g[f(x)]$$
$$=g\ (3x-5)=\frac{3x-5+5}{3}=x$$

$Thus\ g\circ f=I_{x,}\ f\circ g=I_y$

$f\ and\ g\ are\ bijection\ and\ inverses\ to\ each\ other.\ f\ is\ bijective.$

$$x=\frac{y+5}{3}\ \Rightarrow\ f^{-1}(y)=\frac{y+5}{3},$$
$$f^{-1}(x)=\frac{x+5}{3}$$

13. $\boldsymbol{The\ weight\ of\ the\ muscles\ of\ a\ man\ is\ a\ function\ of\ his\ body\ weight\ x\ and\ can\ be\ expressed\ as\ W(x)=0.35x. Determine\ the\ domain\ of\ this\ function.}$

$W(x)=0.35x\ (Note\ that\ x\ is\ positive\ real\ number)$

$Domain: x>0$

14. $\boldsymbol{The\ distance\ of\ an\ object\ falling\ is\ a\ function\ of\ time\ t\ and\ can\ be\ expressed\ \ as\ S(t)=-16t^2. Graph\ the\ function\ and\ determine\ if\ it\ is\ one-to-one.}$

$$S(t) = -16t^2$$

$$Suppose\ S(t_1) = S(t_2)$$

$$-16{t_1}^2 = -16{t_2}^2$$

$$t_1 = t_2\ .Hence\ it\ is\ one - one.$$

$$S(t) = -16t^2$$

$when\ t = -2 \Rightarrow S(-2) = -16(-2)^2$
$S(-2) = -16(4) = -64$

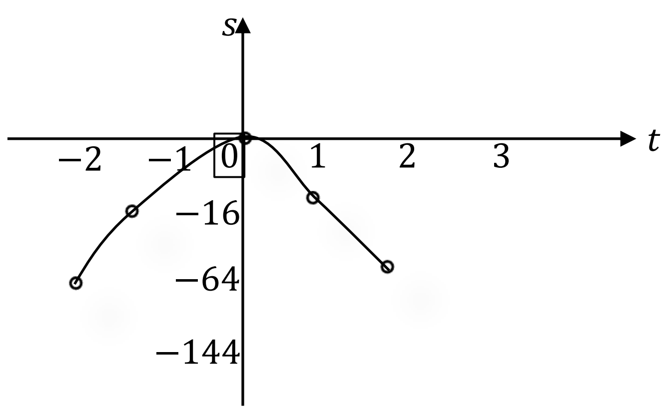
$when\ t = -1 \Rightarrow S(1) = -16(-1)^2$
$S(1) = -16$

$when\ t = 0 \Rightarrow S(0) = -16\ (0)^2$
$S(0) = 0$

$when\ t = 1 \Rightarrow S(1) = -16(1)^2$
$S(1) = -16$

$when\ t = 2 \Rightarrow S(2) = -16(2)^2$
$S(2) = -16 \times 4 = -64$

t	-1	-2	0	1	2
s	-16	-64	0	-16	-64



15. $\boldsymbol{The\ \ total\ cost\ of\ airfare\ on\ a\ given\ route\ is\ comprised\ of\ the\ base\ cost\ C\ and\ the\ \ fuel\ surcharge\ S\ in\ rupee. Both\ C\ and\ S\ are\ functions\ of\ the\ mileage\ m, C(m) = 0.4m + 50\ and\ S(m) = 0.03m.\ Determine\ a\ function\ for\ the\ total\ cost\ of\ a\ ticket\ in\ terms\ of\ the\ mileage\ and\ find\ the\ airfare\ for\ flying\ 1600\ miles.}$

$$C(m) = 0.4m + 50\ and\ S(m) = 0.03m$$

$$Total\ Cost\ T(m) = C(m) + s(m)$$

$$= 0.4m + 50 + 0.03m$$

$$T(m) = 0.43m + 50$$

$$Where\ m = 1600$$

$$T(m) = (0.43)(1600) + 50$$

$$T(m) = 43 \times 16 + 50$$
$$= 688 + 50$$
$$T(m) = Rs. 738$$

16. ***A salesperson whose annual earnings can be represented by the function*** $\boldsymbol{A(x) = 30,000 + 0.04x}$***, where x is the rupee value of the merchandise he sells. His son is also in sales and his earnings are represented by the function*** $\boldsymbol{S(x) = 25000 + 0.05x}$***. Find*** $\boldsymbol{(A+S)(x)}$ ***and determine the total family income if they each sell Rupees*** $\boldsymbol{1,50,00,000}$ ***worth of merchandise.***

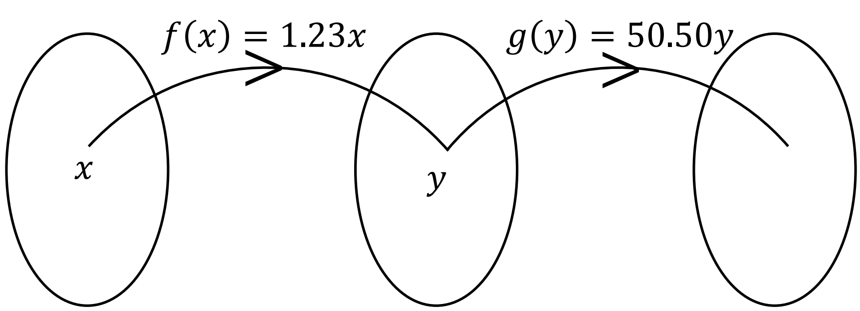
$$A(x) = 30,000 + 0.04x \quad and \quad S(x) = 25,000 + 0.05x$$
$$(A+S)x = A(x) + S(x)$$
$$= 30,000 + 0.04x + 25,000 + 0.05x$$
$$= 55,000 + 0.09x$$

$If\ x = 1,50,00,000\ then$

$$(A+S)x = 55,000 + 0.09(1,50,00,000)$$
$$= 55,000 + 1350000$$
$$= 1405000$$

17. ***The function for exchanging American dollars for Sinapore Dollar on a given day is*** $\boldsymbol{f(x) = 1.23x}$***, where x represents the number of American dollars. on the same day the function for exchanging Singapore Dollar to Indian Rupee is*** $\boldsymbol{g(y) = 50.50y}$***, where y represent the number of Singapore dollars. Write a function which will give the exchange rate of American dollars in terms of Indian rupee***

$$f(x) = 1.23x \quad (x\ is\ number\ of\ American\ dollars)$$
$$f(x)\ is\ Singapore\ dollar.\ g(y) = 50.50y$$
$$y\ is\ the\ Singapore\ dollar.\ g(y) is\ Indian\ rupee.$$



$(x\ is\ American\ dollars)$ $\quad y\ is\ Singapore\ dollar.$ $\quad g(y) is\ Indian\ rupee.$

$Exchaning\ rate\ of\ American\ dollar\ in\ terms\ of\ Indian\ rupee\ is$

$$(g\circ f)x = g[f(x)]$$
$$= g(1.23x) = 50.50(1.23x)$$
$$= 62.115x \ (x\ is\ American\ dollar)$$
$$(g\circ f)x\ is\ Indian\ rupees.$$

18. The owner of a small restaurant can prepare a particular meal at a cost of Rupees 100. He estimates that if the menu price of the meal is x rupees, then the number of customers who will order that meal at that price in an evening is given by the function $D(x) = 200 - x$. Express his day revenue, total cost and profit on this meal as function of x.

$$Number\ of\ customers = 200 - x$$
$$Cost\ of\ one\ meal = 100$$
$$Total\ cost = 100(200 - x)$$
$$Revenue\ on\ one\ meal = x$$
$$Total\ revenue = x(200 - x)$$
$$Profit = Revenue - cost$$
$$= x(200 - x) - 100(200 - x)$$
$$= (200 - x)(x - 100)$$

19. The formula for converting from Fahrenheit to Celsius temperatures is $y = \frac{5x}{9} - \frac{160}{9}$. Find the inverse of this function and determine whether the inverse is also a function.

$$If\ f(x) = \frac{5x - 160}{9}\ to\ find\ f^{-1}(x)$$

$$y = f(x)$$
$$f^{-1}(y) = x$$

$$Let\ y = \frac{5x - 160}{9}$$
$$9y = 5x - 160 \Rightarrow 5x - 160 = 9y$$
$$5x = 9y + 160$$
$$x = \frac{9y + 160}{5} \Rightarrow f^{-1}(y) = \frac{9y + 160}{5}\ Replacing\ y\ by\ x$$
$$f^{-1}(x) = \frac{9x + 160}{5}$$
$$f^{-1}(x) = \frac{9x + 160}{5}$$
$$f^{-1}(x) = f^{-1}(y)$$

$$f(x) = f(y)$$

$$\frac{5x - 160}{\cancel{9}} = \frac{5y - 160}{\cancel{9}}$$

$$5x - \cancel{160} = 5y - \cancel{160} \quad \Longrightarrow \quad x = y$$

hence it is one − one function

hence inverse function is also a function

$$y = \frac{5x}{9} - \frac{160}{9}$$

$$y = \frac{5x - 160}{9}$$

Suppose $f(x) = f(y)$

$$\frac{5x - 160}{\cancel{9}} = \frac{5y - 160}{\cancel{9}}$$

$$5x - \cancel{160} = 5y - \cancel{160} \quad \Longrightarrow \quad x = y$$

It is one − one function

20. *A simple cipher takes a number and codes it, using thefunction $f(x) = 3x - 4$. Find the inverse of this function, determine whether the inverse is also a function and verify the symmetrical property about the line $y = x$ (by drawingthe lines)*

$$f(x) = 3x - 4$$

$f(x) = y$

$x = f^{-1}(y)$

Let $y = 3x - 4 \implies y + 4 = 3x$

$$\frac{y+4}{3} = x \quad \Longrightarrow \quad x = \frac{y+4}{3}$$

$$f^{-1}(y) = \frac{y+4}{3}$$

By replasing y as x $\quad f^{-1}(x) = \dfrac{x+4}{3}$

$$f^{-1}(x) = f^{-1}(y)$$

$$f(x) = f(y)$$

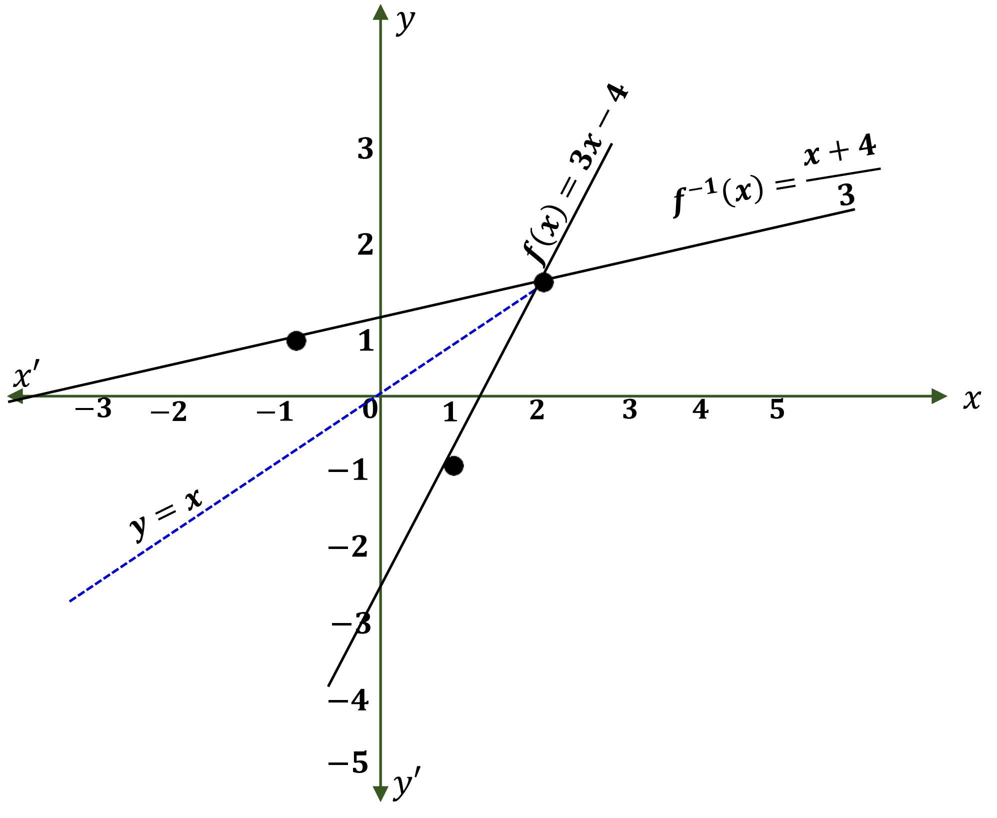
$3x - 4 = 3y - 4 \implies x = y$ $\therefore$ *inverse is also a function*

$y = 3x - 4$

x	-1	0	1	2
y	-7	-4	-1	2

$f^{-1}(x) = \dfrac{x+4}{3}$

x	-1	2
y	1	2



$EXERCISE : 1.4$

Transformation of Functions

$\boldsymbol{1. Translations \quad 2. Reflections}$

$\boldsymbol{3. Dilations\ (expands)\ or\ compresses}$

$\boldsymbol{1. Translations}$

$The\ graph\ of\ y = f(x) + c\ vertical\ shift\ up\ c\ units$

$The\ graph\ of\ y = f(x) - c\ vertical\ shift\ down\ c\ units$

$The\ graph\ of\ y = f(x + c)\ horizontal\ shift\ left\ c\ units$

$The\ graph\ of\ y = f(x - c)\ horizontal\ shift\ right\ c\ units$

$\boldsymbol{2. Reflections}$

$Reflections\ [Negative]$

$y = -f(x) \longrightarrow Reflections\ about\ x - axis$

$y = f(-x) \longrightarrow Reflections\ about\ y - axis$

$Negative\ multiple\ of\ y \longrightarrow reflect\ over\ x$

$Negative\ multiple\ of\ x \longrightarrow reflect\ over\ y$

$\boldsymbol{3. Dilations\ (Expands)\ or\ Compresses}$

$y = a\, f(x) \longrightarrow vertical\ stretch\ or\ shink$

$a > 1 \rightarrow Vertical\ shink$

$0 < a < 1 \rightarrow Vertical\ stretch$

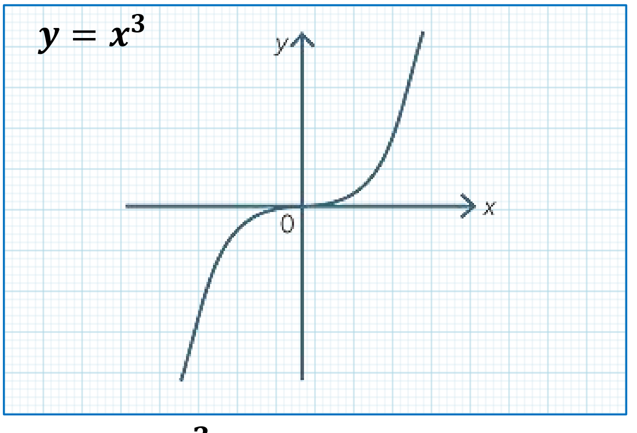
$y = f(bx) \longrightarrow Horizontal\ stretch\ or\ shink$

$b > 1 \rightarrow Horizontal\ shink$

$0 < b < 1 \rightarrow Horizontal\ stretch$

$\boldsymbol{1. For\ the\ curve\ y = x^3\ given\ in\ figure\ draw, (i)\ y = -x^3 (ii)\ y = x^3 + 1}$
$\boldsymbol{(iii)\ y = x^3 - 1\ (iv) y = (x + 1)^3\ with\ the\ same\ scale.}$

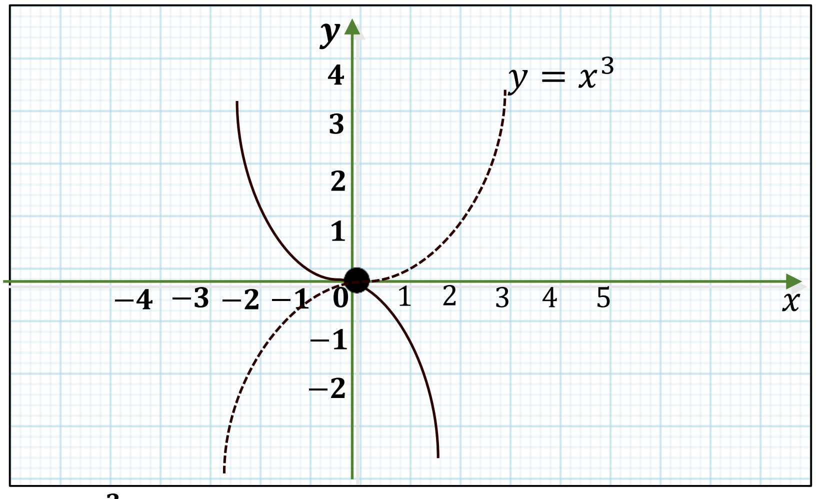
$\boldsymbol{y = x^3}$



$\boldsymbol{y = -x^3}$

$y = -f(x) \longrightarrow Reflections\ about\ x - axis$

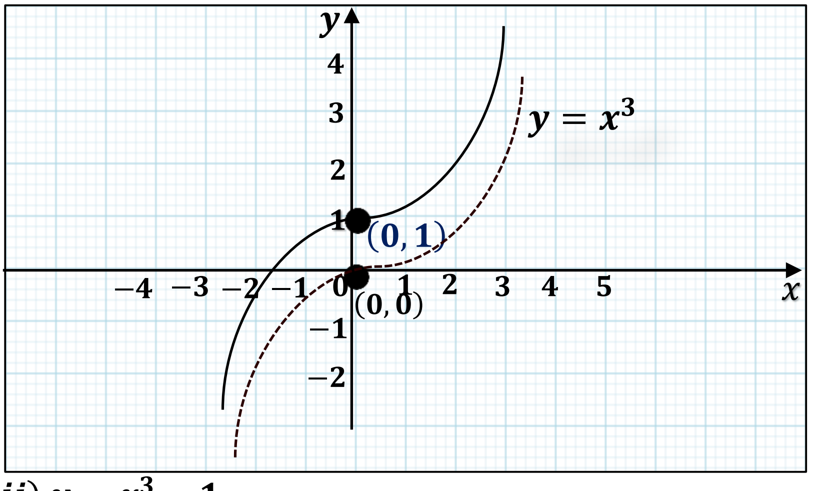
$Negative\ multiple\ of\ y\ \longrightarrow reflect\ over\ x$



$(ii)\ \boldsymbol{y = x^3 + 1.}$

$Vertical\ Translation$

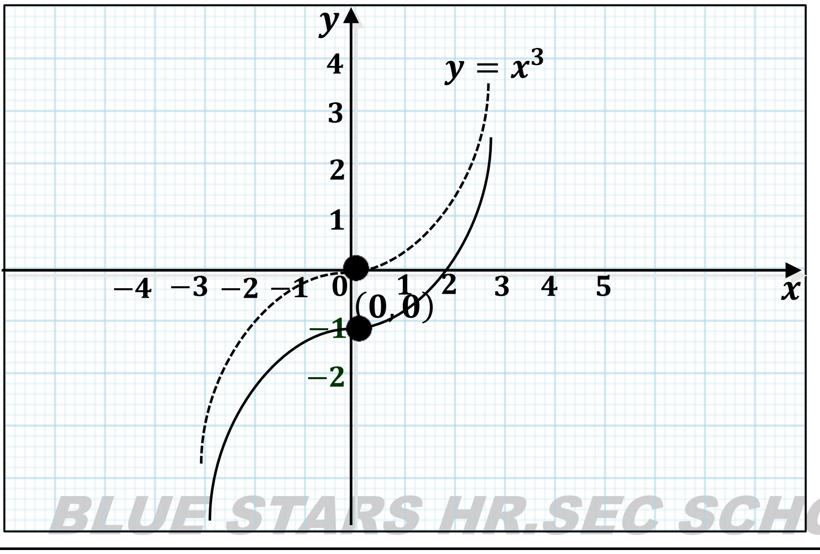
$The\ graph\ of\ y = x^3 + 1\ is\ the\ graph\ of\ y = f(x)\ shifted\ up\ 1\ units$



$(iii)\ \boldsymbol{y = x^3 - 1.}$

$Vertical\ Translation$

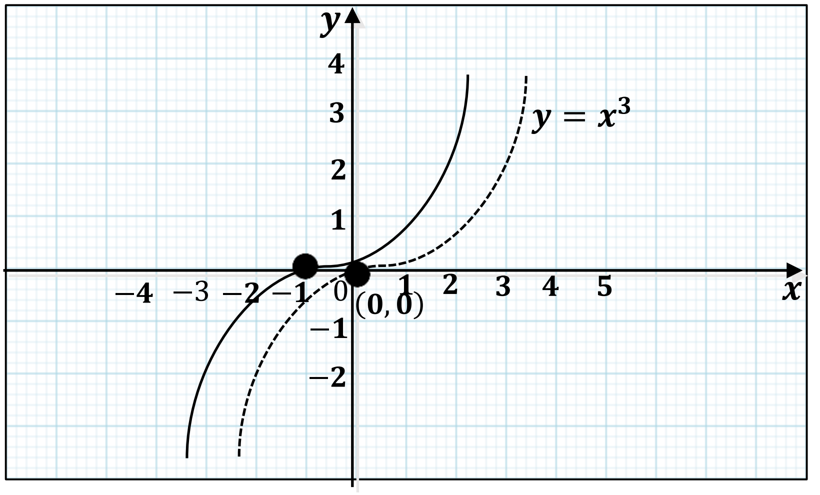
$The\ graph\ of\ y = x^3 - 1\ is\ the\ graph\ of\ y = f(x)\ shifted\ down\ 1\ units$



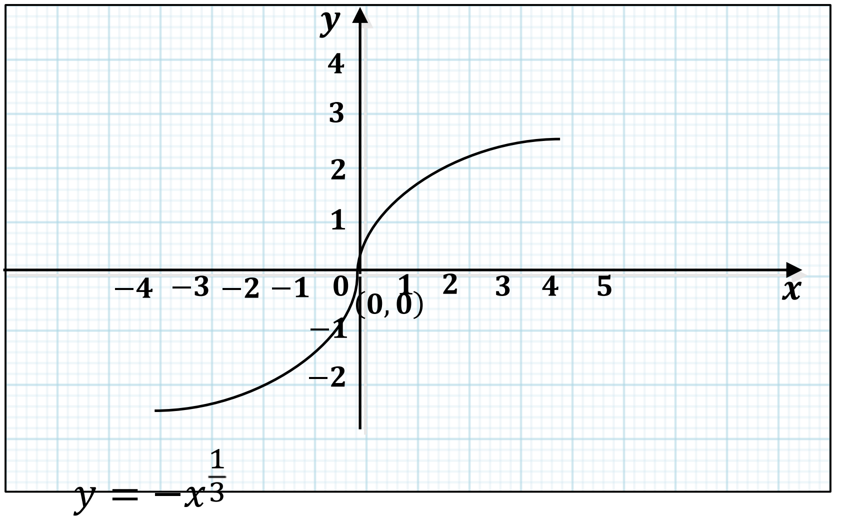
$(iv)\ y = (x+1)^3$

Horizontal Translation

The graph of $y = f(x+1)$ *is the graph of* $y = f(x)$ *shifted left* 1 *units*



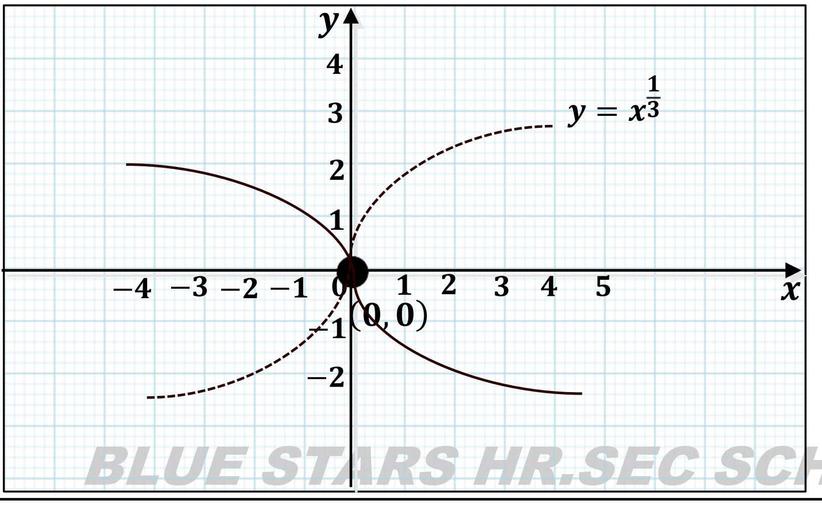
2. ***For the curve*** $y = x^{\frac{1}{3}}$



$y = -x^{\frac{1}{3}}$

$y = -f(x) \longrightarrow$ *Reflections about* $x - axis$

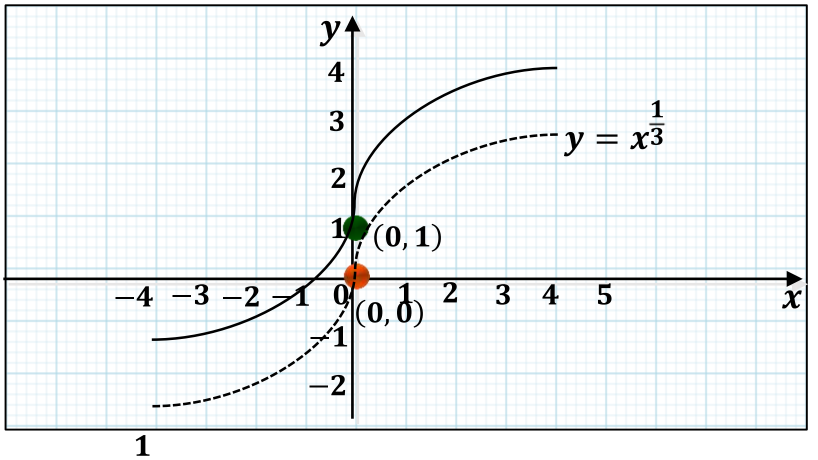
Negative multiple of $y \longrightarrow$ *reflect over* x



$$\boldsymbol{y = x^{\frac{1}{3}} + 1}$$

Vertical Translation

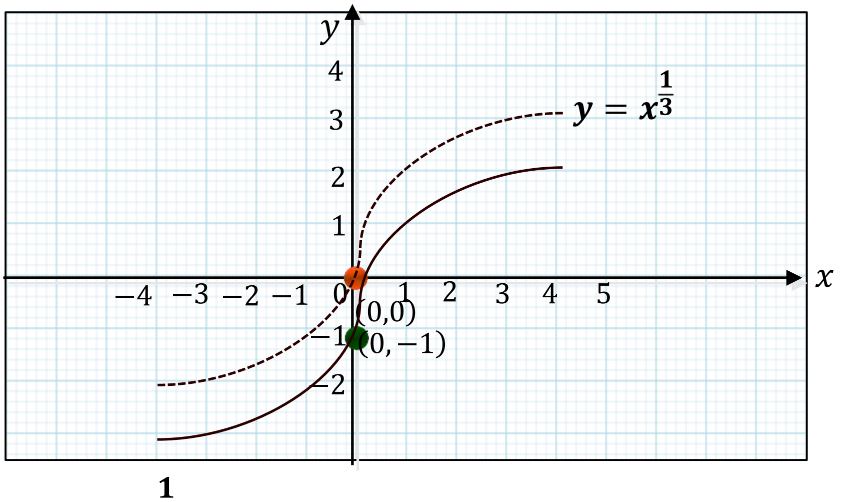
The graph of $y = x^{\frac{1}{3}} + 1$ *is the graph of* $y = f(x)$ *shifted up* 1 *units*



$$\boldsymbol{y = x^{\frac{1}{3}} - 1}$$

Vertical Translation

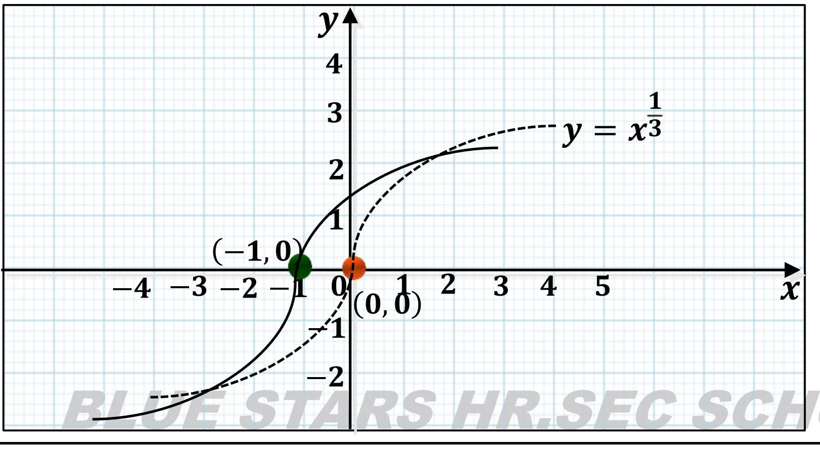
The graph of $y = x^{\frac{1}{3}} - 1$ *is the graph of* $y = f(x)$ *shifted down* 1 *units*



$$\boldsymbol{y = (x + 1)^{\frac{1}{3}}}$$

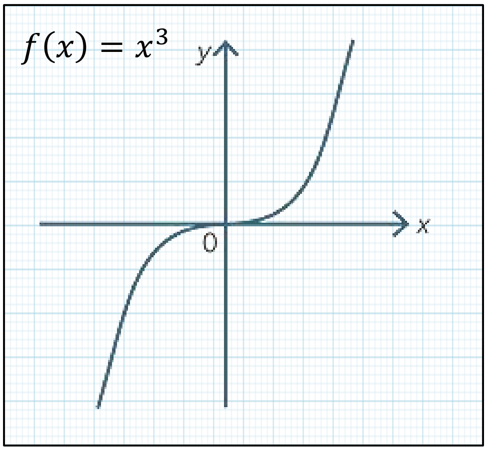
Horizontal Translation

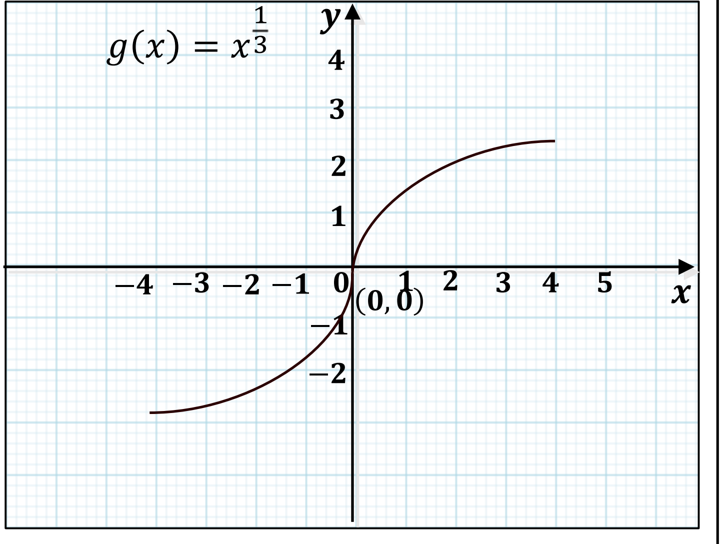
The graph of $y = (x + 1)^{\frac{1}{3}}$ *is the graph of* $y = f(x)$ *shifted left* 1 *units*



3. $\boldsymbol{Graph\ the\ functions\ f(x) = x^3\ and\ g(x) = \sqrt[3]{x}\ on\ the\ same\ co-ordinate\ plane.\ Find\ fog\ and\ graph\ it\ on\ the\ plane\ as\ well.\ Explain\ your\ results.}$

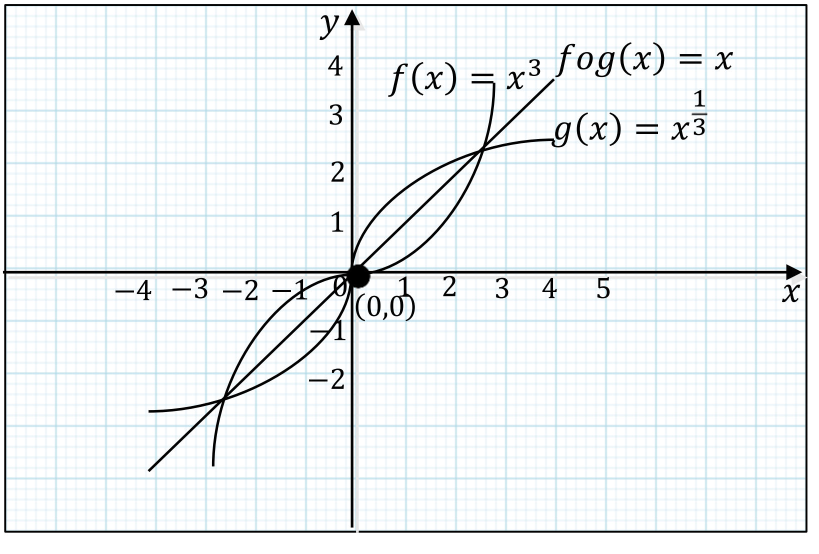
$\boldsymbol{Given\ functions\ are f(x) = x^3\ and\ g(x) = \sqrt[3]{x}.}$





$$fog(x) = f[g(x)] = f\left(x^{\frac{1}{3}}\right) = \left(x^{\frac{1}{3}}\right)^3 = x$$

$$fog = x$$



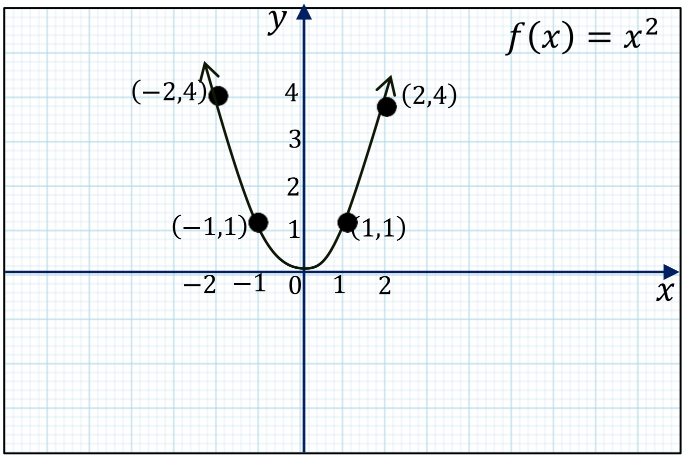
4. $\boldsymbol{Write\ the\ steps\ to\ obtain\ the\ graph\ of\ the\ function\ y = 3(x-1)^2 + 5\ from\ the\ graph\ y = x^2.}$

$Step 1$: $Draw\ the\ Graph\ of\ y = x^2$

$Step\ 2$: $Draw\ the\ Graph\ of\ y = (x-1)^2\ is\ the\ graph\ of\ y = x^2$ $shifted\ right\ 1\ units$

$Step\ 3$: $Draw\ the\ Graph\ of\ y = 3(x-1)^2\ vertical\ shink$

$Step\ 4$: $Draw\ the\ Graph\ of\ y = 3(x-1)^2 + 5\ vertical\ shift\ up\ 5\ units$

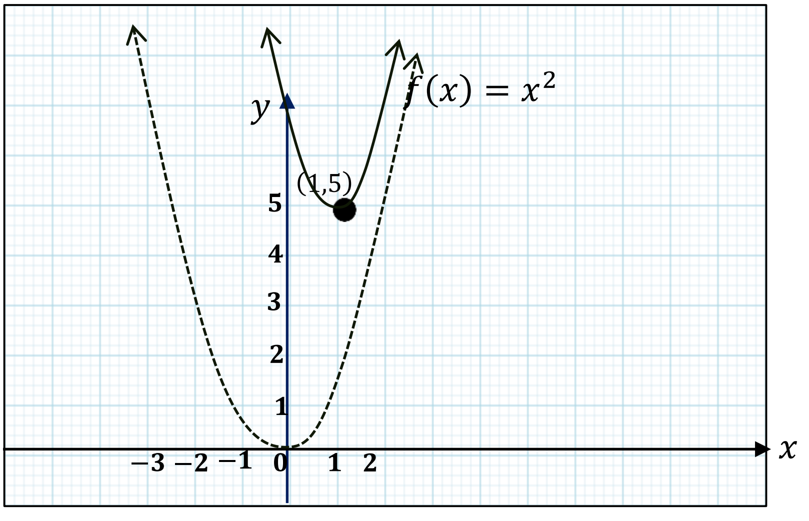


$\boldsymbol{y = 3(x-1)^2 + 5}$

$$y = 3\boxed{(x-1)^2} + 5$$

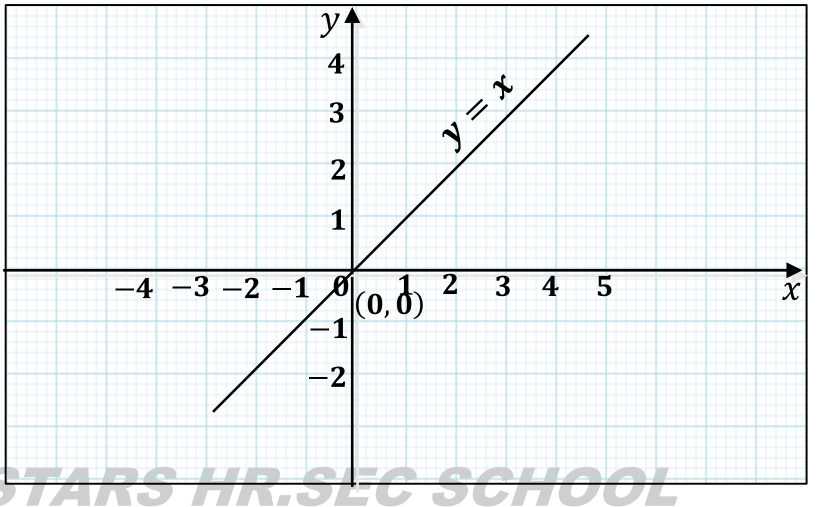
vertical shink *shift 5 units up*

shift 1 units right

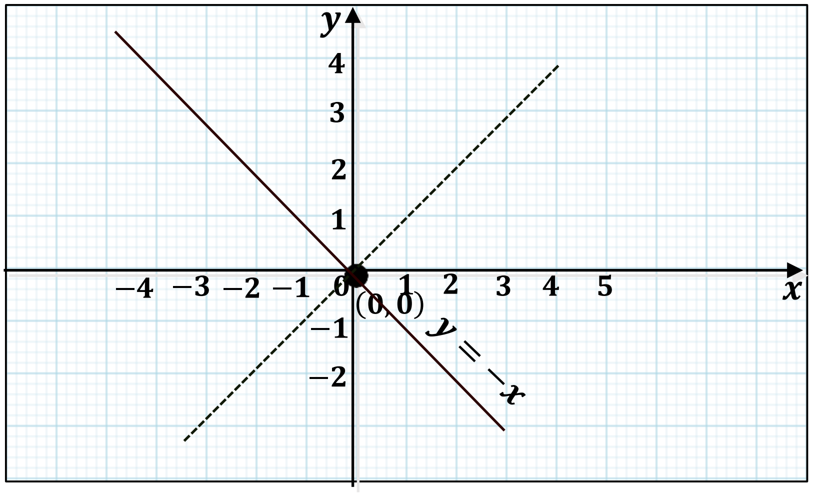


6. From the line $y = x$, draw (i) $y = -x$ (ii) $y = 2x$ (iii) $y = x + 1$ (iv) $y = \frac{1}{2}x + 1$ (v) $2x + y + 3 = 0$.

Graph of $y = x$

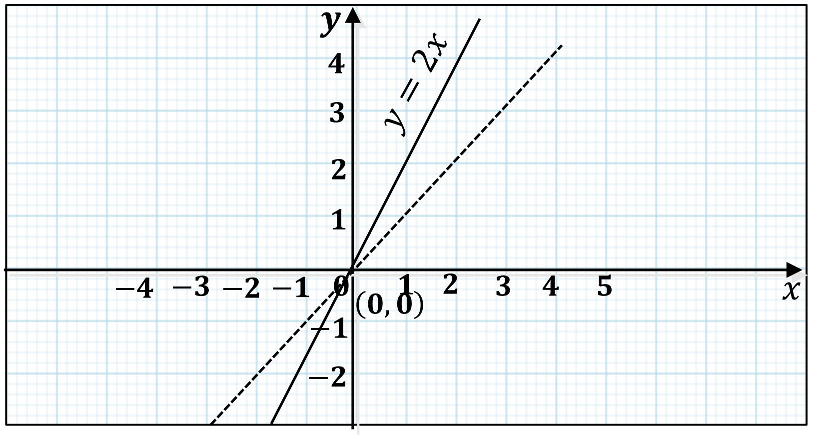


$(i)\ y = -x$



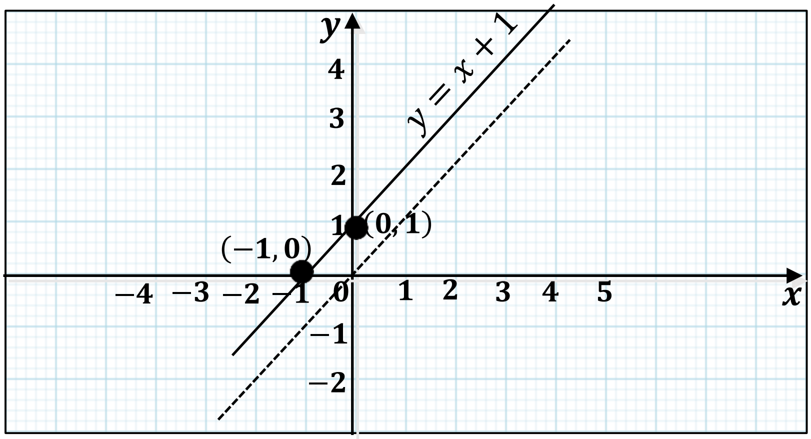
$(ii)\ y = 2x$

$y = a\, f(x) \longrightarrow vertical\ shink\ , since\ a > 1$



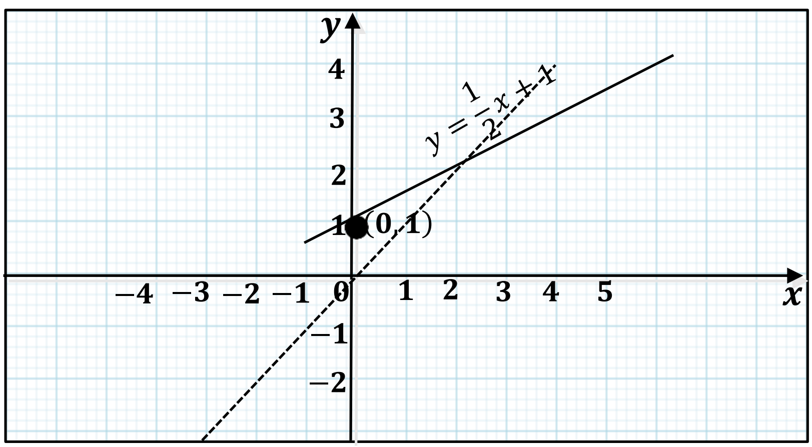
$(iii)\ \boldsymbol{y = x + 1}$

The graph of $y = x + 1$, *vertical shift up* 1 *units*



$(iv)\ \boldsymbol{y = \frac{1}{2}x + 1}$ → *vertical shift up* 1 *units*

→ *vertical expand*



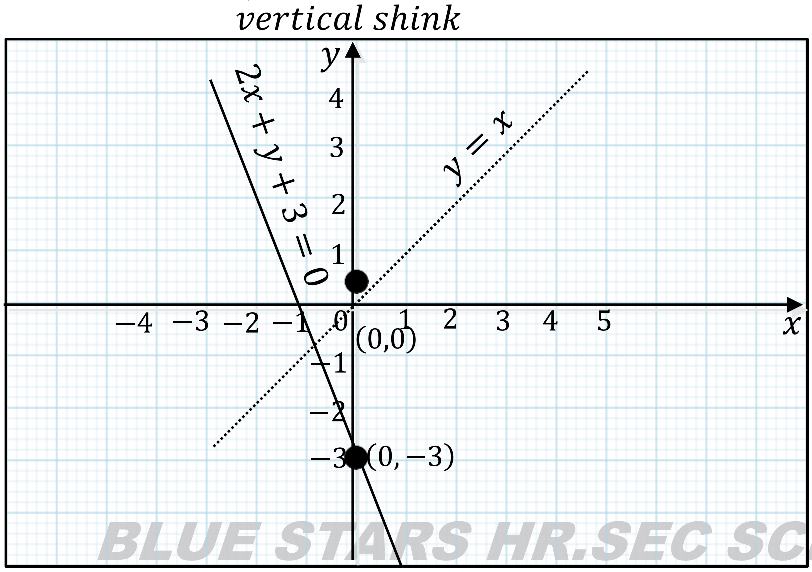
$(v)\ \boldsymbol{2x + y + 3 = 0.}$

$y = -2x - 3$

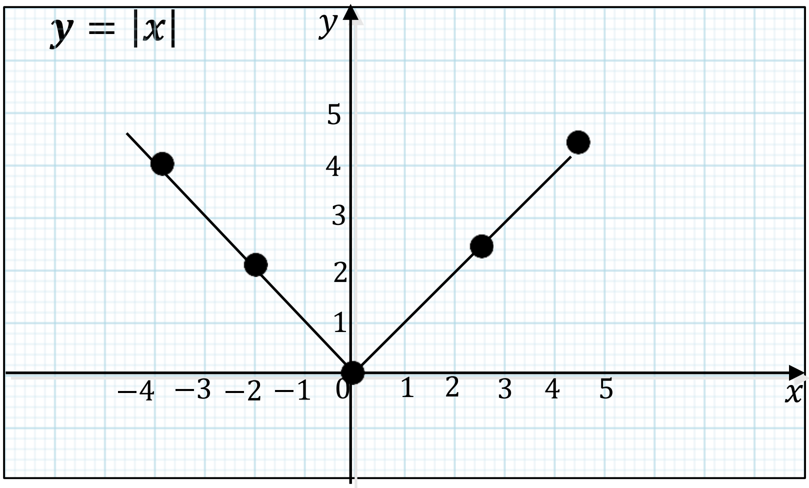
Reflections about $y -$ *axis*

vertical shift 3 *units down*

vertical shink

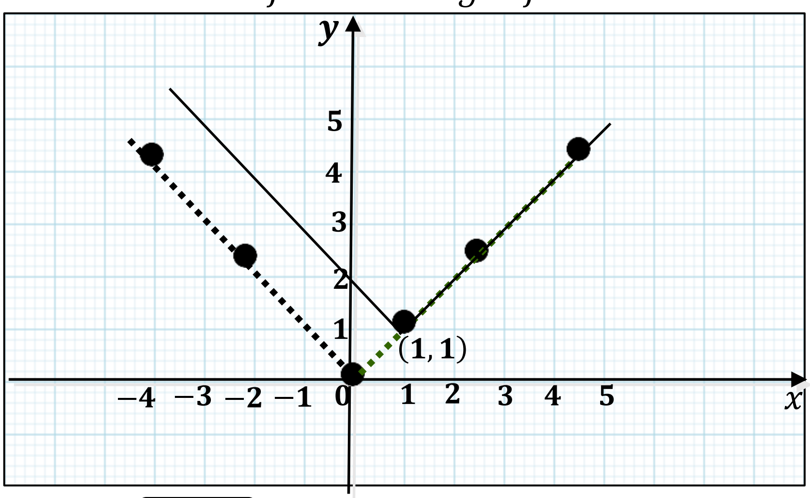


7. From the curve $y = |x|$, **draw** $(i)\ y = |x - 1| + 1$ $(ii)\ y = |x - 1| - 1$ $(iii)\ y = |x + 2| - 3$.



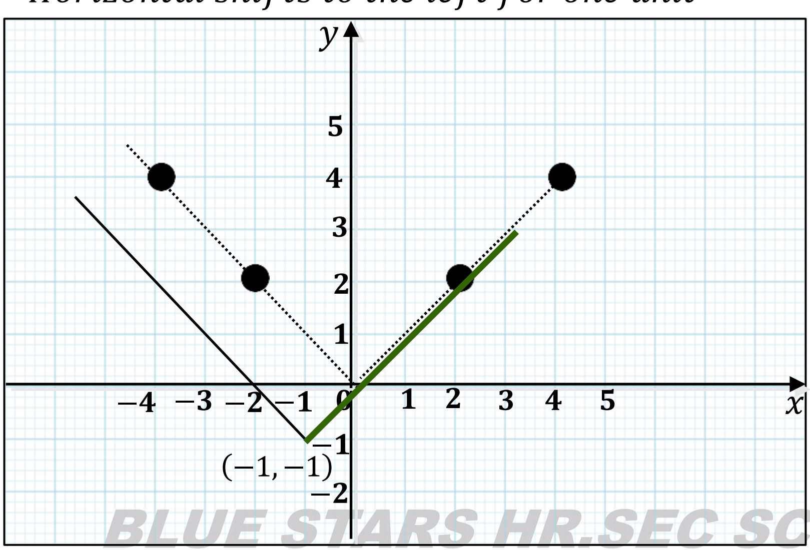
$(i)\ y = \boxed{|x - 1|} + 1 \longrightarrow$ *Vertical shifts to the up for one unit*

Horizontal shifts to the right for one unit



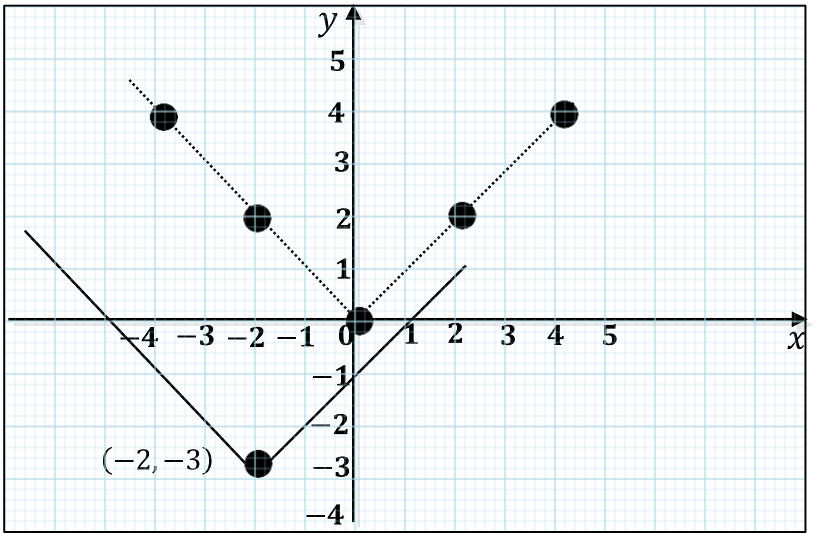
$(ii)\ y = \boxed{|x + 1|} - 1 \longrightarrow$ *Vertical shifts to the down for one unit*

Horizontal shifts to the left for one unit



$(iii)\ y = \boxed{|x+2|} - 3 \longrightarrow$ *Vertical shifts to the down for 3 unit*

Horizontal shifts to the left for 2 unit

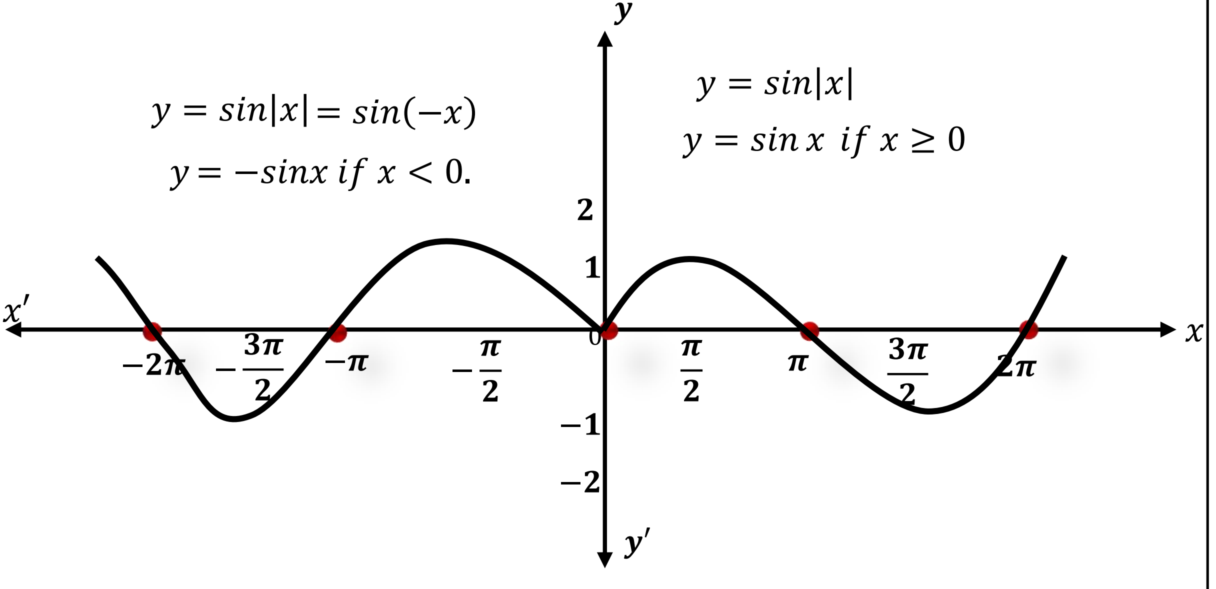


8. From the curve $y = \sin x$, draw $y = \sin|x|$ (Hint : $\sin(-x) = -\sin x$.)

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

$y = \sin|x| = \sin x \text{ if } x \geq 0$

$y = \sin|x| = \sin(-x) = -\sin x \text{ if } x < 0.$

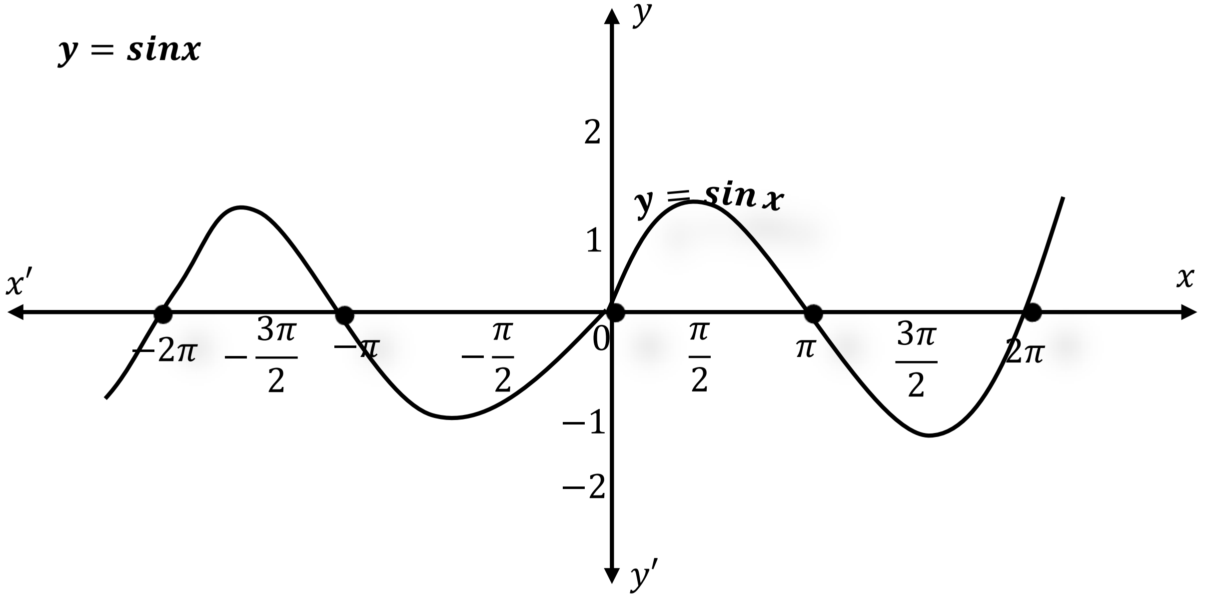


5. From the curve $y = \sin x$, graph the functions.

$(i)\ y = \sin(-x)$ $(ii)\ y = -\sin(-x)$, $(iii)\ y = \sin\left(\frac{\pi}{2} + x\right)$ which is $\cos x$.

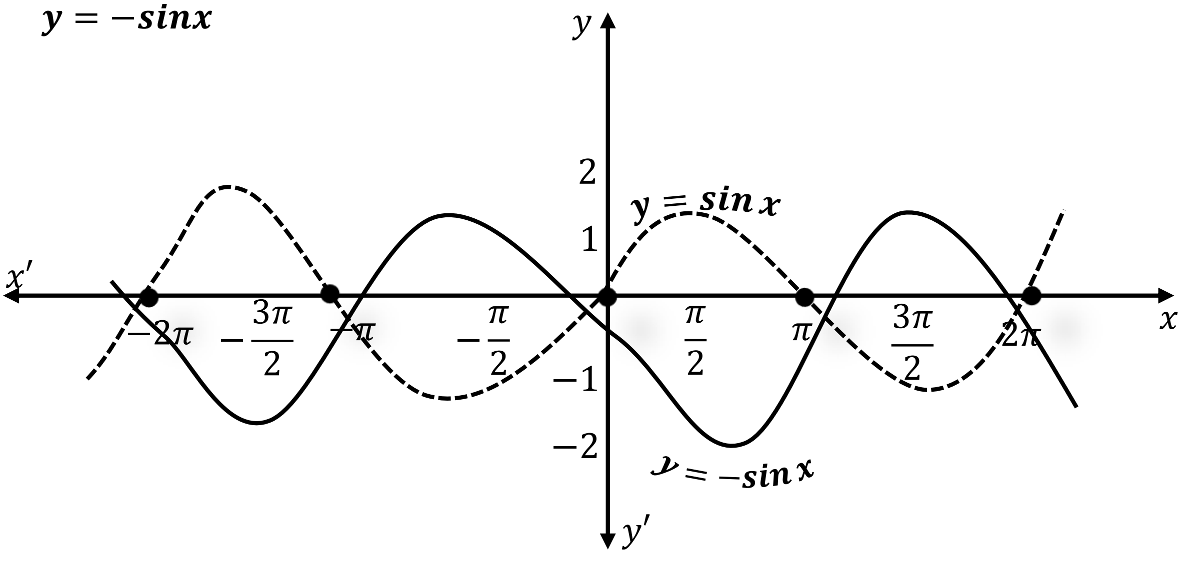
$(iv)\ y = \sin\left(\frac{\pi}{2} - x\right)$ which is also $\cos x$. (*refer trigonometry*)

$y = sinx$


y
2
$y = sin\,x$
1
x'
x
-2π
$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
-1
-2
y'

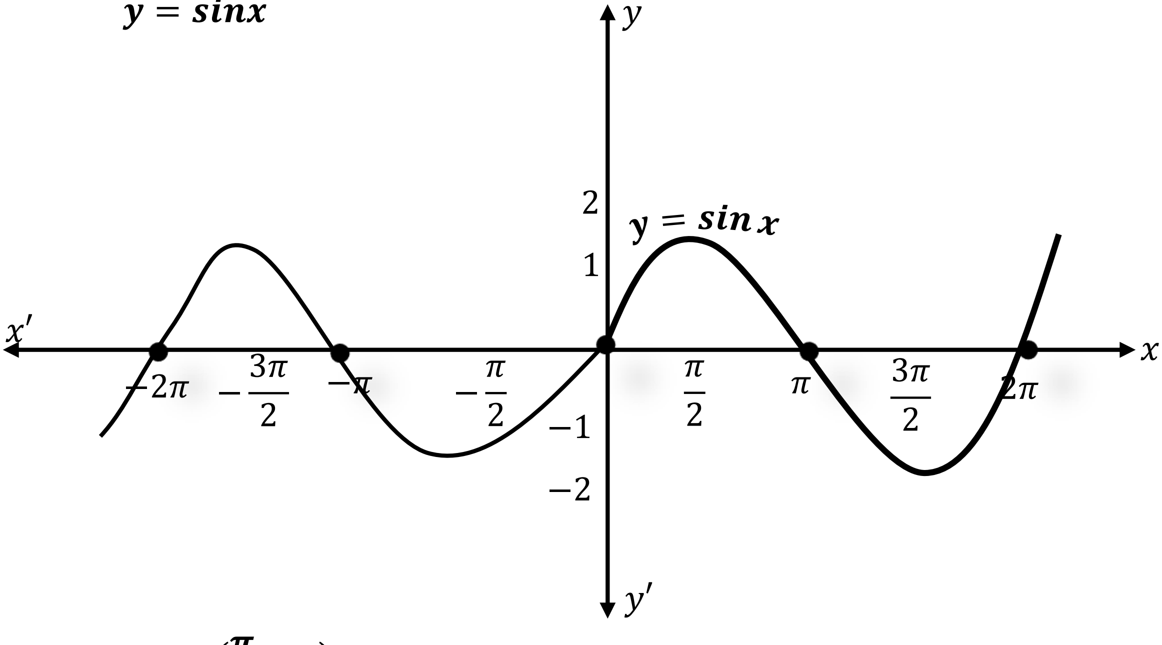
$(i)\ y = sin(-x)$

$y = -sinx$

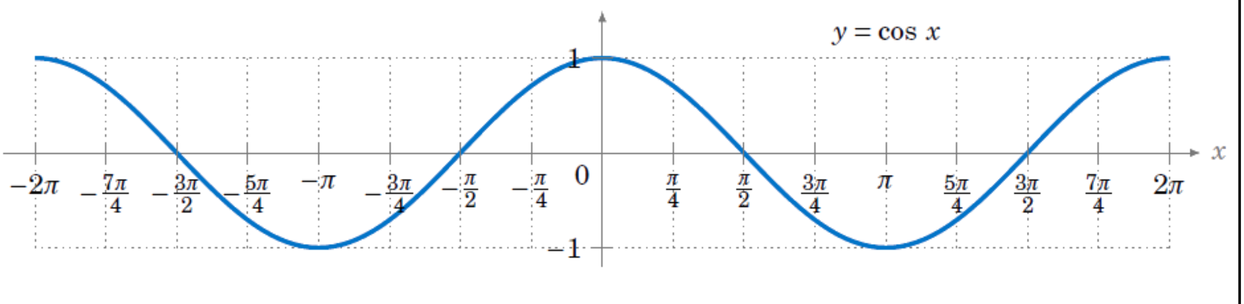


$(ii)\ y = -sin(-x)$

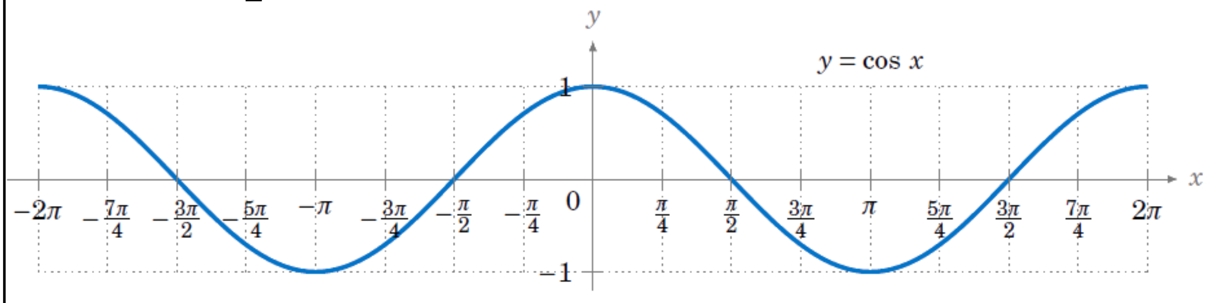
$y = sinx$



$(iii)\ y = sin\left(\frac{\pi}{2} + x\right)$ *which is cosx.*



$(iv)\ y = \sin\left(\frac{\pi}{2} - x\right)$ *which is also* $\cos x$.



BASIC ALGEBRA

EXERCISE : 2.1

2.2 REAL NUMBER SYSTEM

Rational Numbers

$\mathbb{N} = \{1,2,3,\dots\}$ *is enough for counting objects.*

In order to deal with loss or debts, we enlarged $\mathbb{N}$ *to the set of all* integers $\mathbb{Z} = \{\dots,-4,-3,-2,-1,0,1,2,\dots\}$, *which consists of the natural numbers, zero and the negatives of natural numbers.*

$\mathbb{W} = \{0,1,2,3,\dots\}$ *as the set of whole numbers*

It differs from $\mathbb{N}$ *by just one element, namely, zero*

Imagine dividing a cake into five equal parts, which is equivalent to $5x = 1$. *But this equation cannot be solved within* $\mathbb{Z}$.

Hence we have enlarged $\mathbb{Z}$ *to the set*

$\mathbb{Q} = \left\{x = \frac{m}{n} | m,n \in \mathbb{Z}, n \neq 0\right\}, x \in \mathbb{Q}$ *as a rational number.*

Some example of rational numbers are $-5, \frac{-7}{3}, 0, \frac{22}{7}, 7, 12$.

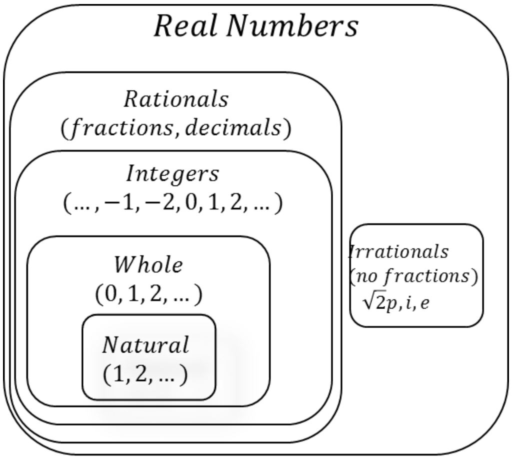
Rational numbers are precisely the set of terminating or infinite periodic decimals.

For example, $-5.0, -2.333\dots, \frac{25}{99}$

$= 0.252525\dots,$

$\frac{2}{3} = 0.66666\dots,$

$7.145278362312312312 31\dots$ *are rational numbers.*



$\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

2.2.3 Irrational Numbers

***Theorem 2.1**: $\sqrt{2}$ **is not a rational number**.*

Proof: *Suppose that* $\sqrt{2}$ *is a rational number.*

Let $\sqrt{2} = \frac{m}{n}$, *where m and n are positive integers with no common factors greater than* 1.

$$\sqrt{2} = \frac{m}{n} \Longrightarrow \sqrt{2}\,n = m \quad \underset{\text{squaring on both sides}}{\Longrightarrow} \quad 2n^2 = m^2$$

$m^2 = 2n^2$ *which implies that* m^2 *is even and hence* m *is even.*

Let $m = 2k \Longrightarrow 2n^2 = 4k^2$

$\cancel{2}n^2 = \overset{2}{\cancel{4}}k^2$ *which gives* $n^2 = 2k^2$ *Thus,* n *is also even.*

It follows, that m and n are even numbers having a common factor 2.

Thus, we arrived at a contradiction. Hence, $\sqrt{2}$ *is an irrational number.*

2.3 Absolute Value

2.3.1 *Definition and Properties*

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

(i) For any $x \in \mathbb{R}$, we have $|x| = |-x|$ and thus, $|x| = |y|$ if and only $x = y$ *or* $x = -y$.

(ii) $|x - a| = r$ *if and only if* $x - a = r$ or $x - a = -r$.

1. Classify each element of $\left\{\sqrt{7}, -\frac{1}{4}, 0, 3.14, 4, \frac{22}{7}\right\}$**, as a member of** $\mathbb{N}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}$ **or** $\mathbb{Z}$**.**

$\sqrt{7}$ *is an irrational number*

$\sqrt{7} \in \mathbb{R} - \mathbb{Q}$,

$-\frac{1}{4}$ *is a negative rational number,* $-\frac{1}{4} \in \mathbb{Q}$

0 *is an integers,* $0 \in \mathbb{Z}$,

3.14 *is an rational number,* $3.14 \in \mathbb{Q}$

$4 \in \mathbb{N}, \mathbb{Z}, \mathbb{Q}$,

$\frac{22}{7}$ *is an rational number,* $\frac{22}{7} \in \mathbb{Q}$

2. Prove that $\sqrt{3}$ is an irrational number.

Suppose that $\sqrt{3}$ is rational, then it can be written as $\frac{m}{n}$

where m and n are natural numbers with no common factor other than 1.

$$\sqrt{3} = \frac{m}{n} \implies \sqrt{3}\,n = m \implies m^2 = 3n^2$$

squaring on both sides

Since m^2 is divisible by 3 it follows that, $m = 3k$ for some natural number k.

Hence $3n^2 = (3k)^2 \implies \cancel{3}n^2 = \cancel{9}k^2$ 3

$$\boxed{n^2 = 3k^2}$$

Since n^2 is divisible by 3, it follows that $n = 3k$ for some natural number k.

Thus both m and n have a common factor 3 is a contradiction.

Hence $\sqrt{3}$ cannot be a rational number.

3. *Are there two distinct irrational numbers such that their difference is a rational number? Justify.*

If $a, b, c \in \mathbb{Q}$

Let $a + \sqrt{b}$ *and* $c + \sqrt{b}$ *are two distinct irrational number*

$$\left(a + \sqrt{b}\right) - \left(c + \sqrt{b}\right) = a + \cancel{\sqrt{b}} - c - \cancel{\sqrt{b}}$$
$$= a - c \quad \textit{is rational number.}$$

4. Find two irrational numbers such that their sum a rational number. Can you find two irrational numbers whose product is a rational number.

Let $a + \sqrt{b}$ *and* $a - \sqrt{b}$ *are two irrational number*

Sum of two irrational : $= a + \cancel{\sqrt{b}} + a - \cancel{\sqrt{b}}$

$$= a + a = 2a \quad \textit{is rational number.}$$

Product of two irrational : $\left(a - \sqrt{b}\right)\left(a + \sqrt{b}\right)$

$$= a^2 - \left(\sqrt{b}\right)^2 = a^2 - b \quad \textit{is rational number.}$$

5. *Find a positive number smaller than* $\frac{1}{2^{1000}}$ *Justify.*

Given : $\frac{1}{2^{1000}}$

$$\boxed{\frac{1}{2^1} > \frac{1}{2^2} > \frac{1}{2^3} \ldots}$$

$$1000 < 1001 \implies \frac{1}{2^{1000}} > \frac{1}{2^{1001}}$$

$\frac{1}{2^{1001}}$ *is less than* $\frac{1}{2^{1000}}$

Hence a positive number smaller than $\frac{1}{2^{1000}}$ is $\frac{1}{2^{1001}}$.

$EXERCISE : 2.2$

2.3.2 Equations Involving Absolute Value

Example 2.1: $\boldsymbol{Solve\ |2x-17| = 3\ for\ x.}$

$$|2x-17| = 3 \implies 2x-17 = \pm 3$$

$$2x = \pm 3 + 17 \implies 2x = 3+17,\ 2x = -3+17$$

$$2x = 20,\ 2x = 14 \implies x = 10,\ x = 7$$

Example 2.2: $\boldsymbol{Solve\ 3\,|x-2| + 7 = 19\ for\ x.}$

$$3|x-2| + 7 = 19 \implies 3|x-2| = 19-7$$

$$\cancel{3}|x-2| = \cancel{12}\ 4 \implies |x-2| = 4$$

$$x-2 = \pm 4 \implies x-2 = 4, x-2 = -4$$

$$x = 4+2, x = -4+2$$

$$x = 6,\ x = -2$$

$Therefore\ the\ solutions\ are\ \ x = -2\ and\ x = 6.$

Example 2.3: $\boldsymbol{Solve\ |2x-3| = |x-5|.}$

$$\boxed{\begin{array}{c}|u| = |v| \\ u = v\ or\ u = -v\end{array}}$$

$$|2x-3| = |x-5|$$

$$2x-3 = x-5\ \ or\ \ 2x-3 = -(x-5)$$

$$2x - x = -5+3\ \ or\ \ 2x-3 = -x+5$$

$$x = -2\ \ \ \ or\ \ \ \ 2x + x = 5+3$$

$$3x = 8$$

$$x = \frac{8}{3}$$

$Hence,\ x = -2\ and\ x = \dfrac{8}{3}\ are\ solutions.$

2.2.3 Some Results For Absolute Value

(i) $If\ x, y \in \mathbb{R}, |y+x| = |x-y|,\ then\ xy = 0.$

(ii) $For\ any x, y \in \mathbb{R}, |xy| = |x||y|.$

(iii) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|},\ for\ all\ \ x, y \in \mathbb{R}\ and\ y \neq 0.$

(iv) $For\ any\ x, y \in \mathbb{R}, |x+y| \leq |x| + |y|.$

$\boldsymbol{2.3.4\ \ Inequalities\ Involving\ Absolute\ Value}$

$$|x| < r$$

$Case\ (1)$: $If\ x \geq 0,\ |x| < r \Rightarrow x < r.$

$Case\ (2)$: $If\ x < 0,\ |x| < r \Rightarrow -x < r\ i.e, x > -r.$

$|x| < r$ $if\ and\ only\ if\ -r < x < r.\ that\ is\ \ x \in (-r, r).$

$|x| > r$

$consider\ |x| > r.\ if\ r < 0\ then\ every\ x \in \mathbb{R}\ satisfies\ the\ inequality$

$There\ are\ two\ possibilities\ for\ r \geq 0$

$Case\ (1)$: $If\ x \geq 0,\ |x| > r \Rightarrow x > r.$

$Case\ (2)$: $If\ x < 0, |x| > r \Rightarrow -x > r\ i.e, x < -r.$

$that\ is\ x \in (-\infty, -r) \cup (r, \infty).$

$\boldsymbol{|x - a| \leq r}$

$There\ are\ two\ possibilities$

$Case\ (1)$: $If\ x \geq 0, |x - a| \leq r \Rightarrow x - a \leq r.$

$Case\ (2)$: $If\ x < 0, |x - a| \leq r \Rightarrow -(x - a) \leq r\ i.e, (x - a) \geq -r$

$|x - a| \leq r\ if\ and\ only\ if\ -r \leq x - a \leq r$

$-r \leq x - a \leq r$

$-r + a \leq x - a + a \leq r + a$

$-r + a \leq x \leq r + a \Rightarrow a - r \leq x \leq a + r$

$x \in [a - r, a + r].$

when the Absolute Value has no solution

$$\left.\begin{matrix} |x| = -r \\ |x| < -r \end{matrix}\right. \Rightarrow |x| \leq -r$$

$consider\ |x| > -r.\ then\ every\ x \in \mathbb{R}\ satisfies\ the\ inequality$

$|x| > -r, x \in \mathbb{R}$

$\boldsymbol{|x - a| \geq r}$

$There\ are\ two\ possibilities$

$Case\ (1)$: $If\ x \geq 0, |x - a| \geq r \Rightarrow x - a \geq r.$

$Case\ (2)$: $If\ x < 0, |x - a| \geq r \Rightarrow -(x - a) \geq r\ i.e, x - a \leq -r$

$|x - a| \geq r\ if\ and\ only\ if\ x - a \leq -r\ or\ x - a \geq r.$

$x - a + a \leq -r + a\ or\ x - a + a \geq r + a$

$x \leq a - r\ or\ x \geq a + r$

$if\ and\ only\ if\ x \in (-\infty, a-r] \cup [a+r, \infty)$.

Example 2.4: $\boldsymbol{Solve\ |x-9| < 2\ for\ x}$.

$$|x-9| < 2$$

$$x-9<2 \quad or \quad -(x-9) < 2$$

$$x-9 > -2$$

$$-2 < x-9 < 2$$

$Adding\ 9\ through\ out\ the\ inequality,$

$$-2+9 < x-9+9 < 2+9$$

$$7 < x < 11$$

Example 2.5 : $\boldsymbol{Solve \left|\frac{2}{x-4}\right| > 1, x \neq 4}$.

$$\left|\frac{2}{x-4}\right| > 1 \Rightarrow \quad \frac{2}{|x-4|} > 1$$

$$2 > |x-4| \Rightarrow |x-4| < 2$$

$$x-4<2 \quad or \quad -(x-4) < 2$$

$$x-4<2 \quad or \quad x-4 > -2$$

$$-2 < x-4 < 2$$

$\boxed{\left|\frac{x}{y}\right| = \frac{|x|}{|y|}}$

$Adding\ 4\ through\ out\ the\ inequality,$

$$-2+4 < x-4+4 < 2+4$$

$$2 < x < 6 \quad where \quad x \neq 4.$$

$$x \in (2,4) \cup (4,6) \ or\ (2,6) - \{4\}$$

1. $\boldsymbol{Solve\ for\ x}$: **(i)** $\boldsymbol{|3-x| < 7}$, **(ii)** $\boldsymbol{|4x-5| \geq -2}$ $\boldsymbol{(iii) \left|3 - \frac{3}{4}x\right| \leq \frac{1}{4}}$

(iv) $\boldsymbol{|x| - 10 < -3}$

$\boldsymbol{(i)\ |3-x| < 7}$

$$|3-x| < 7 \Rightarrow 3-x < 7 \quad or \quad -(3-x) < 7$$

$$3-x > -7$$

$$-7 < 3-x < 7$$

$$7 > x-3 > -7$$

$Adding\ 3\ through\ out\ the\ inequality,$

$$7+3 > x-3+3 > -7+3$$

$$10 > x > -4 \Rightarrow -4 < x < 10$$

(ii) $\boldsymbol{|4x-5| \geq -2}$

$consider\ |x| > -r.\ then\ every\ x \in \mathbb{R}\ satisfies\ the\ inequality$

$$|x| > -r, x \in \mathbb{R}$$

$$|4x - 5| \geq -2 \implies x \in \mathbb{R}$$

(iii) $\left|\mathbf{3 - \frac{3}{4}x}\right| \leq \mathbf{\frac{1}{4}}$

$$\left|3 - \frac{3}{4}x\right| \leq \frac{1}{4}$$

$$3 - \frac{3}{4}x \leq \frac{1}{4} \quad or \quad -\left(3 - \frac{3}{4}x\right) \leq \frac{1}{4}$$

$$3 - \frac{3}{4}x \geq -\frac{1}{4}$$

$$-\frac{1}{4} \leq 3 - \frac{3}{4}x \leq \frac{1}{4}$$

Adding -3 *through out the inequality,*

$$-\frac{1}{4} - 3 \leq 3 - \frac{3}{4}x - 3 \leq \frac{1}{4} - 3$$

$$\frac{-13}{4} \leq -\frac{3}{4}x \leq \frac{-11}{4}$$

$$\frac{13}{4} \geq \frac{3}{4}x \geq \frac{11}{4}$$

multiply by 4

$$13 \geq 3x \geq 11 \quad \implies \quad \frac{13}{3} \geq \frac{3x}{3} \geq \frac{11}{3}$$

$\div 3$

$$\frac{13}{3} \geq x \geq \frac{11}{3} \implies \frac{11}{3} \leq x \leq \frac{13}{3}$$

(iv) $\mathbf{|x| - 10 < -3}$

$$|x| - 10 < -3 \implies |x| < -3 + 10$$

$$|x| < 7 \implies x < 7 \; or -x < 7$$

$$x < 7 \; or \; x > -7 \implies -7 < x < 7$$

2. ***Solve*** $\mathbf{\frac{1}{|2x-1|} < 6}$ ***and express the solution using the interval notation.***

$$\frac{1}{|2x-1|} < 6 \implies |2x - 1| > \frac{1}{6}$$

$$2x - 1 > \frac{1}{6} \quad or \quad -(2x - 1) > \frac{1}{6}$$

$$2x - 1 > \frac{1}{6} \quad or \quad 2x - 1 < -\frac{1}{6}$$

$$2x > \frac{1}{6} + 1 \ or \ 2x < -\frac{1}{6} + 1$$

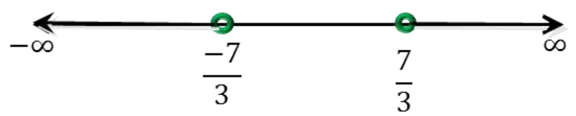
$$2x > \frac{7}{6} \ or \ 2x < \frac{5}{6}$$

$$x > \frac{7}{12} or \ x < \frac{5}{12} \Rightarrow x \in \left(-\infty, \frac{5}{12}\right] \cup \left[\frac{7}{12}, \infty\right)$$

3. *Solve* $-3|x| + 5 \leq -2$ *and graph the solution set in a number line*.

$$-3|x| + 5 \leq -2 \Rightarrow -3|x| \leq -5 - 2$$

$$-3|x| \leq -7 \Rightarrow 3|x| \geq 7$$



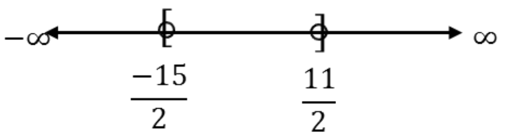
$$|x| \geq \frac{7}{3} \Rightarrow x \geq \frac{7}{3} \ or \ -x \geq \frac{7}{3}$$

$$x \geq \frac{7}{3} \ or \ x \leq -\frac{7}{3}$$

$$x \in \left(-\infty, \frac{-7}{3}\right] \cup \left[\frac{7}{3}, \infty\right)$$

4. *Solve* $2|x+1| - 6 \leq 7$ and graph the solution set in a number line

$$2|x+1| - 6 \leq 7$$



$$2|x+1| \leq 7 + 6 \Rightarrow 2|x+1| \leq 13$$

$$|x+1| \leq \frac{13}{2} \Rightarrow x + 1 \leq \frac{13}{2} \ or \ -(x+1) \leq \frac{13}{2}$$

$$x + 1 \leq \frac{13}{2} \ or \ x + 1 \geq -\frac{13}{2} \quad \Rightarrow \frac{-13}{2} \leq x + 1 \leq \frac{13}{2}$$

$$-\frac{13}{2} - 1 \leq x + 1 - 1 \leq \frac{13}{2} - 1$$

$$\frac{-13-2}{2} \leq x \leq \frac{13-2}{2} \Rightarrow \frac{-15}{2} \leq x \leq \frac{11}{2} \quad \therefore x \in \left[-\frac{15}{2}, \frac{11}{12}\right]$$

5. *Solve*: $\frac{1}{5}|10x - 2| < 1$

$$\frac{1}{5}|10x - 2| < 1 \Rightarrow |10x - 2| < 5$$

$$10x - 2 < 5 \ or \ -(10x - 2) < 5$$

$$-3 < 10x < 7 \Rightarrow -\frac{3}{10} < x < \frac{7}{10}$$

$$\div 10$$

6. *Solve* $|5x - 12| < -2$.

The Absolute value has no solution $|x - a| < -r$

$|5x - 12| < -2$ *has no solution*

$EXERCISE : 2.3$

2.4 Linear Ienqualities

$f(x) = ax + b, a, b \in \mathbb{R}$ *are constants, is called a linear function*

its graph is a straight line

$$f(x) = ax + b = 0.$$

$$x = \frac{-b}{a}$$

For example to describe a statement like "A tower is not taller fifty feet."

If x denotes the height of the tower in feet, then the above statement can be expressed as $x \le 50$.

Example 2.6: ***Our monthly electricity bill contains a basic charge, which does not change with number of units used, and a charge that depends only on how many units we use. Let us say Electricity Board charges Rs. 110 as basic charge and charges Rs. 4 for each unit we use. If a person wants to keep his electricity bill below Rs. 250, then what should be his electricity usage?***

Let x denote the number of units used. Note that $x \ge 0$.

Then, his electricity bill is $Rs. 110 + 4x$.

The person wants his bill to below Rs. 250. Let us solve the inequality

$$110 + 4x < 250.$$

$$110 + 4x < 250 \implies 4x < 250 - 110$$

$$\cancel{4}x < \cancel{140}^{35} \implies x < 35$$

$$\therefore 0 \le x < 35$$

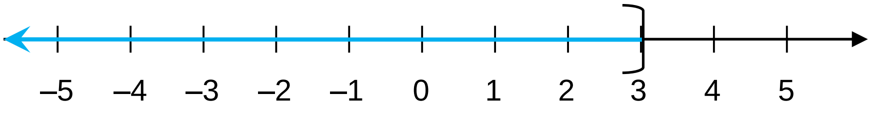
The person should keep his usage below 35 units in order to keep his bill below Rs. 250.

Example 2.7: ***Solve*** $\mathbf{3x - 5 \le x + 1}$ **for** $\boldsymbol{x}$.

$$3x - 5 \le x + 1 \implies 3x - x \le 5 + 1$$

$$\cancel{2}x \le \underset{3}{\cancel{6}} \implies x \le 3$$

The solution set is $(-\infty, 3]$.

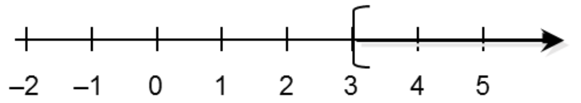


Example 2.8: ***Solve the linear inequalities*** $\mathbf{3x - 9 \ge 0}$, $\mathbf{4x - 10 \le 6}$ **for** $\boldsymbol{x}$.

$$3x - 9 \ge 0$$

$$\cancel{3}x \ge \cancel{9}^{3} \implies x \ge 3$$

$x \in [3, \infty)$



Also

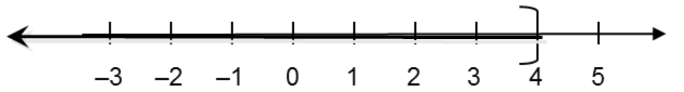
$4x - 10 \leq 6$

$4x \leq 10 + 6 \implies 4x \leq 16$

$\div 4$

$x \leq 4$

$x \in (-\infty, 4]$.



so the solution set of $3x - 9 \geq 0, 4x - 10 \leq 6$ *is the intersection of* $[3, \infty)$ *and* $(-\infty, 4]$.

The intersection of $[3, \infty)$ *and* $(-\infty, 4]$ *is* $[3,4]$

Example 2.9: ***A girl A is reading a book having 446 pages and she has already finished reading 271 pages. She wants to finish reading this book within a week. What is the minimum number of pages she should read per day to complete reading the book within a week?***

Let x denote the number of pages the girl should read per day.

A girl A is reading a book having 446 pages and

$7x + 271 \geq 446.$

$7x \geq 446 - 271 \implies 7x \geq 175 \implies x \geq 25$

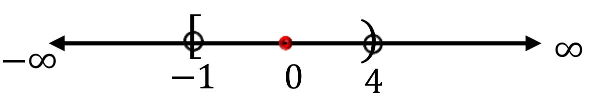
Hence $x \geq 25$; *which implies that she should read at least* 25 *pages per day.*

1. *Represent the following inequalities in the interval notation*:

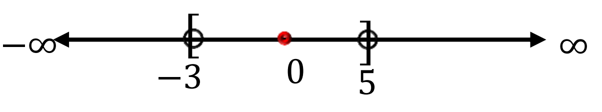
(a) $x \geq -1$ **and** $x < 4$, **(b)** $x \leq 5$ **and** $x \geq -3$

(c) $x < -1$ **or** $x < 3$, **(d)** $-2x > 0$ **or** $3x - 4 < 11$

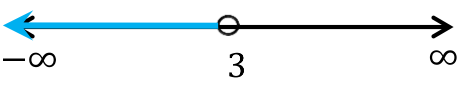
(a) $x \in [-1, 4)$



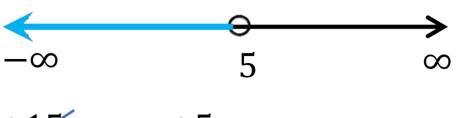
(b) $x \in [-3, 5]$



(c) $x \in (-\infty, 3)$



(d) $-2x > 0$



$-x > 0 \implies x < 0$

$3x - 4 < 11 \implies 3x < 11 + 4 \implies \cancel{3}x < \cancel{15} \implies x < 5$

$\therefore x \in (-\infty, 5)$

2. *Solve* $23x < 100$ ***when*** (i) ***x is a natural number,*** (ii) ***x is an integer.***

$23x < 100$

$$x < \frac{100}{23} \implies x < 4.34$$

(i) When x is a natural number $x = \{1, 2, 3, 4\}$

(ii) When x is an integer then $x = \{-\infty, \ldots -3, -2, -1, 0, 1, 2, 3, 4\}$

3. Solve $-2x \geq 9$ when (i) x is a real number, (ii) x is an integer, (iii) x is a natural number.

$$-2x \geq 9$$

$$2x \leq -9 \implies x \leq \frac{-9}{2}$$

(i) Where x is a real number $\quad x \in \left(-\infty, \frac{-9}{2}\right]$

(ii) Where x is an integer $\; x = \{-\infty \ldots, -7, -6, -5\}$

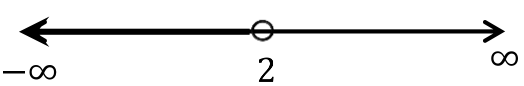
(iii) Where x is a natural number $x =$ no solution

4. Solve: (i) $\dfrac{3(x-2)}{5} \leq \dfrac{5(2-x)}{3}$ (ii) $\dfrac{5-x}{3} < \dfrac{x}{2} - 4$

$$(i) \; \frac{3(x-2)}{5} \leq \frac{5(2-x)}{3} \implies 9(x-2) \leq 25(2-x)$$

$$9x - 18 \leq 50 - 25x \implies 9x + 25x \leq 50 + 18$$

$$\cancel{34}x \leq \cancel{68}^{\,2} \implies x \leq 2$$



$$\therefore x \in (-\infty, 2]$$

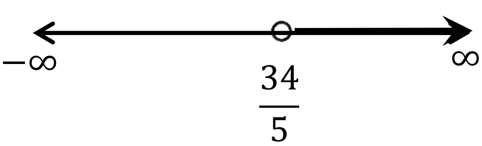
$$(ii) \; \frac{5-x}{3} < \frac{x}{2} - 4$$

$$\frac{5-x}{3} < \frac{x}{2} - 4 \implies \frac{5-x}{3} < \frac{x-8}{2} \implies 2(5-x) < 3(x-8)$$

$$10 - 2x < 3x - 24 \implies 10 + 24 < 3x + 2x$$

$$34 < 5x \implies 5x > 34 \implies x > \frac{34}{5}$$

$$\therefore \left(\frac{34}{5}, \infty\right)$$



5. To secure A grade one must obtain an average of 90 marks or more in 5 subjects each of maximum 100 marks. If one scored 84,87,95,91 in first four subjects, what is the minimum mark one scored in the fifth subject to get A grade in the course?

$Marks\ in\ four\ subjects\ are\ 84, 87, 95, 91.$

$Let\ the\ minimum\ marks\ must\ obtain\ in\ the\ fifth\ subject\ be\ x.$

$$Average\ marks = \frac{84+87+95+91+x}{5} = \frac{357+x}{5}$$

$$\therefore \frac{357+x}{5} \geq 90 \implies 357+x \geq 450$$

$$x \geq 450 - 357 \implies x \geq 93$$

$\therefore Minimum\ marks\ to\ be\ scored\ to\ get\ a\ grade\ is\ 93.$

6. A manufacturer has 600 liters of a 12 percent solution of acid. How many liters of a 30 percent acid solution must be added to it so that the acid content in the resulting mixture will be more then 15 percent but less than 18 percent?

$Let\ x\ be\ the\ number\ of\ liters\ of\ 30\%\ acid$

$(i)\ 30\%\ of\ x\ liters + 12\%\ of\ 600\ liters > 15\%\ (600+x)\ liters$

$$\frac{30}{100} \times x + \frac{12}{100} \times 600 > 15\% \times (600+x)$$

$$\frac{30x}{100} + \frac{7200}{100} > \frac{15(600+x)}{100} \implies \frac{30x+7200}{\cancel{100}} > \frac{15(600+x)}{\cancel{100}}$$

$$30x + 7200 > 9000 + 15x \implies 30x - 15x > 9000 - 7200$$

$$15x > 1800 \implies x > \frac{1800}{15} \implies x > 120$$

$(ii)\ 30\%\ of\ x + 12\%\ of\ 600 < 18\%\ of\ (600+x)$

$$\frac{30}{100} \times x + \frac{12}{100} \times 600 < 18\% \times (600+x)$$

$$\frac{30x}{100} + \frac{7200}{100} < \frac{18(600+x)}{100} \implies \frac{30x+7200}{\cancel{100}} < \frac{18(600+x)}{\cancel{100}}$$

$$30x + 7200 < 18(600+x) \implies 30x + 7200 < 10800 + 18x$$

$$30x - 18x < 10800 - 7200 \implies 12x > 3600$$

$$x < \frac{3600}{12} \implies x < 300 \implies 120 < x < 300$$

7. Find all pairs of consecutive odd natural numbers both of which are larger than 10 and their sum is less than 40.

Let $x\ and\ x+2\ be\ the\ two\ consecutive\ odd\ natural\ numbers.$

$It\ is\ given\ that\ both\ odd\ natural\ numbers\ greater\ than\ 10.$

$$x > 10\ \ and\ x + 2 > 10$$

$Given\ that\ their\ sum\ is\ less\ than\ 40$

$$x + (x+2) < 40 \implies 2x + 2 < 40$$

$$\div 2$$

$x+1<20 \Rightarrow \; x<19$

$x>10$ and $x<19 \;\Rightarrow\; 10<x<19$

$\therefore$ *The numbers* $11, 13, 15, 17$ ($\therefore$ x is an odd number)

$\therefore$ *The required pairs of odd natural numbers are* (11,13),(13,15), (15,17) and (17,19)

8. A model rocket is launched from the ground. The height h reached after t seconds from lift off is given by $h(t) = -5t^2 + 100t, 0 \leq t \leq 20$ At what time the rocket is 495 feet above the ground?

To find time when the rocket is 495 feet above the ground

$0 < h(t) \leq 495$

$0 < -5t^2 + 100t \leq 495 \;\Rightarrow\; -5t^2 + 100t = 495$

$-5t^2 + 100t - 495 = 0 \quad \Rightarrow \quad 5t^2 - 100t + 495 = 0$

Multiply both sides by -1 *Divide throughout by* 5

$t^2 - 20t + 99 = 0 \;\Rightarrow (t-9)(t-11) = 0$

$t-9=0, t-11=0 \Rightarrow t = 9 \text{ or } t = 11$

$\therefore t = 9$ sec *or* 11sec *the rocket is* 495 *feet above the ground*.

9. A plumber can be paid according to the following schemes: In the first scheme he will be paid Rs. 500 plus Rs.70 per hour, and in the second scheme he will paid Rs.120 per hour. If he works x hours, then for what of x does the first scheme give better wages?

Number of hours $= x$

Under first scheme, wage of plumber $= Rs. (500 + 70x)$

Under second scheme wage of plumber $= Rs. 120x$

Let $500 + 70x > 120x \;\Rightarrow\; 500 > 120x - 70x$

$500 > 50x$

$\div 50 \quad \Rightarrow 10 > x \;\; (or)\; x < 10$

$\therefore$ *Number of hours should be less than ten hours*.

10. A and B are working on similar jobs but their monthly salaries differ by more than $Rs. 6000$. If B earns $Rs. 27000$ per month, then what are the possibilities of A's salary per month?

Let monthly salary of A be x . *B's monthly salary* $= 27{,}000$

Difference of the monthly salary $= Rs. 6000$

A's salary $= x + 6000$

$x + 6000 > 27000 \;\Rightarrow\; x > 27000 - 6000 \;\Rightarrow\; x > 21000$

A's salary $= x - 6000$

$x - 6000 < 27000 \Rightarrow\; x < 27000 + 6000 \;\Rightarrow\; x < 33000$

A's salary less than Rs.21, 000 or greater than Rs.33000

EXERCISE : 2.4

2.5 QUADRATIC FUNCTIONS

A function of the form $P(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ are constants, $a \neq 0$, is called a **quadratic function.**

$$let\ us\ solve\ for\ P(x) = 0.$$
$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ *which is called the* ***quadratic formula.***

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad and \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Note:

If α and β are roots of $ax^2 + bx + c$,

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \ and\ \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$ then

$$\alpha + \beta = \frac{-b}{a} \quad and \quad \alpha\beta = \frac{c}{a}$$

$P(x) = 0$ *has two distinct real solutions if* $b^2 - 4ac = 0$ *and* $b^2 - 4ac > 0.$

It does not intersect $x - axis$ *at any point if* $b^2 - 4ac < 0.$

$\Delta = b^2 - 4ac$ *is called the* ***discriminant*** *of thequadratic function* $P(x) = ax^2 + bx + c.$

$$\Delta = \boldsymbol{b^2 - 4ac.}$$

Discriminant	***Nature of roots***	***Parabola***
Positive	*real and distinct*	*intersects x − axis at two points*
Zero	*real and equal*	*touches x − axis at one point*
Negative	*no real roots*	*does not meet x − axis*

Example 2.10: If a and b are the roots of the equation $x^2 - px + q = 0$ find the value of, $1/a + 1/b$.

$$Sum\ of\ the\ roots = p \Longrightarrow a + b = p$$
$$Product\ of\ the\ roots = q \Longrightarrow ab = q$$
$$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{p}{q}$$

Example 2.11: ***Find the complete set of values of a for which the quadratic $x^2 - ax + a + 2 = 0$ has equal roots.***

The quadratic equation $x^2 - ax + a + 2 = 0$ *has equal roots.*

So, its discriminant is zero i.e $b^2 - 4ac = 0$

$$x^2 - ax + a + 2 = 0$$

Where $a = 1, \quad b = -a, \quad c = a + 2$

$$b^2 - 4ac = 0 \Rightarrow (-a)^2 - 4(1)\,(a+2) = 0$$

$$a^2 - 4(a+2) = 0$$

$$a^2 - 4a - 8 = 0$$

Where $A = 1,\ B = -4\ \ C = -8$

$$a = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-8)}}{2(1)}$$

$$a = \frac{4 \pm \sqrt{16+32}}{2} = \frac{4 \pm \sqrt{48}}{2} = \frac{4 \pm \sqrt{4 \times 12}}{2}$$

$$a = \frac{4 \pm 2\sqrt{12}}{2} = \frac{2(2 \pm \sqrt{12})}{2}$$

$$a = 2 \pm \sqrt{12} \Rightarrow a = 2 + \sqrt{12}\,, 2 - \sqrt{12}$$

Example 2.12: ***Find the number of solutions of $x^2 + |x-1| = 1$.***

Case (1) : *For* $x \geq 1, |x-1| = x - 1.$

$$x^2 + |x-1| = 1 \Rightarrow x^2 + x - 1 = 1$$

$$x^2 + x - 1 - 1 = 0 \Rightarrow x^2 + x - 2 = 0$$

Factoring $:(x+2)(x-1) = 0,$

$$x + 2 = 0, x - 1 = 0 \quad \Rightarrow x = -2 \ \ or\ \ 1.$$

As $x \geq 1$, *we obtain* $x = 1.$

Case (2) : *For* $x < 1, \quad |x-1| = -(x-1)$

$$|x-1| = 1 - x \Rightarrow x^2 + |x-1| = 1$$

$$x^2 + 1 - x = 1$$

$$x^2 - x + 1 - 1 = 0 \Rightarrow x^2 - x = 0$$

$$x(x-1) = 0 \Rightarrow x = 0, x - 1 = 0$$

$$x = 0 \ or\ x = 1.$$

As $x < 1$, *we have to choose* $x = 0.$

Thus, the solution set is $\{0,1\}.$

Hence, the equation has two solutions.

1. Construct a quadratic equation with roots 7 and -3.

$Given\ roots: x = 7\ and\ x = -3$

$Factors\ are\ x - 7 = 0\ and\ x + 3 = 0$

$Product\ of\ factors: (x-7)(x+3) = 0$

$$x^2 + 3x - 7x - 21 = 0$$

The required equation is $x^2 - 4x - 21 = 0$

or

$Sum\ of\ the\ roots = 7 + (-3)$

$= 7 - 3 = 4$

$Product\ of\ the\ roots = (7)(-3) = -21$

$\boldsymbol{Quadratic\ equation}$:

$$\boldsymbol{x^2 - (sum\ of\ the\ roots)x + products\ of\ the\ roots = 0}$$

$$x^2 - 4x - 21 = 0$$

2. A quadratic polynomial has one of its zeros $1 + \sqrt{5}$ and it satisfies $p(1) = 2$. Find the quadratic polynomial.

$Given\ one\ of\ its\ root\ is\ 1 + \sqrt{5}\ .other\ root\ is\ 1 - \sqrt{5}$

$Sum\ of\ the\ roots = 1 + \cancel{\sqrt{5}} + 1 - \cancel{\sqrt{5}}$

$= 2$

$Product\ of\ the\ roots = (1 + \sqrt{5})(1 - \sqrt{5})$

$= 1^2 - (\sqrt{5})^2 = 1 - 5$

$= -4$

$\boldsymbol{Quadratic\ polynomial}$:

$$\boldsymbol{x^2 - (sum\ of\ the\ roots)x + products\ of\ the\ roots}$$

$$p(x) = x^2 - 2x - 4$$

$Given: p(1) = 2$

$so\ we\ need\ p(x) = a(x^2 - 2x - 4)$

$p(1) = a(1^2 - 2(1) - 4) \implies 2 = a(1 - 2 - 4)$

$-5a = 2 \quad \therefore a = -\frac{2}{5}$

$$p(x) = -\frac{2}{5}(x^2 - 2x - 4)$$

$\boldsymbol{3. If\ \alpha\ and\ \beta\ are\ the\ roots\ of\ the\ quadratic\ equation\ x^2 + \sqrt{2}x + 3 = 0}$

$\boldsymbol{form\ a\ quadratic\ polynomial\ with\ zeroes \frac{1}{\alpha}, \frac{1}{\beta}}$

let α and $\beta\ be\ the\ roots\ of\ x^2 + \sqrt{2}x + 3 = 0.$

$where\ a = 1, b = \sqrt{2}\ and\ c = 3$

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}$$

$$\alpha+\beta=\frac{-\sqrt{2}}{1}=-\sqrt{2} \text{ and } \alpha\beta=\frac{3}{1}=3$$

To form a quadratic polynomial whose roots are $\frac{1}{\alpha},\frac{1}{\beta}$

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\beta+\alpha}{\alpha\beta} \Longrightarrow \frac{1}{\alpha}+\frac{1}{\beta}=\frac{-\sqrt{2}}{3}$$

$$\frac{1}{\alpha}\times\frac{1}{\beta}=\frac{1}{\alpha\beta}=\frac{1}{3} \Longrightarrow \frac{1}{\alpha\beta}=\frac{1}{3}$$

Quadratic equation :

$$\boldsymbol{x^2-(sum\ of\ the\ roots\ )x+product\ of\ the\ roots=0}$$

The required equation is $x^2-\left(-\frac{\sqrt{2}}{3}\right)x+\frac{1}{3}=0$

$$\times 3$$

$$3x^2+\sqrt{2}x+1=0$$

4. If one root of $\mathbf{k(x-1)^2=5x-7}$ is double the other root, show that $\mathbf{k=2}$ or $\mathbf{-25}$.

$$k(x-1)^2=5x-7 \Longrightarrow k(x^2-2x+1)=5x-7$$

$$kx^2-2kx+k=5x-7 \Longrightarrow kx^2-2kx-5x+k+7=0$$

$$kx^2-x(2k+5)+(k+7)=0$$

$$a=k, b=-(2k+5), c=k+7$$

Let the roots be α and 2α

$$sum\ of\ the\ roots: \alpha+2\alpha=-\frac{b}{a}$$

$$\alpha+2\alpha=\frac{2k+5}{k} \Longrightarrow 3\alpha=\frac{2k+5}{k} \Longrightarrow \alpha=\frac{2k+5}{3k}$$

$$product\ of\ the\ roots: \alpha\times 2\alpha=\frac{c}{a}$$

$$2\alpha^2=\frac{k+7}{k} \Longrightarrow 2\left(\frac{2k+5}{3k}\right)^2=\frac{k+7}{k}$$

$$\frac{2(2k+5)^2}{(3k)^2}=\frac{k+7}{k} \Longrightarrow \frac{2\left[(2k)^2+2(2k)(5)+5^2\right]}{9\cancel{k^2}k}=\frac{k+7}{\cancel{k}}$$

$$\frac{2(4k^2+20k+25)}{9k} = k+7 \implies 8k^2+40k+50 = 9k(k+7)$$

$$8k^2+40k+50 = 9k^2+63k \implies 8k^2+40k+50-9k^2-63k = 0$$

$$-k^2-23k+50 = 0 \implies k^2+23k-50 = 0 \implies (k-2)(k+25) = 0$$

$$\boxed{k = 2 \text{ (or) } -25}$$

5. If the difference of the roots of the equation $2x^2-(a+1)x+a-1=0$ is equal to their product then prove that $a=2$.

$$2x^2-(a+1)x+a-1=0$$

$$a=2, b=-(a+1), c=a-1$$

$$(\alpha-\beta)^2 = (\alpha+\beta)^2-4\alpha\beta$$
$$(\alpha-\beta)^2 = \alpha^2+\beta^2+2\alpha\beta-4\alpha\beta$$
$$(\alpha-\beta)^2 = \alpha^2+\beta^2-2\alpha\beta$$

Let the roots be α and β

$$\alpha+\beta = -\frac{b}{a} \implies \alpha+\beta = \frac{a+1}{2}$$

$$\alpha\beta = \frac{c}{a} \implies \alpha\beta = \frac{a-1}{2}$$

Given difference of the root = product of the roots

$$\alpha-\beta = \alpha\beta$$

$$(\alpha-\beta)^2 = (\alpha\beta)^2 \implies (\alpha+\beta)^2-4\alpha\beta = (\alpha\beta)^2$$

$$\left(\frac{a+1}{2}\right)^2 - 4\left(\frac{a-1}{2}\right) = \left(\frac{a-1}{2}\right)^2$$

$$\frac{(a+1)^2}{4} - 2(a-1) = \frac{(a-1)^2}{4} \implies \frac{(a+1)^2-8(a-1)}{\cancel{4}} = \frac{(a-1)^2}{\cancel{4}}$$

$$(a+1)^2-8(a-1) = (a-1)^2 \implies \cancel{a^2}+2a+\cancel{1}-8a+8 = \cancel{a^2}-2a+\cancel{1}$$

$$2a-8a+8 = -2a \implies 2a-8a+8+2a = 0$$

$$-4a+8 = 0 \implies -4a+8 = 0$$

$$4a = 8 \implies a = 2$$

6. Find the condition that one of the roots of ax^2+bx+c may be

(i) negative of the other, (ii) thrice the other,
(iii) reciprocal of the other.

Let $ax^2+bx+c=0$

(i) Let the roots be α and $-\alpha$

Sum of the *roots* $= -\frac{b}{a} \implies \alpha+(-\alpha) = -\frac{b}{a}$

$$0 = -\frac{b}{a} \Rightarrow \boxed{b = 0}$$

(ii) *Let the roots be* α and 3α

$$Sum \text{ of the } roots = -\frac{b}{a} \Rightarrow \alpha + 3\alpha = -\frac{b}{a} \Rightarrow 4\alpha = -\frac{b}{a}$$

$$\boxed{\alpha = -\frac{b}{4a}}$$

$$\alpha \times 3\alpha = \frac{c}{a} \Rightarrow 3\alpha^2 = \frac{c}{a} \Rightarrow 3\left(-\frac{b}{4a}\right)^2 = \frac{c}{a}$$

$$\frac{3b^2}{16a^2} = \frac{c}{a} \Rightarrow 3b^2 = 16a^2 \times \frac{c}{a} \Rightarrow 3b^2 = 16ac$$

$(\boldsymbol{iii})$ $\boldsymbol{Reciprocal\ of\ the\ other}$

$$Let\ the\ roots\ be\ \alpha\ and\ \frac{1}{\alpha}$$

$$Product\ the\ roots = \frac{c}{a}$$

$$\alpha\left(\frac{1}{\alpha}\right) = \frac{c}{a} \Rightarrow 1 = \frac{c}{a} \Rightarrow \boxed{a = c}$$

7. If the equation $x^2 - ax + b = 0$ and $x^2 - ex + f = 0$ have one root in common and if the second equation has equal roots then prove that $ae = 2(b + f)$.

Let α and β be the roots of $x^2 - ax + b = 0$ and α be the common root.

$$Sum \text{ of the } roots = -\frac{b}{a} \Rightarrow \boxed{\therefore \alpha + \beta = a} \quad \ldots (1)$$

$$Product \text{ of the } roots = \frac{c}{a} \Rightarrow \boxed{\alpha\beta = b} \quad \ldots (2)$$

Let α and α *be the roots of* $x^2 - ex + f = 0$

$$\alpha + \alpha = e \Rightarrow 2\alpha = e \Rightarrow \boxed{\alpha = \frac{e}{2}} \quad \ldots (3)$$

$$Product \text{ of the } roots = \frac{c}{a} \Rightarrow \alpha \times \alpha = f$$

$$\alpha^2 = f,\ where\ \alpha = \frac{e}{2}$$

$$\frac{e^2}{4} = f \Rightarrow e^2 = 4f$$

$From\ (2)\ \alpha\beta = b$

$sub\ \alpha = \dfrac{e}{2}$

$\dfrac{e}{2} \times \beta = b \Rightarrow \beta = \dfrac{2b}{e}$

$From\ (2)\ \alpha + \beta = a$

$\dfrac{e}{2} + \dfrac{2b}{e} = a \Rightarrow \dfrac{e^2 + 4b}{2e} = a$

$e^2 + 4b = 2ae\ where\ e^2 = 4f$

$4f + 4b = 2ae \Rightarrow 4(f + b) = 2ae$

$2(f + b) = ae$

8. Discuss the nature of roots of
$\boldsymbol{(i) - x^2 + 3x + 1, (ii)\ 4x^2 - x - 2 = 0, (iii)\ 9x^2 + 5x = 0.}$

$\boldsymbol{(i) - x^2 + 3x + 1}$

$a = -1, b = 3, c = 1$

$b^2 - 4ac = 3^2 - 4(-1)(1) = 9 + 4$
$= 13 > 0$

$\therefore The\ roots\ are\ real\ and\ distinct.$

$\boldsymbol{(ii)\ 4x^2 - x - 2 = 0}$

$a = 4, b = -1,\ c = -2$

$b^2 - 4ac = (-1)^2 - 4(4)(-2)$
$= 1 + 32 = 33 > 0$

$\therefore The\ roots\ are\ real\ and\ distinct.$

$\boldsymbol{(iii)\ 9x^2 + 5x = 0.}$

$a = 9, b = 5, c = 0$

$b^2 - 4ac = 5^2 - 4(9)0$
$= 25 - 0 = 25 > 0$

$\therefore The\ roots\ are\ real\ and\ distinct.$

9. Without sketching the graphs, find whether the graphs of the following functions will intersect the x-axis and if so in how many points

$\boldsymbol{(i)\ y = x^2 + x + 2, (ii)\ y = x^2 - 3x - 7, (iii)\ y = x^2 + 6x + 9.}$

$\boldsymbol{(i)\ y = x^2 + x + 2}$

$The\ funtion\ interset\ x - axis\ i.e\ y = 0$

$x^2 + x + 2 = 0$

$a = 1, b = 1, c = 2$

$$b^2 - 4ac = 1^2 - 4(1)(2)$$

$$= 1 - 8 = -7 < 0 \quad \therefore The\ roots\ are\ imaginary$$

Since D is negative, the parabola does not meet the x-axis.

(ii) $\boldsymbol{y = x^2 - 3x - 7}$

$The\ funtion\ intersect\ x - axis\ i.e\ y = 0$

$$x^2 - 3x - 7 = 0$$

$$a = 1, b = -3, c = -7$$

$$b^2 - 4ac = (-3)^2 - 4(1)(-7)$$

$$= 9 + 28 = 37 > 0 \quad \therefore The\ roots\ are\ real\ and\ distinct.$$

Since D is positive the parabola intersection x-axis at two points.

(iii) $\boldsymbol{y = x^2 + 6x + 9}$

$The\ funtion\ intersect\ x - axis\ i.e\ y = 0$

$$x^2 + 6x + 9 = 0$$

$$a = 1, b = 6, c = 9$$

$$b^2 - 4ac = 6^2 - 4(1)(9)$$

$$= 36 - 36$$

$$= 0 \quad \therefore The\ roots\ are\ real\ and\ equal.$$

Since D $= 0$ the parabola touches x-axis at a point

10. Write $\boldsymbol{f(x) = x^2 + 5x + 4}$ in completed square form.

$$f(x) = x^2 + 5x + 4$$

$$f(x) = x^2 + 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 4$$

$$\boxed{\begin{aligned}\left(x + \frac{5}{2}\right)^2 &= x^2 + 2(x)\left(\frac{5}{2}\right) + \left(\frac{5}{2}\right)^2 \\ &= x^2 + 5x + \left(\frac{5}{2}\right)^2\end{aligned}}$$

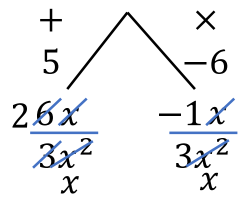
$$f(x) = \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + 4$$

$$f(x) = \left(x + \frac{5}{2}\right)^2 + 4 - \frac{25}{4} \Rightarrow f(x) = \left(x + \frac{5}{2}\right)^2 + \frac{16 - 25}{4}$$

$$f(x) = \left(x + \frac{5}{2}\right)^2 - \frac{9}{4} \Rightarrow f(x) = \left(x + \frac{5}{2}\right)^2 - \left(\frac{3}{2}\right)^2$$

EXERCISE : 2.5

Example 2.13 : ***Solve*** $\mathbf{3x^2 + 5x - 2 \leq 0.}$

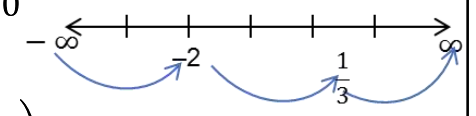


$3x^2 + 5x - 2 \leq 0.$

Let $3x^2 + 5x - 2 = 0$

$(x+2)(3x-1) = 0 \implies x + 2 = 0, 3x - 1 = 0$

$x = -2,\ x = \dfrac{1}{3}$



The intervals are $(-\infty, -2), \left(-2\,, \dfrac{1}{3}\right)$ *and* $\left(\dfrac{1}{3}, \infty\right)$

Interval	*Sign of* $(x+2)$	Sign of $(x - 1/3)$	Sign of $3x^2 + 5x - 2$
$(-\infty, -2)$	$-$	$-$	$+$
$(-2, 1/3)$	$+$	$-$	$-$
$(1/3, \infty)$	$+$	$+$	$+$

The solution set is $[-2, 1/3]$.

Example 2.14 : ***Solve*** $\mathbf{\sqrt{x+14} < x + 2.}$

The function $\sqrt{x+14}$ *is defined for* $\sqrt{x+14} \geq 0.$

$\sqrt{x+14} < x + 2$

Squaring on both sides

$(\sqrt{x+14})^2 < (x+2)^2 \implies x + 14 < (x+2)^2$

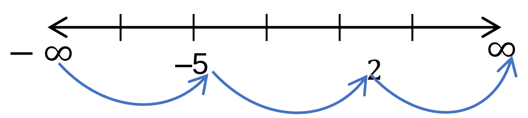
$x + 14 < x^2 + 4x + 2^2 \implies x + 14 < x^2 + 4x + 4$

$0 < x^2 + 4x + 4 - x - 14 \implies x^2 + 3x - 10 > 0.$

Let $x^2 + 3x - 10 = 0.$

Hence, $(x+5)(x-2) = 0.$

$x = -5$ *and* $x = 2.$



The intervals are $(-\infty, -5), (-5\,, 2)$ *and* $(2, \infty)$

Interval	*Sign of* $(x+5)$	Sign of $(x-2)$	Sign of $3x^2 + 5x - 2$
$(-\infty, -5)$	$-$	$-$	$+$
$(-5, 2)$	$+$	$-$	$-$
$(2, \infty)$	$+$	$+$	$+$

The solution set is $(-\infty,-5)\cup(2,\infty)$

$x < -5$ *or* $x > 2$

since $\sqrt{x+14} \geq 0$ *hence the solution set is* $x > 2$

Example 2.15 : ***Solve the equation*** $\boldsymbol{\sqrt{6-4x-x^2} = x+4}$

The given equation is equivalent to the system $x+4 \geq 0 \implies x \geq -4$

$\sqrt{6-4x-x^2} = x+4$

Squaring on both sides

$6-4x-x^2 = (x+4)^2 \implies 6-4x-x^2 = x^2+8x+16$

$x^2+8x+16+x^2+4x-6 = 0 \implies 2x^2+12x+10 = 0$

$\div 2$

$x^2+6x+5 = 0 \implies (x+1)\ (x+5) = 0$

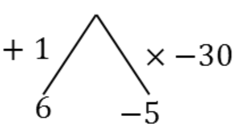
$x+1 = 0, x+5 = 0, \implies x = -1,\ x = -5$

Thus, $x = -1, -5.$

But only $x = -1$ *satisfies both the conditions.*

Hence, $x = -1.$

1. ***Solve*** **:** $\boldsymbol{2x^2+x-15 \leq 0.}$



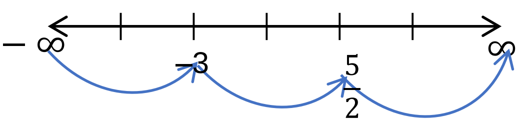
Let $2x^2+x-15 = 0$

$2x^2+6x-5x-15 = 0 \implies 2x(x+3)-5(x+3) = 0$

$(x+3)(2x-5) = 0 \implies x+3 = 0,\ 2x-5 = 0$

$x = -3,\ 2x = 5$

$x = \frac{5}{2}$



The critical points are -3 and $\frac{5}{2}$

The intervals are $(-\infty,-3), \left(-3, \frac{5}{2}\right)$ *and* $\left(\frac{5}{2}, \infty\right)$

Interval	*Sign of* $(x+3)$	*Sign of* $(2x-5)$	*Sign of* $2x^2+x-15$
$(-\infty,-3)$	$-$	$-$	$+$
$\left(-3,\frac{5}{2}\right)$	$+$	$-$	$-$
$\left(\frac{5}{2},\infty\right)$	$+$	$+$	$+$

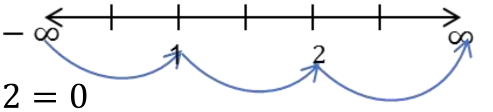
$x^2-3x+2 < 0$ is satisfied in $[1,2]$.

The solution set is $\left[-3, \frac{5}{2}\right]$

2. Solve $-x^2 + 3x - 2 \geq 0.$

$-x^2 + 3x - 2 \geq 0 \implies x^2 - 3x + 2 \leq 0$

$(x-1)(x-2) \leq 0$



Let $(x-1)(x-2) = 0 \implies x - 1 = 0,\ x - 2 = 0$

The critical points are $x = 1, x = 2$

The intervals are $(-\infty, 1), (1, 2)$ *and* $(2, \infty)$

Interval	Sign of $(x-1)$	Sign of $(x-2)$	Sign of $(x^2 - 3x + 2)$
$(-\infty, 1)$	$-$	$-$	$+$
$(1,2)$	$+$	$-$	$-$
$(2, \infty)$	$+$	$+$	$+$

EXERCISE : 2.6

***Definition* 2.1**:

A real number a is said to be a ***zero of the polynomial*** $f(x)$ *iff* $f(a) = 0$.
If $x = a$ *is a zero of* $f(x)$, *then* $x - a$ *is a* ***factor*** $f(x)$.

(i) A polynomial function of degree n can have at most n distinct real zeros.

(ii) It is also possible that a polynomial function like $P(x) = x^2 + 1$ has no real zeros at all.

(iii) Suppose that P(x) is a polynomial function having rational coefficients.

(iv) If $a + b\sqrt{p}$ where $a, b \in \mathbb{Q}$, p is a prime, is a zero of P(x), then its **Conjugate** $a - b\sqrt{p}$ is also zero.

Two important problems relating to polynomial are

(i) Finding zeros of a given polynomial function; and hence factoring the polynomial into linear factors and

(ii) Constructing polynomials with the given zeros and/or satisfying some additional conditions.

1. Find the zeros of the polynomial function $f(x) = 4x^2 - 25$.

$$f(x) = 4x^2 - 25 \Rightarrow f(x) = (2x)^2 - 5^2$$

$$f(x) = (2x - 5)(2x + 5)$$

$$put\ f(x) = 0 \Rightarrow (2x - 5)(2x + 5) = 0$$

$$2x - 5 = 0,\ 2x + 5 = 0$$

$$x = \frac{5}{2},\ x = -\frac{5}{2}$$

Here the zeros of the polynomial are $x = \frac{5}{2}$ *and* $x = -\frac{5}{2}$

2. If $x = -2$ *is one root of* $x^3 - x^2 - 17x = 22$, *then find the other roots of equation*.

If $x = -2$ *is one of the roots of* $x^3 - x^2 - 17x - 22 = 0$

To find other roots of the equation by synthetic division

$$\begin{array}{r|rrrr} & 1 & -1 & -17 & -22 \\ -2 & 0 & -2 & 6 & 22 \\ \hline & 1 & -3 & -11 & \boxed{0} \end{array}$$

$$x^2 - 3x - 11 = 0$$

$$a = 1, b = -3, c = -11$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{9 - 4(1)(-11)}}{2} = \frac{3 \pm \sqrt{9 + 44}}{2} = \frac{3 \pm \sqrt{53}}{2}$$

$\textit{The other roots of the equation are}\ \frac{3+\sqrt{53}}{2}, \frac{3-\sqrt{53}}{2}$

3. *Find the real roots of* $x^4 = 16$.

$$x^4 = 16 \implies x^4 - 16 = 0$$

$$(x^2)^2 - 4^2 = 0 \implies (x^2+4)(x^2-4) = 0 \implies (x^2-2^2)(x^2+4) = 0$$

$$(x+2)(x-2)(x^2+4) = 0$$

$$x+2=0,\ x-2=0,\ \textit{and}\ x^2+4=0$$

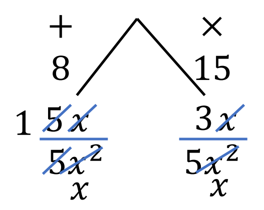
$$x=-2, x=2 \qquad \textit{and}\ x^2=-4$$

$$x = \sqrt{-4}\ \ (\textit{not real})$$

$\therefore$ *The real roots are* $x = 2$, $x = -2$

The other two roots are non real roots.

4. *Solve* $(2x+1)^2 - (3x+2)^2 = 0$.



$$(2x+1)^2 - (3x+2)^2 = 0$$

$$(2x)^2 + 2(2x)(1) + 1^2 - \left[(3x)^2 + 2(3x)(2) + 2^2\right] = 0$$

$$4x^2 + 4x + 1 - (9x^2 + 12x + 4) = 0$$

$$4x^2 + 4x + 1 - 9x^2 - 12x - 4 = 0$$

$$-5x^2 - 8x - 3 = 0 \implies 5x^2 + 8x + 3 = 0$$

$$(5x+3)(x+1) = 0 \implies 5x+3=0,\ x+1=0$$

$$5x = -3\ ,x = -1$$

$$\therefore x = -\frac{3}{5}$$

$\therefore x = -\frac{3}{5}$ and $x = -1$ *are the roots of the given equation.*

4. *Solve* $(2x+1)^2 - (3x+2)^2 = 0$.

$$(2x+1+3x+2)[2x+1-(3x+2)] = 0$$

$$(5x+3)[2x+1-3x-2] = 0$$

$$(5x+1)[-x-1] = 0 \implies -(5x+3)(x+1) = 0$$

$$(5x+3)(x+1) = 0 \implies 5x+3=0,\ x+1=0$$

$$5x = -3\ ,x = -1$$

$$\therefore x = -\frac{3}{5}$$

$\therefore x = -\frac{3}{5}$ *and* $x = -1$ *are the roots of the given equation.*

$EXERCISE : 2.7$

$Method\ of\ Undetermined\ Coefficients$

$Now\ let\ us\ focus\ on\ constructing\ polynomials\ with\ the\ given\ information\ using\ the\ method\ of\ undetermined\ coefficients.$

$That\ is, we\ shall\ determine\ coefficients\ of\ the\ required\ polynomial\ using\ the\ given\ conditions.$

$The\ main\ idea\ here\ is\ that\ two\ polynomials\ are\ equal\ if\ and\ only\ if\ the\ coefficients\ of\ same\ powers\ of\ the\ variables\ in\ the\ two\ polynomials\ are\ equal.$

2.6.1 Division Algorithm

Let $f(x)$ and $g(x)$ be two polynomials with $g(x) \neq 0$.
Then there exists $q(x)$ and $r(x)$ such that
$f(x) = [g(x)q(x)] + r(x) \leftarrow$ division algorithm.

$if\ r(x) = 0$ then $f(x) = q(x)\, r(x)$.

If $g(x) = x - a$ then $r(x)$ should be of degree zero that is a constant.

$r(x)$ is called reminder and $q(x)$ is called quotient.

$f(x) = (x-1)\, q(x) + c$, by putting $x = a$

$$f(a) = 0 + c \Rightarrow c = f(a)$$

Remainder Theorem: If a polynomial $f(x)$ is divided by $(x - a)$ then the remainder is $f(a)$.

Note: If Suppose $f(a) = 0$ then it implies

$f(x) = (x-1)\, q(x)$

$(x-1)$ is 1 factor of $f(x)$.

Factor Theorem:

If $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

$x = a$ is called zero of the polynomial $f(x)$.

Remark:
(i) A polynomial function of degree n can have at most n distinct real zeros.

(ii) It is also possible that a polynomial function has no real zero at all.

Eg. 2.16 : Find a quadratic polynomial $f(x)$ such that, $f(0) = 1, f(-2) = 0\ and\ f(1) = 0$.

$Let\ f(x) = ax^2 + bx + c\ be\ the\ polynomial\ satisfying\ thegiven\ conditions.$

$f(x) = ax^2 + bx + c$

$f(0) = 1$

$put\ x = 0,\ f(0) = a(0)^2 + b(0) + c$

$1 = c \Longrightarrow \; c = 1$

$f(-2) = 0$

$put \;\; x = -2, f(-2) = a(-2)^2 + b(-2) + c$

$0 = 4a - 2b + c \Longrightarrow 4a - 2b + c = 0 \quad \dots(1)$

$f(1) = 0$

$put \;\; x = 1, f(1) = a(1)^2 + b(1) + c$

$0 = a + b + c \;\Longrightarrow\; a + b + c = 0 \dots(2)$

$sub \;\; c = 1 \; in \; (1) \; and \; (2)$

$4a - 2b + c = 0 \Longrightarrow \; 4a - 2b + 1 = 0 \Longrightarrow 4a - 2b = -1 \quad \dots(3)$

$a + b + c = 0 \Longrightarrow a + b + 1 = 0 \Longrightarrow a + b = -1 \dots(4)$

$\boldsymbol{solve \; (3) \; and \; (4)}$

$(3) \Longrightarrow \;\; 4a - \cancel{2b} = -1$

$(4) \times 2 \Longrightarrow \;\; 2a + \cancel{2b} = -2$

$6a = -3$

$a = \frac{-3}{6} \; \Longrightarrow a = -\frac{1}{2}$

$sub \;\; a = -\frac{1}{2} \;\; in \; (4) \; a + b = -1$

$-\frac{1}{2} + b = -1 \;\; \Longrightarrow b = -1 + \frac{1}{2} \;\; \Longrightarrow \;\; b = -\frac{1}{2}$

$f(x) = ax^2 + bx + c$

$a = -\frac{1}{2}, b = -\frac{1}{2} and \; c = 1$

$\boxed{f(x) = -\frac{1}{2}x^2 - \frac{1}{2}x + 1.}$

Eg. 2.16 : Find a quadratic polynomial $\boldsymbol{f(x)}$ such that, $\boldsymbol{f(0) = 1, f(-2) = 0}$ $\boldsymbol{and \; f(1) = 0}$.

$x = -2, x = 1$ *are zero of* $f(x)$.

$f(x) = d(x + 2)(x - 1)$ *for some constant* d.

$f(0) = 1$

$f(0) = d(0 + 2)(0 - 1)$

$1 = 2d(-1) \;\Longrightarrow\; -2d = 1 \;\Longrightarrow\; d = -\frac{1}{2}$

$f(x) = -\frac{1}{2}(x + 2)(x - 1) \;\Longrightarrow\; f(x) = -\frac{1}{2}[x^2 - x + 2x - 2]$

$f(x) = -\frac{1}{2}[x^2 + x - 2] \;\Longrightarrow\; f(x) = -\frac{1}{2}x^2 - \frac{1}{2}x + 1$

Example 2.17: ***Construct a cubic polynomial function having zeros at*** $\boldsymbol{x = \frac{2}{5}, 1+\sqrt{3}}$ ***such that*** $\boldsymbol{f(0) = -8}$.

Given that $\frac{2}{5}$ *and* $1+\sqrt{3}$ *are zeros of* $f(x)$. $1-\sqrt{3}$ *is also a zero of* $f(x)$.

$$f(x) = a\left(x-\frac{2}{5}\right)\left[x-\left(1+\sqrt{3}\right)\right]\left[x-\left(1-\sqrt{3}\right)\right]$$

$$f(x) = a\left(x-\frac{2}{5}\right)\left[x-1-\sqrt{3}\right]\left[x-1+\sqrt{3}\right]$$

$$f(x) = a\left(x-\frac{2}{5}\right)\left[(x-1)^2-(\sqrt{3})^2\right]$$

$$= a\left(x-\frac{2}{5}\right)[x^2-2x+1-3]$$

$$f(x) = a\left(x-\frac{2}{5}\right)(x^2-2x-2)$$

$$\boxed{f(0) = -8}$$

$$put\ x = 0, f(0) = a\left(0-\frac{2}{5}\right)(0^2-2(0)-2)$$

$$-8 = -\frac{2a}{5}(-2) \implies -8 = \frac{4a}{5} \implies -40 = 4a$$

$$4a = -40 \implies a = \frac{-40}{4} \implies \boxed{a = -10}$$

$$f(x) = -10\left(x-\frac{2}{5}\right)(x^2-2x-2)$$

$$f(x) = \left(-10x+\frac{20}{5}\right)(x^2-2x-2) \implies f(x) = (-10x+4)(x^2-2x-2)$$

$$f(x) = -10x^3+20x^2+20x+4x^2-8x-8$$

Thus the required polynomial is $f(x) = -10x^3+24x^2+12x-8$

Example 2.18 : ***Prove that*** $\boldsymbol{ap+q=0}$ ***if*** $\boldsymbol{f(x) = x^3-3px+2q}$ ***is divisible by*** $\boldsymbol{g(x) = x^2+2ax+a^2}$.

$$g(x)\ divides\ f(x) \implies \frac{f(x)}{g(x)} = x+b$$

$$f(x) = (x+b)g(x), b \in \mathbb{R}.$$

$$x^3-3px+2q = (x+b)(x^2+2ax+a^2)$$

$x^3 + 0x^2 - 3px + 2q = (x + b)(x^2 + 2ax + a^2)$

$Equating\ like\ coefficient\ of\ x^2, x\ and\ constant$

$Equating\ \ x^2\ terms:\ bx^2 + 2ax^2 = 0x^2$

$coefficient\ of\ x^2 : b + 2a = 0$

$Equating\ x\ terms:\ a^2x + 2abx = -3px$

$coefficient\ of\ x : a^2 + 2ab = -3p$

$constant\ term:\ a^2b = 2q$

$b + 2a = 0 \implies b = -2a$

$sub\ b = -2a\ in\ a^2 + 2ab = -3p$

$a^2 + 2a(-2a) = -3p \implies a^2 - 4a^2 = -3p$

$-3a^2 = -3p \implies p = a^2$

$sub\ \ b = -2a\ in\ a^2b = 2q$

$a^2(-2a) = 2q \implies -\cancel{2}a^3 = \cancel{2}q \implies q = -a^3$

$q = -a \times a^2 \implies q = -ap \implies \boxed{ap + q = 0}$

Example 2.19 Use the method of undetermined coefficients to find the sum of $\mathbf{1 + 2 + 3 + \cdots + (n - 1) + n}$

$Let\ \ s(n) = n + (n-1) + (n-2) + \ \ldots\ldots + 2 + 1$

$S(n) = n + (n-1) + (n-2) + \cdots\cdots\cdots + n - (n-2) + n - (n-1)$

$$= n\left[\frac{n}{n} + \frac{n-1}{n} + \frac{n-2}{n} + \cdots + \frac{n}{n} - \left(\frac{n-2}{n}\right) + \frac{n}{n} - \left(\frac{n-1}{n}\right)\right]$$

$$S(n) = n\left[1 + \cancel{\frac{n-1}{n}} + \cancel{\frac{n-2}{n}} + \cdots + 1 - \left(\cancel{\frac{n-2}{n}}\right) + 1 - \left(\cancel{\frac{n-1}{n}}\right)\right]$$

$= n\ [1 + 1 + \cdots + 1] \quad since\ \ \frac{n-1}{n} < 1 \quad \frac{n-2}{n} < 1$

$S(n) \le\ n(n)$

$S(n) \le\ n^2$

$Let\ \ S(n) = a + bn + cn^2\ \ Where\ a, b, c,$

$s(n) = 1 + 2 + 3 + \cdots + n$

$s(n+1) = 1 + 2 + \cdots + n + (n+1)$

$S(n+1) = S(n) + (n+1)$

$S(n) = a + bn + cn^2$

$S(n+1) - S(n) = n + 1$

$a + b\,(n+1) + c(n+1)^2 - [a + bn + cn^2] = n + 1$

$$a + b\,(n+1) + c(n^2 + 2n + 1) - a - bn - cn^2 \;= n + 1$$

$$\cancel{a} + \cancel{bn} + b + c\cancel{n^2} + 2cn + c - \cancel{a} - \cancel{bn} - \cancel{cn^2}$$

$$b + 2cn + c \;= n + 1$$

$$2cn + b + c \;= n + 1$$

Equating coefficient of n and constant

$$2c = 1,\; b + c = 1$$

$$c = \frac{1}{2},\quad b + \frac{1}{2} = 1$$

$$b = 1 - \frac{1}{2} \implies b = \frac{1}{2}$$

$$S(n) = a + bn + cn^2$$

$$S(1) = a + b(1) + c(1)^2 \implies S(1) = a + b + c$$

$$since\; S(1) \;= 1$$

$$a + b + c = 1 \implies a + \frac{1}{2} + \frac{1}{2} = 1$$

$$a + 1 = 1 \implies \boxed{a = 0}$$

$$S(n) = a + bn + cn^2 \implies S(n) = 0 + \frac{1}{2}n + \frac{1}{2}n^2$$

$$S(n) = \frac{1}{2}(n + n^2) = \frac{n\,(n+1)}{2}\; n \in N$$

Eg. 2.20: ***Find the roots of the polynomial equation $(x-1)^3(x+1)^2(x+5) = 0$. state their multiplicity***

$$(x-1)^3(x+1)^2(x+5) = 0.$$

$$(x-1)^3 = 0 \implies x - 1 = 0$$

$x = 1$ *Hence, the root of* 1 *with multiplicity* 3

$$(x+1)^2 = 0 \implies x + 1 = 0$$

$x = -1,$ *Hence, the root of* 1 *with multiplicity* 2

$x + 5 = 0, \implies x = -5$ *with multiplicity* 1.

The roots $x = 1, -1, -5$.

Note: When the root has multiplicity 1, it is called a simple root.

Example 2.21: ***Solve*** $\boldsymbol{x = \sqrt{x+20}}$ ***for*** $\boldsymbol{x \in \mathbb{R}}$.

Observe that $\sqrt{x+20}$ *is defined only if* $x + 20 \geq 0$.

By definition, $\sqrt{x+20} \geq 0$ *is positive. So, x is positive.*

$$x = \sqrt{x+20}$$

squaring on both sides.

$$x^2 = x + 20 \implies x^2 - x - 20 = 0$$

$$(x-5)(x+4) = 0 \implies x - 5 = 0,\; x + 4 = 0$$

$$x = 5, x = -4$$

Since, x is positive, the required solution is $x = 5$.

Eg. 2.22: $\boldsymbol{The\ equations\ x^2 - 6x + a = 0\ and\ x^2 - bx + 6 = 0\ have\ one\ root\ in\ common. The\ other\ root\ of\ the\ first\ and\ the\ second\ equations\ are\ integers\ in\ the\ ratio\ 4 : 3.\ Find\ the\ common\ root.}$

Hence, the common root is α

Let $\alpha, 4\beta$ *be the roots of* $x^2 - 6x + a = 0$

$$Product\ of\ the\ root = \frac{c}{a} = a$$

$$\alpha \times 4\beta = a \Rightarrow 4\alpha\beta = a$$

Let $\alpha, 3\beta$ *be the roots of* $x^2 - bx + 6 = 0$.

$$Product\ of\ the\ root = 6$$

$$\alpha \times 3\beta = a \Rightarrow 3\alpha\beta = 6$$

$$\cancel{3}\alpha\beta = \overset{2}{\cancel{6}} \Rightarrow \alpha\beta = 2$$

$$4\alpha\beta = a \Rightarrow 4(2) = a$$

$$\boxed{a = 8}$$

sub $a = 8$ *in* $x^2 - 6x + a = 0$

$$x^2 - 6x + 8 = 0 \Rightarrow (x-2)(x-4) = 0$$

$$\boxed{\frac{4\beta}{3\beta} = \frac{4}{3}}$$

$$x - 2 = 0, x - 4 = 0 \Rightarrow x = 2,\ x = 4$$

If $\alpha = 2$, *in* $3\alpha\beta = 6$

$$3(2)\beta = 6 \Rightarrow 6\beta = 6$$

$$\boxed{\beta = 1}$$

If $\alpha = 4$ *in* $3\alpha\beta = 6$

$$3(4)\beta = 6$$

$$12\beta = 6 \Rightarrow \beta = \frac{6}{12}$$ is not an integers

$$\beta = \frac{1}{2}$$. *Hence, the common root is* 2.

Example 2.23 : $\boldsymbol{Find\ the\ values\ of\ p\ for\ which\ the\ difference\ between\ the\ roots\ of\ the\ equation\ x^2 + px + 8 = 0\ is\ 2.}$

Let α *and* β *be the roots of the equation* $x^2 + px + 8 = 0$

$$sum\ of\ the\ roots: \alpha + \beta = -\frac{b}{a}$$

$$\alpha + \beta = \frac{-p}{1} \Rightarrow \alpha + \beta = -p$$

$$product\ of\ the\ roots: \alpha\beta = \frac{c}{a}$$

$$\alpha\beta = \frac{8}{1} \Longrightarrow \alpha\beta = 8$$

$$Given: \alpha - \beta = 2$$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

$$2^2 = (-p)^2 - 4(8) \Longrightarrow p^2 - 32 = 4$$

$$p^2 = 36 \Longrightarrow p^2 = \sqrt{36} \Longrightarrow p = \pm 6$$

1. **Factorize: $x^4 + 1$.** (Hint: Try completing the square.)

$$x^4 + 1 = (x^2)^2 + 1 + 2x^2 - 2x^2$$

$$= \underbrace{(x^2)^2 + 2x^2 + 1}_{} - 2x^2$$

$$= (x^2 + 1)^2 - \left(\sqrt{2}x\right)^2$$

$$= \left(x^2 + 1 + \sqrt{2}x\right)\left(x^2 + 1 - \sqrt{2}x\right)$$

$$= \left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right)$$

2. If $x^2 + x + 1$ is a factor of the polynomial $3x^3 + 8x^2 + 8x + a$ find the value of a.

$$\begin{array}{r|l}
 & 3x + 5 \\ \hline
x^2 + x + 1 & 3x^3 + 8x^2 + 8x + a \\
 & (-)\quad (-)\quad (-) \\
 & 3x^3 + 3x^2 + 3x \\ \hline
 & 5x^2 + 5x + a \\
 & (-)\quad (-)\quad (-) \\
 & 5x^2 + 5x + 5 \\ \hline
 & 0 \\ \hline
\end{array}$$

$$\therefore a - 5 = 0 \Longrightarrow a = 5$$

That means a should be 5, if the remainder is zero.

$EXERCISE : 2.8$

2.7.1 Rational Inequalities

Example 2.24: $\boldsymbol{Solve\ \frac{x+1}{x+3} < 3}$.

$$\frac{x+1}{x+3} - 3 < 0.$$

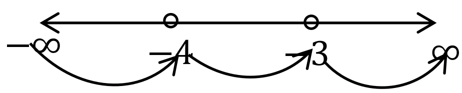
$$\frac{x+1-3(x+3)}{x+3} < 0 \Rightarrow \quad \frac{x+1-3x-9}{x+3} < 0$$

$$\frac{-2x-8}{x+3} < 0 \Rightarrow \underset{\div\, by -2}{\frac{-2(x+4)}{x+3}} < 0 \Rightarrow \quad \frac{x+4}{x+3} > 0 \quad x \neq -3$$

$So\ let\ us\ find\ out\ the\ signs\ of\ x+3\ and\ x+4$

$x^2 - 3x + 2 < 0$ is satisfied in $[1,2]$.

$$\frac{x+4}{x+3} = 0 \Rightarrow x+4 = 0 \Rightarrow x = -4$$



$The\ intervals\ are\ (-\infty, -4), (-4, -3)\ and\ (-3, \infty)$

$Intervals$	$\boldsymbol{x+3}$	$\boldsymbol{x+4}$	$\boldsymbol{\frac{x+4}{x+3}}$
$(-\infty, -4)$	$-$	$-$	$+$
$(-4, -3)$	$-$	$+$	$-$
$(-3, \infty)$	$+$	$+$	$+$
$x = -4$	$-$	0	0

$So\ the\ solution\ set\ is\ given\ by (-\infty, -4) \cup (-3, \infty)$

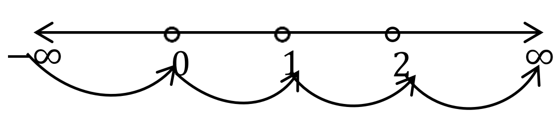
$\boldsymbol{1.\ Find\ all\ values\ of\ x\ for\ which\ \frac{x^3(x-1)}{x-2} > 0}$

$$\frac{x^3(x-1)}{x-2} > 0 \quad , x \neq 2$$

$$Let\ \frac{x^3(x-1)}{x-2} = 0 \Rightarrow \quad x^3(x-1) = 0$$

$$x^3 = 0,\ x - 1 = 0$$

$$x = 0,\ x = 1$$



$The\ intervals\ are\ (-\infty, 0), (0, 1), (1, 2)\ and\ (2, \infty)$

Intervals	x^3	$x-1$	$x-2$	$\dfrac{x^3(x-1)}{x-2}$
$(-\infty, 0)$	$-$	$-$	$-$	$-$
$(0,1)$	$+$	$-$	$-$	$+$
$(1,2)$	$+$	$+$	$-$	$-$
$(2,\infty)$	$+$	$+$	$+$	$+$

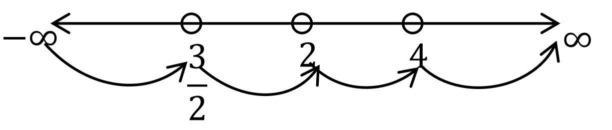
The solution set is $(0,1) \cup (2,\infty)$

2. Find all the values of x that satisfies the inequality $\dfrac{2x-3}{(x-2)(x-4)} < 0$

$$\frac{2x-3}{(x-2)(x-4)} < 0 \quad , x \neq 2, x \neq 4$$

$$Let\ \frac{2x-3}{(x-2)(x-4)} = 0 \implies 2x-3=0$$

$$2x = 3 \implies \boxed{x = \frac{3}{2}}$$



The intervals are $\left(-\infty, \frac{3}{2}\right), \left(\frac{3}{2}, 2\right), (2,4)$ *and* $(4,\infty)$

Intervals	$2x-3$	$x-2$	$x-4$	$\dfrac{2x-3}{(x-2)(x-4)}$
$\left(-\infty, \frac{3}{2}\right)$	$-$	$-$	$-$	$-$
$\left(\frac{3}{2}, 2\right)$	$+$	$-$	$-$	$+$
$(2,4)$	$+$	$+$	$-$	$-$
$(4,\infty)$	$+$	$+$	$+$	$+$

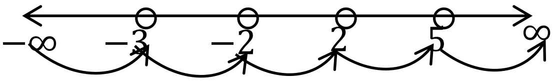
The solution set is $\left(-\infty, \frac{3}{2}\right) \cup (2,4)$

3. Solve $\dfrac{x^2-4}{x^2-2x-15} \leq 0$

$$\frac{x^2-2^2}{x^2-2x-15} \leq 0 \implies \frac{(x+2)\,(x-2)}{(x-5)\,(x+3)} \leq 0 \quad , x \neq -3, x \neq 5$$

$$Let\ \frac{(x+2)(x-2)}{(x-5)(x+3)} = 0 \implies (x+2)(x-2) = 0$$

$x+2=0,\ x-2=0 \Longrightarrow x=-2,\ x=2$



The intervals are $(-\infty,-3)$, $(-3\ ,-2), (-2,2), (2,5)$ *and* $(5,\infty)$

Intervals	$(x+2)$	$(x-2)$	$(x-5)$	$(x+3)$	$\frac{(x+2)(x-2)}{(x-5)(x+3)}$
$(-\infty,-3)$	$-$	$-$	$-$	$-$	$+$
$(-3,-2)$	$-$	$-$	$-$	$+$	$-$
$(-2,2)$	$+$	$-$	$-$	$+$	$+$
$(2,5)$	$+$	$+$	$-$	$+$	$-$
$(5,\infty)$	$+$	$+$	$+$	$+$	$+$

The solution set is $(-3,-2] \cup [2,5)$

EXERCISE : 2.9

Example 2.25. Resolve into partial fractions $\dfrac{x}{(x+3)(x-4)}$

Let $\dfrac{x}{(x+3)(x-4)} = \dfrac{A}{x+3} + \dfrac{B}{x-4}$

$\dfrac{x}{\cancel{(x+3)(x-4)}} = \dfrac{A(x-4)+B(x+3)}{\cancel{(x+3)(x-4)}} \implies x = A(x-4)+B(x+3)$

$\boxed{\begin{array}{l} x-4=0 \implies x=4 \\ x+3=0 \implies x=-3 \end{array}}$

$Put\ x=4,\ 4 = A(4-4)+B(4+3)$

$4 = A(0)+B(7) \implies 7B = 4 \implies \boxed{B=\frac{4}{7}}$

$Put\ x=-3, -3 = A(-3-4)+B(-3+3)$

$-3 = A(-7)+B(0) \implies -7A = -3 \implies \boxed{A = \frac{3}{7}}$

$$\therefore \frac{x}{(x+3)(x-4)} = \frac{\frac{3}{7}}{x+3} + \frac{\frac{4}{7}}{x-4} = \frac{3}{7(x+3)} + \frac{4}{7(x-4)}$$

Example 2.26. Resolve into partial fractions $\dfrac{2x}{(x-1)(x^2+1)}$

$Let\ \dfrac{2x}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$ $\boxed{x-1=0 \implies x=1}$

$\dfrac{2x}{\cancel{(x-1)(x^2+1)}} = \dfrac{A(x^2+1)+(Bx+C)\,(x-1)}{\cancel{(x-1)(x^2+1)}}$

$2x = A(x^2+1)+(Bx+C)\,(x-1)$

Put $x=1,\ 2(1) = A[(1)^2+1]+(B\times 1+C)\,(1-1)$

$2 = A(1+1)+(B+C)\,(0) \implies 2 = 2A+0$

$2 = 2A \implies A = 1$

$2x = A(x^2+1)+(Bx+C)\,(x-1)$

$put\ x=0, 2(0) = A(0^2+1)+(B\times 0+C)\,(0-1)$

$0 = A(1)+(0+C)(-1) \implies 0 = A-C\ \ where\ A=1$

$0 = 1-C \implies \boxed{C=1}$

$Equating\ the\ Co-efficient\ of\ x^2\ in\ (1)\ we\ get,$

$0 = A+B \implies 0 = 1+B \implies \boxed{B=-1}$

$\dfrac{2x}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}\ \ where A=1, B=-1, C=1$

$$\therefore \frac{2x}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{-1x+1}{x^2+1}$$

$$= \frac{1}{x-1} + \frac{1-x}{x^2+1}$$

***Example*: 2.27** Resolve into partial fractions : $\frac{\boldsymbol{x+1}}{\boldsymbol{x^2(x-1)}}$

Let $\frac{x+1}{x^2(x-1)} = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x^2}$

$$\frac{x+1}{\cancel{x^2(x-1)}} = \frac{Ax^2 + Bx(x-1) + C(x-1)}{\cancel{x^2(x-1)}} \qquad \boxed{x-1=0 \Rightarrow x=1}$$

$$x+1 = Ax^2 + Bx(x-1) + C(x-1)$$

$Put\ x = 1,\ 1+1 = A(1)^2 + B(1)(1-1) + C(1-1)$

$2 = A + B(1)(0) + C(0) \Rightarrow 2 = A + 0 + 0$

$2 = A \Rightarrow \boxed{A = 2}$

$Put\ x = 0,\ x+1 = Ax^2 + Bx(x-1) + C(x-1)$

$0+1 = A(0)^2 + B(0)(0-1) + C(0-1) \Rightarrow 1 = 0 + B(0)(-1) + C(-1)$

$1 = 0 + 0 - C \Rightarrow 1 = -C \Rightarrow \boxed{C = -1}$

$put\ x = -1, x+1 = Ax^2 + Bx(x-1) + C(x-1)$

$-1+1 = A(-1)^2 + B(-1)(-1-1) + C(-1-1)$

$0 = A + B(-1)(-2) + C(-2) \Rightarrow 0 = A + 2B - 2C$

$0 = A + 2B - 2C$

$sub\ A = 2\ and\ C = -1$ $\Rightarrow 0 = 2 + 2B - 2(-1) \Rightarrow 0 = 2 + 2B + 2$

$0 = 4 + 2B \Rightarrow -4 = 2B \Rightarrow 2B = -4 \Rightarrow \boxed{B = -2}$

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x^2}$$

$where\ A = 2, B = -2, C = -1$

$$\frac{x+1}{x^2(x-1)} = \frac{2}{x-1} + \frac{-2}{x} + \frac{-1}{x^2} \Rightarrow \boxed{\frac{x+1}{x^2(x-1)} = \frac{2}{x-1} - \frac{2}{x} - \frac{1}{x^2}}$$

1. ***Resolve into partial fractions*** $\frac{\boldsymbol{1}}{\boldsymbol{x^2 - a^2}}$

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} \Rightarrow \frac{1}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a}$$

$$\frac{1}{x^2-a^2} = \frac{A(x+a) + B(x-a)}{(x-a)(x+a)} \Rightarrow \frac{1}{\cancel{x^2-a^2}} = \frac{A(x+a) + B(x-a)}{\cancel{x^2-a^2}}$$

$$1 = A(x+a) + B(x-a) \ \ldots (1)$$

$Put\ x = a\ in\ (1)$

$$1 = A(a+a) + B(a-a) \implies 1 = 2aA + B(0) \implies 2aA = 1$$

$$\boxed{A = \frac{1}{2a}}$$

$Put\ x = -a\ in\ (1)$

$$1 = A(-a+a) + B(-a-a) \implies 1 = A(0) + B(-2a)$$

$$1 = B(-2a) \implies \frac{-1}{2a} = B \implies \boxed{B = -\frac{1}{2a}}$$

$$\therefore \frac{1}{x^2 - a^2} = \frac{\frac{1}{2a}}{x-a} - \frac{\frac{1}{2a}}{x+a}$$

$$\boxed{\frac{1}{x^2 - a^2} = \frac{1}{2a(x-a)} - \frac{1}{2a(x+a)}}$$

2. Resolve into partial fractions $\dfrac{\mathbf{3x+1}}{\mathbf{(x-2)(x+1)}}$

$$\frac{3x+1}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)} \implies \frac{3x+1}{\cancel{(x-2)(x+1)}} = \frac{A(x+1) + B(x-2)}{\cancel{(x-2)(x+1)}}$$

$$3x+1 = A(x+1) + B(x-2) \ \ldots (1)$$

$Put\ x = -1\ in\ (1)\ \ 3(-1) + 1 = A(-1+1) + B(-1-2)$

$$-3+1 = A(0) + B(-3) \implies -3+1 = B(-3) \implies -2 = -3B$$

$$\boxed{B = \frac{2}{3}}$$

$Put\ x = 2\ in\ (1),\ \ 3(2) + 1 = A(2+1) + B(2-2)$

$$6+1 = A(2+1) + B(0) \implies 7 = 3A$$

$$\boxed{A = \frac{7}{3}}$$

$$\frac{3x+1}{(x-2)(x+1)} = \frac{\frac{7}{3}}{x-2} + \frac{\frac{2}{3}}{x+1} \implies \frac{3x+1}{(x-2)(x+1)} = \frac{7}{3(x-2)} + \frac{2}{3(x+1)}$$

3. Resolve into partial fractions $\dfrac{\boldsymbol{x}}{\mathbf{(x^2+1)(x-1)(x+2)}}$

$$\frac{x}{(x^2+1)(x-1)(x+2)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)} + \frac{D}{(x+2)}$$

$$\frac{x}{\cancel{(x^2+1)(x-1)(x+2)}} = \frac{(Ax+B)(x-1)(x+2)+C(x^2+1)(x+2)+D(x^2+1)(x-1)}{\cancel{(x^2+1)(x-1)(x+2)}}$$

$$x = (Ax+B)(x-1)(x+2)+C(x^2+1)(x+2)+D(x^2+1)(x-1) \ldots (1)$$

$Put\ x = 1\ in\ (1)$

$$1 = 0 + C(1^2+1)(1+2) + 0 \Rightarrow 1 = C(1+1)(3)$$

$$1 = C(2)(3) \Rightarrow 1 = 6C \Rightarrow \boxed{C = \frac{1}{6}}$$

$Put\ x = -2\ in\ (1)$

$$-2 = 0 + 0 + D[(-2)^2+1](-2-1) \Rightarrow -2 = D(4+1)(-3)$$

$$-2 = D(5)(-3) \Rightarrow -2 = -15D$$

$$\boxed{D = \frac{2}{15}}$$

$Put\ x = 0\ in\ (1)$

$$0 = (0+B)(-1)(2) + C(0+1)(0+2) + D(0+1)(0-1)$$

$$0 = (B)(-1)(2) + C(1)(2) + D(1)(-1)$$

$$0 = -2\mathrm{B} + 2\mathrm{C} - \mathrm{D}$$

$$where\ C = \frac{1}{6}\ and\ D = \frac{2}{15}$$

$$0 = -2\mathrm{B} + 2\left(\frac{1}{6}\right) - \frac{2}{15} \Rightarrow 2B = \frac{2}{6} - \frac{2}{15} \Rightarrow 2B = \frac{1}{3} - \frac{2}{15}$$

$$2B = \frac{5-2}{15} \Rightarrow 2B = \frac{3}{15} \Rightarrow 2B = \frac{1}{5}$$

$$\mathrm{B} = \frac{1}{5} \times \frac{1}{2} \Rightarrow \boxed{B = \frac{1}{10}}$$

$$0 = \mathrm{A} + \mathrm{C} + \mathrm{D}$$

$Equating\ the\ Co-efficient\ of\ x^3\ in\ (1)$

$$A = -C - D \Rightarrow A = -\frac{1}{6} - \frac{2}{15} \Rightarrow A = \frac{-5-4}{30}$$

$$A = \frac{-9}{30} \Rightarrow \boxed{A = \frac{-3}{10}}$$

$$\therefore \frac{x}{(x^2+1)(x-1)(x+2)} = \frac{\frac{-3}{10}x + \frac{1}{10}}{x^2+1} + \frac{\frac{1}{6}}{x-1} - \frac{\frac{2}{15}}{x+2}$$

$$= \frac{\frac{-3x+1}{10}}{x^2+1} + \frac{\frac{1}{6}}{x-1} - \frac{\frac{2}{15}}{x+2}$$

$$= \frac{-3x+1}{10(x^2+1)} + \frac{1}{6(x-1)} + \frac{2}{15(x+2)}$$

$$\therefore \frac{x}{(x^2+1)(x-1)(x+2)} = \frac{1-3x}{10(x^2+1)} + \frac{1}{6(x-1)} + \frac{2}{15(x+2)}$$

4. *Resolve into partial fractions* $\boldsymbol{\frac{x}{(x-1)^3}}$

$$\frac{x}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

$$\frac{x}{\cancel{(x-1)^3}} = \frac{A(x-1)^2 + B(x+1) + C}{\cancel{(x-1)^3}}$$

$$x = A(x-1)^2 + B(x-1) + C \ldots (1)$$

Putting $x = 1$ *in* (1)

$$1 = A(1-1)^2 + B(1-1) + C$$

$$1 = A(0) + B(0) + C$$

$$\boxed{C = 1}$$

Putting $x = 0$ *in* (1) $\quad 0 = A(0-1)^2 + B(0-1) + C$

$$0 = A(-1)^2 + B(-1) + C \implies 0 = A - B + C$$

$$0 = A - B + 1 \implies A - B = -1 \ldots (2)$$

Equating the Co − efficient of x^2 *we get*

$$A = 0$$

Substituting $A = 0$ in (2) $\quad 0 - B = -1$

$$\boxed{B = 1}$$

$$\therefore \frac{x}{(x-1)^3} = \frac{0}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3}$$

$$\boxed{\frac{x}{(x-1)^3} = \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3}}$$

5. *Resolve into partial fractions* $\boldsymbol{\frac{1}{x^4-1}}$

$$\frac{1}{x^4-1} = \frac{1}{(x^2+1)(x^2-1)} \implies \frac{1}{x^4-1} = \frac{1}{(x^2+1)(x+1)(x-1)}$$

$$\frac{1}{x^4-1}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1}+\frac{D}{x-1}$$

$$\frac{1}{\cancel{x^4-1}}=\frac{(Ax+B)(x+1)(x-1)+C(x^2+1)(x-1)+D(x^2+1)(x+1)}{\cancel{(x^2+1)(x+1)}(x-1)}$$

$$1=(Ax+B)(x+1)(x-1)+C(x^2+1)(x-1)+D(x^2+1)(x+1)\ldots(1)$$

Putting $x=1$ in we get,

$$1=0+0+D(1^2+1)(1+1)\Longrightarrow 1=D(2)(2)\Longrightarrow 4D=1$$

$$\boxed{D=\frac{1}{4}}$$

Putting $x=-1$ in (1) we get,

$$1=0+C[(-1)^2+1](-1-1)+0\Longrightarrow 1=C(1+1)(-2)$$

$$1=C(2)(-2)\Longrightarrow 1=-4C$$

$$\boxed{C=-\frac{1}{4}}$$

Equating the Co – efficient of x^3 we get

$$0=A+C+D\Longrightarrow A=-C-D \text{ where } C=\frac{-1}{4} \text{ and } D=\frac{1}{4}$$

$$A=\frac{1}{4}-\frac{1}{4}=0\Longrightarrow \boxed{A=0}$$

Putting $x=0$ in (1) we get,

$$1=B(0+1)(0-1)+C(0+1)(0-1)+D(0+1)(0+1)$$

$$1=B(1)(-1)+C(1)(-1)+D(1)(1)\Longrightarrow 1=-B-C+D$$

$$\text{where } C=-\frac{1}{4} \text{ and } D=\frac{1}{4}$$

$$1=-B+\frac{1}{4}+\frac{1}{4}\Longrightarrow 1=-B+\frac{2}{4}\Longrightarrow 1=-B+\frac{1}{2}$$

$$B=-1+\frac{1}{2}\Longrightarrow \boxed{B=-\frac{1}{2}}$$

$$\therefore \frac{1}{x^4-1}=\frac{0x-\frac{1}{2}}{x^2+1}+\frac{-\frac{1}{4}}{x+1}+\frac{\frac{1}{4}}{x-1}=\frac{-\frac{1}{2}}{x^2+1}-\frac{\frac{1}{4}}{x+1}+\frac{\frac{1}{4}}{x-1}$$

$$\frac{1}{x^4-1}=-\frac{1}{2(x^2+1)}-\frac{1}{4(x+1)}+\frac{1}{4(x-1)}$$

6. Resolve into partial fractions $\dfrac{(x-1)^2}{x^3+x}$

$$\frac{(x-1)^2}{x^3+x} = \frac{(x-1)^2}{x(x^2+1)} \implies \frac{(x-1)^2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\frac{(x-1)^2}{\cancel{x(x^2+1)}} = \frac{A(x^2+1)+(Bx+c)x}{\cancel{x(x^2+1)}}$$

$$(x-1)^2 = A(x^2+1) + (Bx+C)(x) \quad \dots(1)$$

Putting $x = 0$ *in* (1) *we get,* $(0-1)^2 = A(0^2+1) + (B\times 0 + C)(0)$

$$(-1)^2 = A(1) + 0 \implies \boxed{A = 1}$$

Putting $x = 1$ *in* (1) *we get,* $(1-1)^2 = A(1^2+1) + (B\times 1 + C)\times 1$

$$0 = A(2) + (B+C)(1) \implies 2A + B + C = 0 \dots(2)$$

Equating the Co-efficient of x^2 *in* (1)

$$1 = A + B \implies 1 = 1 + B \implies \boxed{B = 0}$$

Substituting $A = 1, B = 0$ *in* (2) *we get,* $0 = 2 + 0 + C \implies \boxed{C = -2}$

$$\therefore \frac{(x-1)^2}{x^3+x} = \frac{1}{x} + \frac{0x-2}{x^2+1} \implies \frac{(x-1)^2}{x^3+x} = \frac{1}{x} - \frac{2}{x^2+1}$$

7. Resolve into partial fractions $\dfrac{x^2+x+1}{x^2-5x+6}$

Since the degree of the numerator is equal to the degree of the denominator, let us divide the numerator by the denominator.

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2-5+6 & x^2+x+1 \\ & (-)\ (+)\ \ (-) \\ & x^2-5x+6 \\ \hline & \qquad 6x-5 \end{array}$$

$$\therefore \frac{x^2+x+1}{x^2-5x+6} = 1 + \frac{6x-5}{x^2-5x+6} \dots(1)$$

Take $\dfrac{6x-5}{x^2-5x+6} = \dfrac{6x-5}{(x-3)(x-2)} \implies \dfrac{6x-5}{x^2-5x+6} = \dfrac{A}{(x-3)} + \dfrac{B}{(x-2)}$

$$\frac{6x-5}{x^2-5x+6} = \frac{A(x-2)+B(x-3)}{(x-3)(x-2)}$$

$$6x-5 = A(x-2) + B(x-3) \dots(2)$$

$Putting\ x = 2\ in\ (2)\ we\ get, 6(2) - 5 = A(0) + B(2-3)$

$12 - 5 = B(-1) \Longrightarrow 7 = B(-1)$

$7 = -B \Longrightarrow \boxed{B = -7}$

$Putting\ x = 3\ in\ (2)\ we\ get,$

$6(3) - 5 = A(3-2) + B(3-3) \Longrightarrow 18 - 5 = A(1) + 0$

$13 = A(1) \Longrightarrow \boxed{A = 13}$

$$\therefore \frac{6x-5}{x^2-5x+6} = \frac{13}{x-3} - \frac{7}{x-2}$$

Substituting this in (1) we get, $\therefore \dfrac{x^2+x+1}{x^2-5x+6} = 1 + \dfrac{6x-5}{x^2-5x+6}$

$$\frac{x^2+x+1}{x^2-5x+6} = 1 + \frac{13}{x-3} - \frac{7}{x-2}$$

8. $\boldsymbol{\dfrac{x^3+2x+1}{x^2+5x+6}}$

Since the numerator′s degree is more than the denominator′s degree, let us divide the numerator by the denominator.

$$\begin{array}{r|l}
 & \quad x - 5 \\
\hline
x^2+5x+6 & x^3 + 2x + 1 \\
 & (-)\ \ (-)\ \ (-) \\
 & x^3 + 5x^2 + 6x \\
\hline
 & \quad -5x^2 - 4x + 1 \\
 & \quad (+)\ \ (+)\ \ (+) \\
 & \quad -5x^2 - 25x - 30 \\
\hline
 & \qquad \underline{21x + 31}
\end{array}$$

$$\therefore \frac{x^2+2x+1}{x^2+5x+6} = x - 5 + \frac{21x+31}{x^2+5x+6} \quad \ldots(1)$$

$$Consider\ \frac{21x+31}{x^2+5x+6} = \frac{21x+31}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2}$$

$$21x + 31 = A(x+2) + B(x+3) \quad \ldots(2)$$

$Putting\ x = -2\ in\ (2)\ we\ get,\ 21(-2) + 31 = A(-2+2) + B(-2+3)$

$-11 = A(0) + B(1) \Longrightarrow -11 = B(1)$

$\boxed{B = -11}$

$Putting\ x = -3\ in\ (2)\ we\ get, 21(-3) + 31 = A(-3+2) + B(-3+3)$

$-32 = A(-1) + B(0) \Longrightarrow -32 = A(-1) \Longrightarrow -32 = -A$

$\boxed{A = 32}$

$$\therefore \frac{21x+31}{x^2+5x+6} = \frac{32}{x+3} - \frac{11}{x+2}$$

Substituting this in (1) we get,

$$\frac{x^2+2x+1}{x^2+5x+6} = (x-5) + \frac{32}{x+3} - \frac{11}{x+2}$$

9. solve $\dfrac{\boldsymbol{x+12}}{\boldsymbol{(x+1)^2\ (x-2)}}$

$$\frac{x+12}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

$$\frac{x+12}{(x+1)^2(x-2)} = \frac{A(x+1)(x-2)+B(x-2)+C(x+1)^2}{(x+1)^2(x-2)}$$

$$x+12 = A(x+1)(x-2)+B(x-2)+C(x+1)^2 \ \dots(1)$$

Putting $x=-1$ *in* (1)

$$-1+12 = A(-1+1)(-1-2)+B(-1-2)+C(-1+1)^2$$

$$11 = A(0)(-3)+B(-3)+C(0)^2$$

$$11 = B(-3) \Rightarrow 3B = -11 \Rightarrow \boxed{B = -\frac{11}{3}}$$

Putting $x=2$ *in (1) we get,*

$$2+12 = A\,(2+1)\,(2-2)+B\,(2-2)+C\,(2+1)^2$$

$$14 = A(3)(0)+B(0)+C(3)^2 \Rightarrow 14 = C(9)$$

$$9C = 14 \Rightarrow \boxed{C = \frac{14}{9}}$$

Equating the Co – efficient of x^2 *in (1) we get,*

$$0 = A + C \Rightarrow A = -C \Rightarrow \boxed{A = -\frac{14}{9}}$$

$$\therefore \frac{x+12}{(x+1)^2\ (x-2)} = \frac{\frac{-14}{9}}{x+1} - \frac{\frac{11}{3}}{(x+1)^2} + \frac{\frac{14}{9}}{x-2}$$

$$= -\frac{14}{9(x+1)} - \frac{11}{3(x+1)^2} + \frac{14}{9(x-2)}$$

10. solve $\dfrac{\boldsymbol{6x^2-x+1}}{\boldsymbol{x^3+x^2+x+1}}$

$$\frac{6x^2-x+1}{x^3+x^2+x+1} = \frac{6x^2-x+1}{x^2(x+1)+1(x+1)} = \frac{6x^2-x+1}{(x^2+1)(x+1)}$$

$$\frac{6x^2 - x + 1}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

$$6x^2 - x + 1 = (Ax+B)(x+1) + C(x^2+1) \quad \ldots(1)$$

Putting $x = -1$ in (1) we get,

$$6(-1)^2 + 1 + 1 = (A(-1)+B)(-1+1) + C\left((-1)^2+1\right)$$

$$6 + 1 + 1 = (-A+B)(0) + C(1+1) \Rightarrow 8 = C(2) \Rightarrow 8 = 2C$$

$$2C = 8 \Rightarrow \boxed{C = 4}$$

Equation the Co − efficient of x^2 in (1) we get,

$$6 = A + C \Rightarrow 6 = A + 4 \Rightarrow A = 6 - 4$$

$$\boxed{A = 2}$$

Putting $x = 0$ in (1) we get, $6(0) - 0 + 1 = (A \times 0 + B)(0+1) + C(0+1)$

$$1 = B + C \Rightarrow 1 = B + 4 \Rightarrow 1 - 4 = B$$

$$\boxed{B = -3}$$

$$\therefore \frac{6x^2 - x + 1}{x^3 + x^2 + x + 1} = \frac{2x-3}{x^3+1} + \frac{4}{x+1}$$

11. *Resolve into partial fractions* $\boldsymbol{\dfrac{2x^2 + 5x - 11}{x^2 + 2x - 3}}$

Since the numerator*'s degree is equal to the denominatorsdegree, let us divide the numerator by the denominator.*

$$\begin{array}{r|l} & \quad 2 \\ \hline x^2 + 2x - 3 & 2x^2 + 5x - 11 \\ & (-) \\ & 2x^2 + 4x - 6 \\ \hline & \quad \underline{x - 5} \end{array}$$

$$\therefore \frac{2x^2 + 5x - 11}{x^2 + 2x - 3} = 2 + \frac{x-5}{x^2+2x-3} \quad \ldots(1)$$

$$\text{Consider } \frac{x-5}{x^2+2x-3} = \frac{x-5}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

$$\therefore x - 5 = A(x-1) + B(x+3) \quad \ldots(2)$$

Putting $x = 1$ in (2) we get $1 - 5 = A(1-1) + B(1+3)$

$$-4 = B(4) \Rightarrow 4B = -4 \Rightarrow \boxed{B = -1}$$

Put $x = -3$ in (2) we get, $-3 - 5 = A(-3-1) + B(-3+3)$

$$-8 = A(-4) + B(0) \Rightarrow -8 = -4A \Rightarrow 4A = 8$$

$$A = \frac{8}{4} \implies \boxed{A = 2}$$

$$\therefore \frac{x-5}{x^2+2x-3} = \frac{2}{x+3} - \frac{1}{x-1}$$

12. *Resolve into partial fractions* $\boldsymbol{\dfrac{7+x}{(1+x)(1+x^2)}}$

$$\frac{7+x}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{x^2+1}$$

$$\frac{7+x}{\cancel{(1+x)(1+x^2)}} = \frac{A(x^2+1)+(Bx+c)(x+1)}{\cancel{(1+x)(x^2+1)}}$$

$$x+7 = A(x^2+1) + (Bx+C)(x+1) \ldots (1)$$

Put $x = -1$ *in* (1)

$$-1+7 = A[(-1)^2+1] + (B(-1)+C)(-1+1)$$

$$6 = A(2) + (-B+C)(0) \implies 6 = A(2) \implies 2A = 6$$

$$A = \frac{6}{2} \implies \boxed{A = 3}$$

Put $x = 0$ *in* (1), $0+7 = A[(0)^2+1] + (B(0)+C)(0+1)$

$$7 = A(1) + C(1) \implies 7 = A + C \ \text{ where } A = 3$$

$$7 = 3 + C \implies \boxed{C = 4}$$

Equating the Co-efficient of x^2 *in* (1) *we get,*

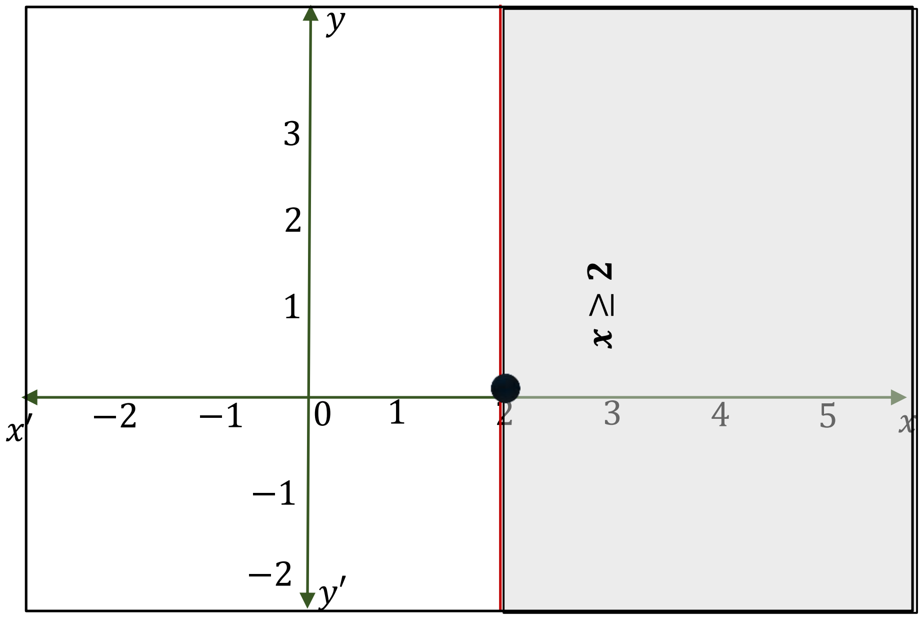
$$0 = A + B \implies 0 = 3 + B \implies \boxed{B = -3}$$

$$\therefore \frac{7+x}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{x^2+1}$$

$$= \frac{3}{1+x} + \left(\frac{-3x+4}{x^2+1}\right)$$

$EXERCISE : 2.5$

Example 2.28 Shade the region given by the inequality $x \geq 2$



Example 2.29 Shade the region given by the linear inequality $x + 2y > 3$

$x + 2y > 3$

$Let\ x + 2y = 3$

$x - intercept$

$put\ y = 0$

$x + 2(0) = 3 \Longrightarrow x = 3$

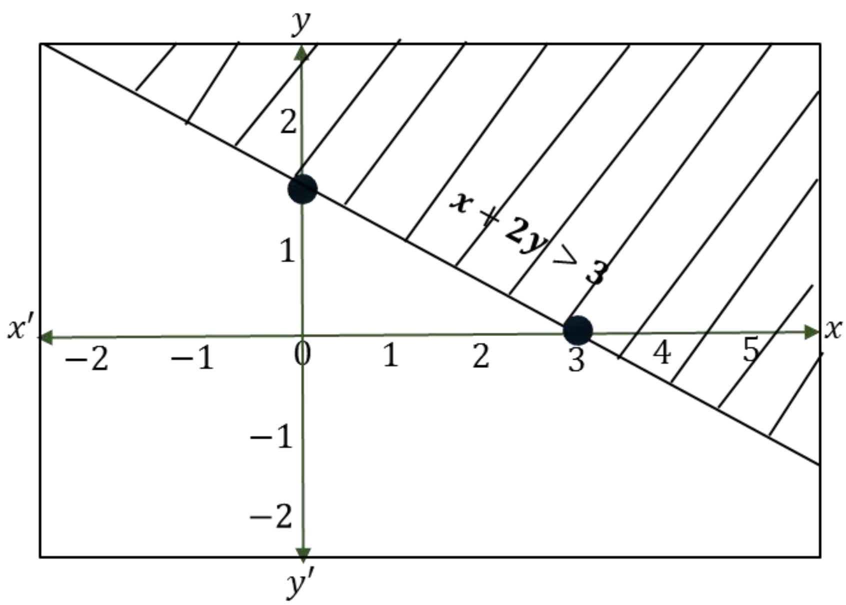
$y - intercept$

$put\ x = 0$

$0 + 2y = 3 \Longrightarrow y = \frac{3}{2}$

$y = 1.5$

x	3	0
y	0	1.5

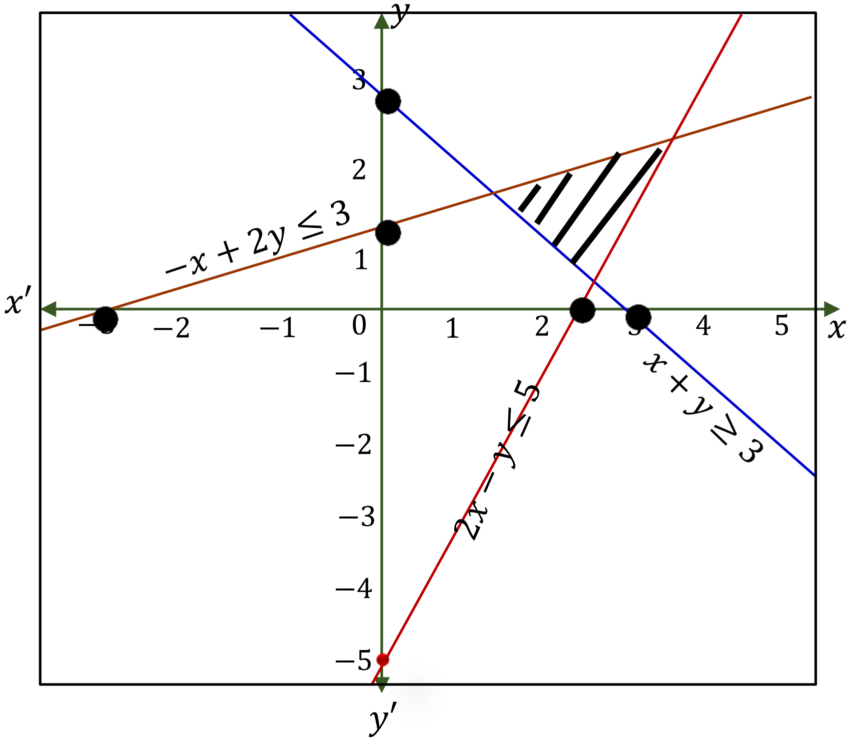


Example 2.29 Shade the region given by the linear inequality

$x + y \geq 3, 2x - y \leq 5, -x + 2y \leq 3.$

$x + y \geq 3$ $Let\ x + y = 3$ $x - intercept$ $put\ y = 0$ $x + 0 = 3 \Longrightarrow x = 3$	$2x - y \leq 5$ $Let\ 2x - y = 5$ $x - intercept$ $put\ y = 0$ $2x - 0 = 5 \Longrightarrow x = \frac{5}{2}$	$-x + 2y \leq 3$ $Let\ -x + 2y = 3$ $x - intercept$ $put\ y = 0$ $-x + 2(0) = 3 \Longrightarrow x = -3$

$y - intercept$ $put\ x = 0$ $0 + y = 3 \Rightarrow y = 3$ $y = 3$	$y - intercept$ $put\ x = 0$ $0 - y = 5 \Rightarrow y = -5$ $y = -5$	$y - intercept$ $put\ x = 0$ $0 + 2y = 3 \Rightarrow y = \frac{3}{2}$ $y = 1.5$



$x + y \geq 3$

x	3	0
y	0	3

$2x - y \leq 5$

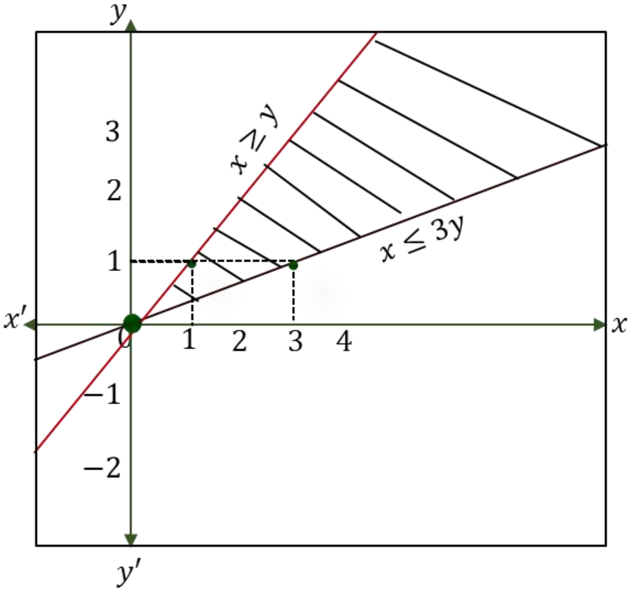
x	2.5	0
y	0	−5

$-x + 2y \leq 3$

x	−3	0
y	0	1.5

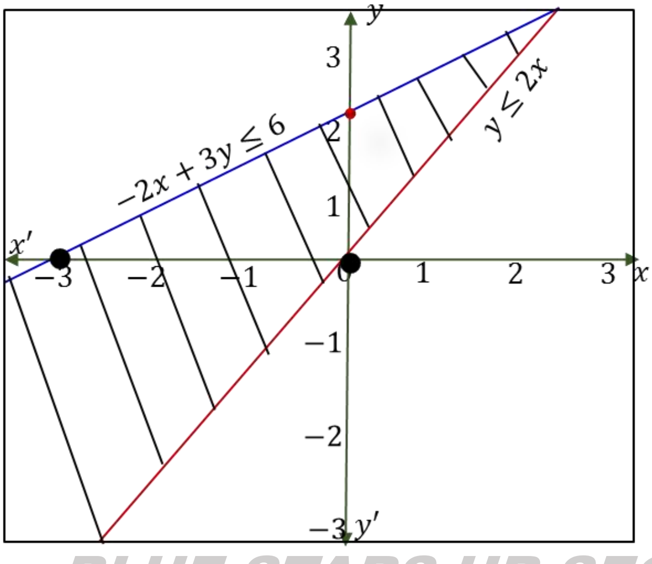
Ex:1 Shade the region given by the linear inequality $x \leq 3y, x \geq y$

$x \leq 3y$ $Let\ x = 3y$ $x - intercept$ $put\ y = 0$ $x = 3(0) \Rightarrow x = 0$ $y - intercept$ $put\ x = 0$ $0 = 3y \Rightarrow y = 0$ $y = 0$	$x \geq y$ $Let\ x = y$ $x - intercept$ $put\ y = 0$ $x = 0$ $y - intercept$ $put\ x = 0$ $y = 0$



Ex:2 Shade the region given by the linear inequality $y \geq 2x$, $-2x + 3y \leq 6$

$y \geq 2x$ $Let\ y = 2x$ $x - intercept$ $put\ y = 0$ $0 = 2x \Rightarrow x = 0$ $y - intercept$ $put\ x = 0$ $y = 2(0) \Rightarrow y = 0$ $y = 0$	$-2x + 3y \leq 6$ $Let - 2x + 3y = 6$ $x - intercept$ $put\ y = 0$ $-2x + 3(0) = 6$ $-2x = 6 \Rightarrow x = -3$ $y - intercept$ $put\ x = 0$ $-2(0) + 3y = 6$ $3y = 6 \Rightarrow y = 2$

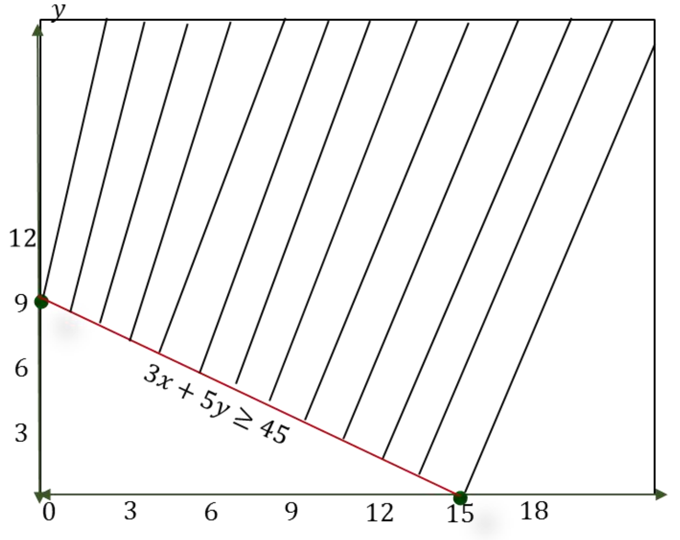


Ex:3 Shade the region given by the linear inequality $3x + 5y \geq 45$, $x \geq 0, y \geq 0$

$3x + 5y \geq 45$

$Let\ 3x + 5y = 45$

$x - intercept$ | $y - intercept$
put $y = 0$ | put $x = 0$
$3x + 5(0) = 45$ | $3(0) + 5y = 45$
$3x = 45 \implies x = 15$ | $5y = 45 \implies y = 9$



Ex:4 Shade the region given by the linear inequality $2x + 3y \leq 35$, $y \geq 2, x \geq 5$

$2x + 3y \geq 35$

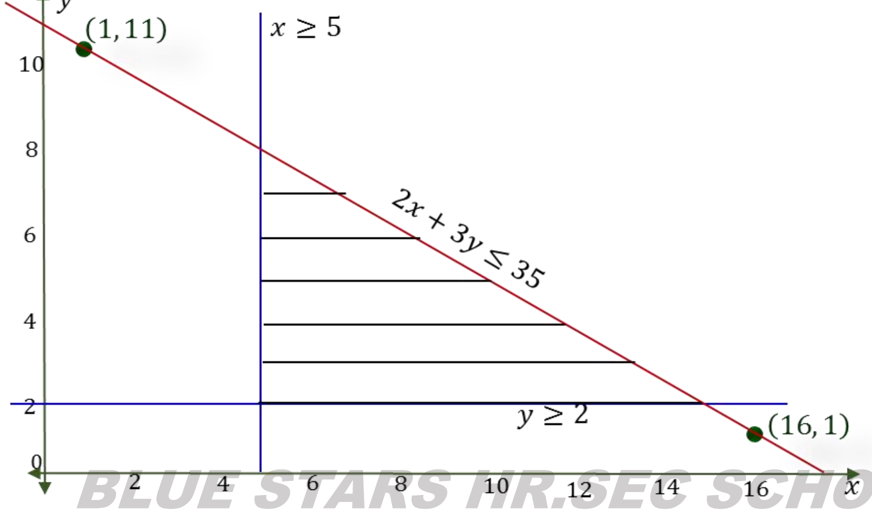
$Let\ 2x + 3y = 35$

put $y = 1$ $\implies 2x + 3(1) = 35 \implies 2x + 3 = 35$

$2x = 32 \implies x = 16$

put $x = 1 \implies 2(1) + 3y = 35 \implies 2 + 3y = 35$

$3y = 33 \implies y = 11$



Ex:4 Shade the region given by the linear inequality $2x + 3y \leq 35$, $y \geq 2, x \geq 5$

$2x + 3y \geq 35$

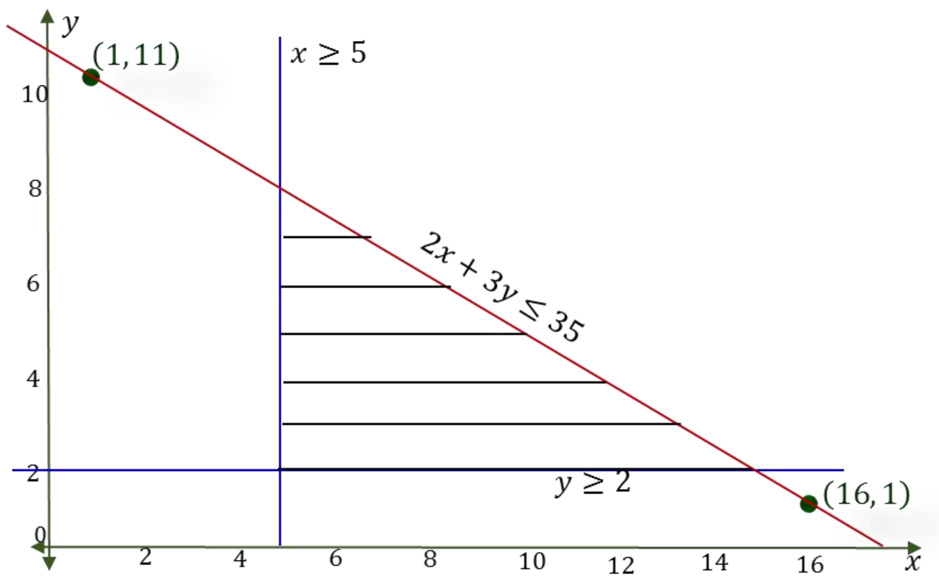
$Let\ 2x + 3y = 35$

$put\ y = 1$ $\Rightarrow$ $2x + 3(1) = 35 \Rightarrow 2x + 3 = 35$

$2x = 32 \Rightarrow x = 16$

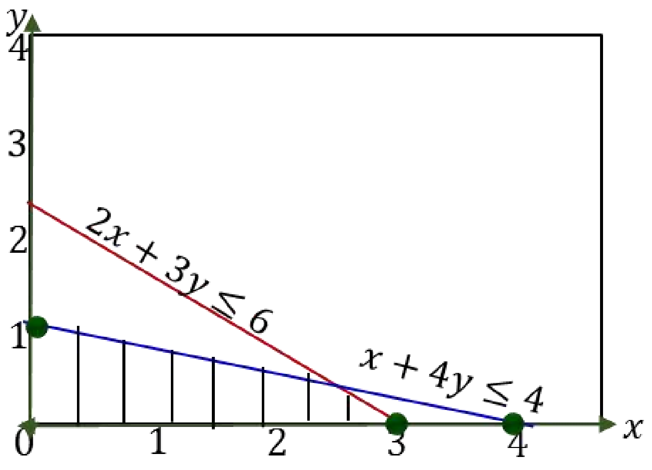
$put\ x = 1 \Rightarrow 2(1) + 3y = 35 \Rightarrow 2 + 3y = 35$

$3y = 33 \Rightarrow y = 11$



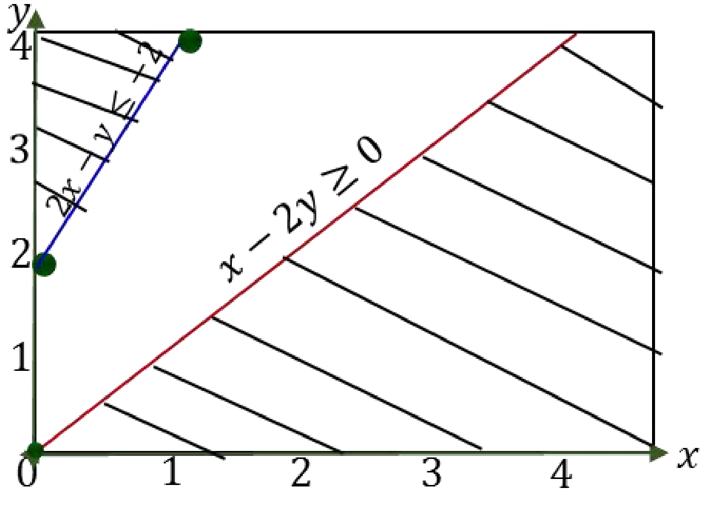
Ex:5 Shade the region given by the linear inequality $2x + 3y \leq 6$, $x + 4y \leq 4, x \geq 0, y \geq 0$

$2x + 3y \leq 6$ $Let\ 2x + 3y = 6$ $x - intercept$ $put\ y = 0$ $2x + 3(0) = 6$ $2x = 6 \Rightarrow x = 3$ $y - intercept$ $put\ x = 0$ $2(0) + 3y = 6$ $3y = 6 \Rightarrow y = 2$	$x + 4y \geq 4$ $Let\ x + 4y = 4$ $x - intercept$ $put\ y = 0$ $x + 4(0) = 4$ $x = 4$ $y - intercept$ $put\ x = 0$ $0 + 4y = 4$ $4y = 4 \Rightarrow y = 1$



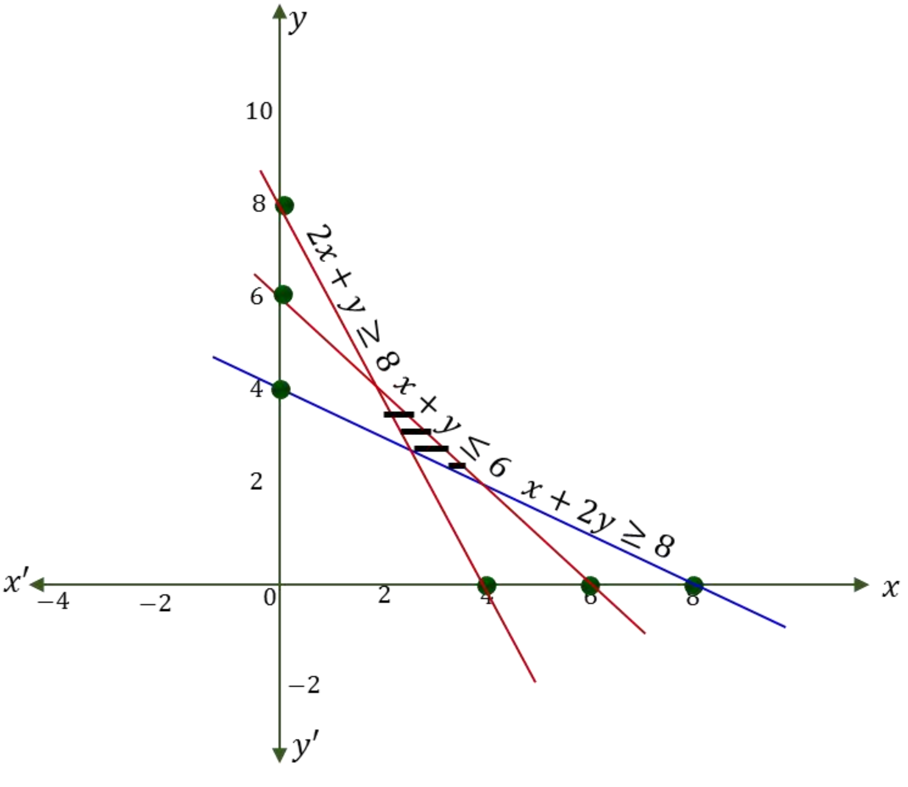
Ex:6 Shade the region given by the linear inequality $x - 2y \geq 0$, $2x - y \leq -2, x \geq 0, y \geq 0$

$x - 2y \geq 0$ $Let\ x - 2y = 0$ $x - intercept$ $put\ y = 0$ $x - 2(0) = 0$ $x = 0$ $y - intercept$ $put\ x = 0$ $0 - 2y = 0$ $2y = 0 \implies y = 0$	$2x - y \leq -2$ $Let\ 2x - y = -2$ $put\ y = 4$ $2x - 4 = -2$ $2x = -2 + 4$ $2x = 2$ $x = 1$ $put\ x = 0$ $2(0) - y = -2$ $-y = -2 \implies y = 2$



Ex:7 Shade the region given by the linear inequality
$$2x + y \geq 8,\ x + 2y \geq 8,\ x + y \leq 6.$$

$2x + y \geq 8$ $Let\ 2x + y = 8$ $x - intercept$ $put\ y = 0$ $2x + 0 = 8 \implies x = 4$ $y - intercept$ $put\ x = 0$ $2(0) + y = 8$ $y = 8$	$x + 2y \geq 8$ $Let\ x + 2y = 8$ $x - intercept$ $put\ y = 0$ $x + 0 = 8 \implies x = 8$ $y - intercept$ $put\ x = 0$ $0 + 2y = 8$ $y = 4$	$x + y \leq 6$ $Let\ x + y = 6$ $x - intercept$ $put\ y = 0$ $x + 0 = 6 \implies x = 6$ $y - intercept$ $put\ x = 0$ $0 + y = 6$ $y = 6$



$EXERCISE : 2.11$

Example 2.31 (i) ***Simplify***: $(x^{\frac{1}{2}}y^{-3})^{\frac{1}{2}}$; ***where***, $x, y \geq 0$.

(ii) ***Simplify***: $\sqrt{x^2 - 10x + 25}$.

(i) $Since\ x, y \geq 0,$

$$(x^{\frac{1}{2}}y^{-3})^{\frac{1}{2}} = \left(\frac{x^{\frac{1}{2}}}{y^3}\right)^{\frac{1}{2}} = \frac{x^{\frac{1}{4}}}{y^{\frac{3}{2}}}$$

$$(ii)\sqrt{x^2 - 10x + 25} = \sqrt{x^2 - 2(x)(5) + 5^2} = \sqrt{(x-5)^2}$$

$$= |x - 5|$$

Example 2.32 ***Rationalize the denominator of*** $\dfrac{\sqrt{5}}{(\sqrt{6}+\sqrt{2})}$.

$$\frac{\sqrt{5}}{(\sqrt{6}+\sqrt{2})} = \frac{\sqrt{5}}{\sqrt{6}+\sqrt{2}} \times \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}$$

$Multiplying\ both\ numerator\ and\ denominator\ by\ \sqrt{6}-\sqrt{2}$

$$= \frac{\sqrt{5}(\sqrt{6}-\sqrt{2})}{(\sqrt{6})^2-(\sqrt{2})^2} = \frac{\sqrt{5}(\sqrt{6}-\sqrt{2})}{6-2} = \frac{\sqrt{30}-\sqrt{10}}{4}$$

Example 2.33 ***Find the square root of*** $7 - 4\sqrt{3}$.

$\sqrt{7-4\sqrt{3}} = a + b\sqrt{3}$ $where\ a, b\ are\ rational.$

$Squaring\ on\ both\ sides$

$$7 - 4\sqrt{3} = (a + b\sqrt{3})^2 \Longrightarrow 7 - 4\sqrt{3} = a^2 + 2a(b\sqrt{3}) + (b\sqrt{3})^2$$

$$7 - 4\sqrt{3} = a^2 + 2a\,b\sqrt{3} + 3b^2 \Longrightarrow a^2 + 3b^2 + 2\,a\,b\sqrt{3} = 7 - 4\sqrt{3}$$

$$a^2 + 3b^2 = 7,\ \cancel{2}ab = -\cancel{4}^{-2}$$

$$\boxed{a = -\frac{2}{b}}$$

$$sub\ \ a = -\frac{2}{b} in\ a^2 + 3b^2 = 7 \Longrightarrow \left(-\frac{2}{b}\right)^2 + 3b^2 = 7 \Longrightarrow \frac{4}{b^2} + 3b^2 = 7$$

$$\frac{4+3b^4}{b^2} = 7 \Longrightarrow 4 + 3b^4 = 7b^2 \Longrightarrow 3b^4 - 7b^2 + 4 = 0$$

$$3(b^2)^2 - 7b^2 + 4 = 0$$

$Let\ b^2 = x$

$3x^2 - 7x + 4 = 0.$

$A = 3, B = -7,\ \ c = 4$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \implies x = \frac{-7 \pm \sqrt{49 - 4\,(3)(4)}}{2(3)} = \frac{7 \pm \sqrt{49 - 48}}{6}$$

$$x = \frac{7 \pm 1}{6} \implies x = \frac{7+1}{6},\ \ \frac{7-1}{6} \implies b^2 = \frac{8}{6},\ \frac{6}{6} \implies b^2 = \frac{4}{3},\ b^2 = 1$$

$$b = \sqrt{\frac{4}{3}},\quad b = \sqrt{1} \implies \mathrm{b} = \pm\frac{2}{\sqrt{3}},\ \mathrm{b} = \pm 1$$

$Since\ b\ is\ a\ rational, we\ have\ \ b = \pm 1$

$$\boxed{a = -\frac{2}{b}}$$

$if\ \ b = 1\ then\ \ a = -\frac{2}{1}$

$a = -2,$

$if\ \ b = -1\ then\ a = \frac{-2}{-1}$

$a = 2$

$\sqrt{7 - 4\sqrt{3}} = a + b\sqrt{3}\ \ Since\ \sqrt{7 - 4\sqrt{3}} > 0$

$hence\ the\ values\ of\ a = 2\ and\ b = -1$

$$\sqrt{7 - 4\sqrt{3}} = 2 - \sqrt{3}$$

1. $Simplify(i)\ (125)^{\frac{2}{3}}$

$$(125)^{\frac{2}{3}} = (5^3)^{\frac{2}{3}}$$

$$= 5^{3\times\frac{2}{3}} = 5^2 = 25$$

$(ii)\ 16^{-\frac{3}{4}}$

$$16^{-\frac{3}{4}} = (2^4)^{-\frac{3}{4}}$$

$$= 2^{4\times\frac{-3}{4}} = (2)^{-3} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$(iii)\ (-1000)^{-\frac{2}{3}}$

$$\boxed{\therefore (a^m)^n = a^{mn}}$$

$$(-1000)^{-\frac{2}{3}} = (-10^3)^{-\frac{2}{3}}$$

$$= (-10)^{3\times\frac{-2}{3}}$$

$$= (-10)^{-2} = \left(-\frac{1}{10}\right)^2 = \frac{1}{100}$$

$(iv)\ \left(3^{-6}\right)^{\frac{1}{3}}$

$$= 3^{-6\times\frac{1}{3}} = 3^{-2} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$(v)\ \dfrac{(27)^{-\frac{2}{3}}}{(27)^{-\frac{1}{3}}} = (27)^{-\frac{2}{3}+\frac{1}{3}}$

$\boxed{\dfrac{a^m}{a^n} = a^{m-n}}$

$= (27)^{\frac{-2+1}{3}} = (27)^{-\frac{1}{3}} = (3^3)^{-\frac{1}{3}} = 3^{-1}$

$= \dfrac{1}{3}$

2. Evaluate $\left(\left[(256)^{\frac{-1}{2}}\right]^{\frac{-1}{4}}\right)^3$

$\boxed{(a^m)^n = a^{mn}}$

$\left(\left[(256)^{-\frac{1}{2}}\right]^{-\frac{1}{4}}\right)^3 = (256)^{-\frac{1}{2}\times-\frac{1}{4}\times 3} = (256)^{\frac{3}{8}}$

$= (2^8)^{\frac{3}{8}} = 2^3 = 8$

3. $\boldsymbol{If\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)^2 = \dfrac{9}{2}, then\ find\ the\ value\ of\left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right) for\ x > 1.}$

$\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)^2 = \dfrac{9}{2}$

$\left(x^{\frac{1}{2}}\right)^2 + \left(x^{-\frac{1}{2}}\right)^2 + 2x^{\frac{1}{2}}x^{-\frac{1}{2}} = \dfrac{9}{2} \Longrightarrow x + x^{-1} + 2x^{\frac{1}{2}-\frac{1}{2}} = \dfrac{9}{2}$

$x + \dfrac{1}{x} + 2x^0 = \dfrac{9}{2} \Longrightarrow x + \dfrac{1}{x} + 2 = \dfrac{9}{2}$

$x + \dfrac{1}{x} = \dfrac{9}{2} - 2 \Longrightarrow x + \dfrac{1}{x} = \dfrac{9-4}{2} \Longrightarrow \boxed{x + \dfrac{1}{x} = \dfrac{5}{2}}$

$\left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)^2 = \left(x^{\frac{1}{2}}\right)^2 + \left(x^{-\frac{1}{2}}\right)^2 - 2x^{\frac{1}{2}}x^{-\frac{1}{2}}$

$\left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)^2 = x + x^{-1} - 2 \Longrightarrow \left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)^2 = x + \dfrac{1}{x} - 2$

$\left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)^2 = \dfrac{5}{2} - 2 \Longrightarrow \left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)^2 = \dfrac{5-4}{2}$

$\left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)^2 = \dfrac{1}{2} \Longrightarrow x^{\frac{1}{2}} - x^{-\frac{1}{2}} = \sqrt{\dfrac{1}{2}}$

$x^{\frac{1}{2}} - x^{-\frac{1}{2}} = \pm\dfrac{1}{\sqrt{2}} \Longrightarrow x^{\frac{1}{2}} - x^{-\frac{1}{2}} = \dfrac{1}{\sqrt{2}}$ *Since* $x > 1$

4. $\boldsymbol{Simplify\ and\ hence\ find\ the\ value\ of\ n: \dfrac{3^{2n}9^2 3^{-n}}{3^{3n}} = 27.}$

Given: $\dfrac{3^{2n}9^2 3^{-n}}{3^{3n}} = 27$

$\boxed{a^m \times a^n = a^{m+n}}$

$\dfrac{3^{2n-n}\times 9^2}{3^{3n}} = 27 \Longrightarrow \dfrac{3^n \times (3^2)^2}{3^{3n}} = 27$

$\boxed{\dfrac{a^m}{a^n} = a^{m-n}\ \&\ (a^m)^n = a^{mn}}$

$$3^{n-3n} \times (3^4) = 27 \Longrightarrow 3^{-2n} \times 3^4 = 27$$

$$3^{-2n+4} = 3^3$$

Equating the powers both sides we get,

$$-2n + 4 = 3 \Longrightarrow -2n = 3 - 4$$

$$-2n = -1 \Longrightarrow 2n = 1$$

$$\boxed{n = \frac{1}{2}}$$

5. Find the radius of the spherical tank whose volume is $\frac{32\pi}{3}$ units.

Let r be the radius of the spherical tank.

$$\text{Volume of the spherical tank} = \frac{32\pi}{3}$$

$$\frac{4}{\cancel{3}}\cancel{\pi}r^3 = \frac{32\cancel{\pi}}{\cancel{3}} \Longrightarrow 4r^3 = 32$$

$$r^3 = \frac{32}{4} \Longrightarrow r^3 = 8 \Longrightarrow r^3 = 2^3$$

$$\boxed{r = 2}$$

Radius of the spherical tank is 2 unit.

6. Simplify by rationalising the denominator $\frac{7+\sqrt{6}}{3-\sqrt{2}}$

$$\boxed{\begin{aligned}\sqrt{12} &= \sqrt{4\times 3}\\ &= 2\sqrt{3}\end{aligned}}$$

$$\frac{7+\sqrt{6}}{3-\sqrt{2}} = \frac{7+\sqrt{6}}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$$

$$= \frac{(7+\sqrt{6})(3+\sqrt{2})}{3^2 - (\sqrt{2})^2} = \frac{21 + 7\sqrt{2} + 3\sqrt{6} + \sqrt{12}}{9-2}$$

$$= \frac{21 + 7\sqrt{2} + 3\sqrt{6} + 2\sqrt{3}}{7}$$

7. Simplify $\frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2}$

$$\text{Given } \frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2} \ldots (1)$$

Multiplying each term by the conjugate of the denominator

$$\frac{1}{3-\sqrt{8}} = \frac{1}{3-\sqrt{8}} \times \frac{3+\sqrt{8}}{3+\sqrt{8}} = \frac{3+\sqrt{8}}{3^2 - (\sqrt{8})^2} = \frac{3+\sqrt{8}}{9-8} = \frac{3+\sqrt{8}}{1}$$

$$\frac{1}{3-\sqrt{8}} = 3 + \sqrt{8}$$

$$\frac{1}{\sqrt{8}-\sqrt{7}} = \frac{1}{\sqrt{8}-\sqrt{7}} \times \frac{\sqrt{8}+\sqrt{7}}{\sqrt{8}+\sqrt{7}} = \frac{\sqrt{8}+\sqrt{7}}{(\sqrt{8})^2 - (\sqrt{7})^2} = \frac{\sqrt{8}+\sqrt{7}}{8-7} = \frac{\sqrt{8}+\sqrt{7}}{1}$$

$$\boxed{\frac{1}{\sqrt{8}-\sqrt{7}}=\sqrt{8}+\sqrt{7}}$$

$$\frac{1}{\sqrt{7}-\sqrt{6}}=\frac{1}{\sqrt{7}-\sqrt{6}}\times\frac{\sqrt{7}+\sqrt{6}}{\sqrt{7}+\sqrt{6}}=\frac{\sqrt{7}+\sqrt{6}}{\left(\sqrt{7}\right)^2-\left(\sqrt{6}\right)^2}=\frac{\sqrt{7}+\sqrt{6}}{7-6}=\frac{\sqrt{7}+\sqrt{6}}{1}$$

$$\boxed{\frac{1}{\sqrt{7}-\sqrt{6}}=\sqrt{7}+\sqrt{6}}$$

$$\frac{1}{\sqrt{6}-\sqrt{5}}=\frac{1}{\sqrt{6}-\sqrt{5}}\times\frac{\sqrt{6}+\sqrt{5}}{\sqrt{6}+\sqrt{5}}=\frac{\sqrt{6}+\sqrt{5}}{\left(\sqrt{6}\right)^2-\left(\sqrt{5}\right)^2}=\frac{\sqrt{6}+\sqrt{5}}{6-5}=\frac{\sqrt{6}+\sqrt{5}}{1}$$

$$\boxed{\frac{1}{\sqrt{6}-\sqrt{5}}=\sqrt{6}+\sqrt{5}}$$

$$\frac{1}{\sqrt{5}-2}=\frac{1}{\sqrt{5}-2}\times\frac{\sqrt{5}+2}{\sqrt{5}+2}=\frac{\sqrt{5}+2}{\left(\sqrt{5}\right)^2-2^2}=\frac{\sqrt{5}+2}{5-4}=\frac{\sqrt{5}+2}{1}$$

$$\boxed{\frac{1}{\sqrt{5}-2}=\sqrt{5}+2}$$

Substituting all these values in (1) *we get,*

$$\frac{1}{3-\sqrt{8}}-\frac{1}{\sqrt{8}-\sqrt{7}}+\frac{1}{\sqrt{7}-\sqrt{6}}-\frac{1}{\sqrt{6}-\sqrt{5}}+\frac{1}{\sqrt{5}-2}$$

$$=3+\sqrt{8}-\left(\sqrt{8}+\sqrt{7}\right)+\sqrt{7}+\sqrt{6}-\left(\sqrt{6}+\sqrt{5}\right)+\sqrt{5}+2$$

$$=3+\cancel{\sqrt{8}}-\cancel{\sqrt{8}}-\cancel{\sqrt{7}}+\cancel{\sqrt{7}}+\cancel{\sqrt{6}}-\cancel{\sqrt{6}}-\cancel{\sqrt{5}}+\cancel{\sqrt{5}}+2=5$$

$$\frac{1}{3-\sqrt{8}}-\frac{1}{\sqrt{8}-\sqrt{7}}+\frac{1}{\sqrt{7}-\sqrt{6}}-\frac{1}{\sqrt{6}-\sqrt{5}}+\frac{1}{\sqrt{5}-2}=5$$

8. $\boldsymbol{If\ x=\sqrt{2}+\sqrt{3}\ find\ \frac{x^2+1}{x^2-2}}$

$$x=\sqrt{2}+\sqrt{3}$$

$$x^2=\left(\sqrt{2}+\sqrt{3}\right)^2=\left(\sqrt{2}\right)^2+\left(\sqrt{3}\right)^2+2\left(\sqrt{2}\right)\left(\sqrt{3}\right)=2+3+2\sqrt{6}$$

$$x^2=5+2\sqrt{6}$$

$$\therefore\frac{x^2+1}{x^2-2}=\frac{5+2\sqrt{6}+1}{5+2\sqrt{6}-2}=\frac{6+2\sqrt{6}}{3+2\sqrt{6}}$$

$$=\frac{6+2\sqrt{6}}{3+2\sqrt{6}}\times\frac{3-2\sqrt{6}}{3-2\sqrt{6}}=\frac{\left(6+2\sqrt{6}\right)\left(3-2\sqrt{6}\right)}{3^2-\left(2\sqrt{6}\right)^2}=\frac{18-12\sqrt{6}+6\sqrt{6}-4\,(6)}{9-4(6)}$$

$$=\frac{18-6\sqrt{6}-24}{9-24}=\frac{-6-6\sqrt{6}}{-15}=\frac{-6\left(1+\sqrt{6}\right)}{-15}=\frac{2\left(1+\sqrt{6}\right)}{5}$$

$EXERCISE: 2.12$

WHAT IS AN EXPONENT?

An exponent refers to the number of times a number is multiplied by itself.

For Example : $3 \times 3 \times 3 = 3^3$

$$4 \times 4 \times 4 \times 4 \times 4 = 4^5$$

When an exponent is zero, as in a^0, *the expression is always equal to* 1.

To prove: $a^0 = 1$

$$\boxed{\begin{array}{l} \dfrac{a^a}{a^a} = 1 \Rightarrow a^{a-a} = 1 \\ \quad a^0 = 1 \end{array}}$$

When an exponent is a negative number, the result is always a fraction. Fractions consist of a numerator over a denominator. In this instance, the numerator is always 1.

$$a^{-m} = \frac{1}{a^m}$$

The exponential function $f(x) = a^x$ *with a base* $a \neq 1$

Properties of Logarithm

(i) $a^{\log_a x} = x$ *for all* $x \in (0, \infty)$ *and* $\log_a(a^y) = y$ *for all* $y \in \mathbb{R}$.

Product Rule

$$\log_a(xy) = \log_a x + \log_a y.$$

Quotient Rule

$$(iii)\ \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

Power Rule

$$(iv)\ \log_a x^r = r \log_a x,\ r \in \mathbb{R}$$

Change of base formula

$$(v)\ log_b\, x = \frac{log_a\, x}{log_a\, b} \quad \text{with } a \text{ and } b \text{ as bases,}$$

Example 2.34: Find the logarithm of 1728 to the base $2\sqrt{3}$.

Let $\log_{2\sqrt{3}} 1728 = x$

$(2\sqrt{3})^x = 1728 \Rightarrow (2\sqrt{3})^x = 2^6 \times 3^3$

$(2\sqrt{3})^x = 2^6\left[\left(\sqrt{3}\right)^2\right]^3$

2	1728
2	864
2	432
2	216
2	108
2	54
3	27
3	9
3	3
	1

$(2\sqrt{3})^x = 2^6(\sqrt{3})^6 \implies (2\sqrt{3})^x = (2\sqrt{3})^6$

$x = 6$

$\log_{2\sqrt{3}} 1728 = 6.$

Example 2.35: $\boldsymbol{If\ the\ logarithm\ of\ 324\ to\ base\ a\ is\ 4, then\ find\ a.}$

$\log_a 324 = 4$

$a^4 = 324 \implies a^4 = 3^4 \times 2^2$

$a^4 = 3^4 \times \left[\left(\sqrt{2}\right)^2\right]^2 \implies a^4 = 3^4 \times \left(\sqrt{2}\right)^4$

$a^4 = (3\sqrt{2})^4 \implies a = 3\sqrt{2}$

Example 2.36: $\boldsymbol{Prove\ \log\frac{75}{16} - 2\log\frac{5}{9} + \log\frac{32}{243} = \log 2}$.

$L.H.S = \log\frac{75}{16} - 2\log\frac{5}{9} + \log\frac{32}{243}$

$= \log 75 - \log 16 - 2(\log 5 - \log 9) + \log 32 - \log 243$

$= log\, 75 - log\, 16 - 2log\, 5 + 2log\, 9 + \log 32 - \log 243$

$= log\, 3 \times 25 - log\, 16 - 2log\, 5 + 2log\, 9 + \log 2 \times 16 - \log 81 \times 3$

$= \log 3 + \log 25 - \log 16 - 2\log 5 + 2\log 9 + \log 2 + \log 16 - (\log 81 + \log 3)$

$= log\, 3 + log\, 25 - 2\, log\, 5 + 2\, log\, 9 + log\, 2 - log\, 81 - log\, 3$

$= \log 5^2 - 2\log 5 + 2\log 9 + \log 2 - \log 9^2$

$= 2\log 5 - 2\log 5 + 2\log 9 + \log 2 - 2\log 9 = \log 2$

Example 2.37: $\boldsymbol{If\ \log_2 x + \log_4 x + \log_{16} x = \frac{7}{2}, find\ the\ value\ of\ x}$.

$\log_2 x + \log_4 x + \log_{16} x = \frac{7}{2}$

$\frac{1}{log_x 2} + \frac{1}{log_x 4} + \frac{1}{log_x 16} = \frac{7}{2}$ $(change\ of\ base\ rule)$

$\frac{1}{log_x 2} + \frac{1}{2\, log_x 2} + \frac{1}{4\, log_x 2} = \frac{7}{2}$

$log_x 4 = log_x 2^2$
$log_x 4 = 2\, log_x 2$
$log_x 16 = log_x 2^4$
$log_x 16 = 4\, log_x 2$

$where\ a = \log_x 2$

$\frac{1}{a} + \frac{1}{2a} + \frac{1}{4a} = \frac{7}{2} \implies \frac{4+2+1}{4a} = \frac{7}{2} \implies \frac{7}{4a} = \frac{7}{2} \implies 4a = 2$

$a = \frac{1}{2} \implies \log_x 2 = \frac{1}{2}$ $which\ gives$ $x^{\frac{1}{2}} = 2.$

$x = 2^2 = 4 \implies x = 4$

Example 2.38: $\boldsymbol{Solve\ \ x^{\log_3 x} = 9.}$

$$Let\ \log_3 x = y$$

$$x = 3^y$$

$$3^{y\, log_3 3^y} = 9 \Rightarrow \ 3^{y\times y} = 9$$

$$3y^2 = 9 \Rightarrow \ y^2 = 2 \ \Rightarrow \ y = \sqrt{2},\ \ -\sqrt{2}.$$

$$y = \sqrt{2},\ -\sqrt{2}\ in\ x = 3^y$$

$$Hence, x = 3^{\sqrt{2}}, 3^{-\sqrt{2}}$$

Example 2.39: $\boldsymbol{Compute \log_3 5 \log_{25} 27}$

$$log_3 5\, log_{25} 27 = log_3 5\, log_{25} 3^3$$

$$= log_3 5 \times 3\, log_{25} 3$$

$$= 3 \log_{25} 5$$

$$= \frac{3}{log_5 25} = \frac{3}{2\, log_5 5} = \frac{3}{2}$$

$$log_3 5 \times 3\, log_{25} 3$$

$$= 3 log_3 5 \times log_{25} 3$$

$$= 3 \times log_{25} 5$$

Example 2.40: $\boldsymbol{Given\ that \log_{10} 2 = 0.30103,\ \log_{10} 3 = 0.47712 (app),}$
$\boldsymbol{find\ the\ number\ of\ digits\ in\ 2^8 . 3^{12}.}$

$$N = 2^8 . 3^{12}\ \ has\ n+1\ digits.$$

$$Taking\ logarithm\ to\ the\ base\ 10,$$

$$N = 10^n \times b\ where\ \ 1 \le b < 10.$$

$$\log N = \log(10^n b)$$

$$= n \log 10 + \log b$$

$$logN = n + \log b$$

$$\log N = \log 2^8 3^{12} = \log 2^8 + log 3^{12}$$

$$= 8\, log\, 2 + 12\, log\, 3$$

$$logN = 8 \times 0.30130 + 12 \times 0.47712$$

$$logN = 8.133368$$

$$n + \log b = 8.13368.\ Since\ 1 \le b < 10$$

$$The\ numberof\ digits\ is\ 9.$$

2. Compute $\boldsymbol{\log_9 27 - \log_{27} 9}$

$$= \log_9 27 - \log_{27} 9$$

$$= log_9 3^3 - log_{27} 3^2 = 3\, log_9 3 - 2\, log_{27} 3$$

$$= \frac{3}{log_3 9} - \frac{2}{log_3 27} = \frac{3}{log_3 3^2} - \frac{2}{log_3 3^3} \qquad [\because \log_3 3 = 1]$$

1. Let $b > 0$ and $b \neq 1$. Express $y = b^x$ in logarithmic form. Also state the domain and range of the logarithmic function.

$Given\ \ y = b^x$

$Converting\ this\ into\ logarithmic\ form, we\ get$

$\log_b y = x,\ (0,\infty), (-\infty,\infty)$

2. Compute $\log_9 27 - \log_{27} 9$

$Given\ \ \log_9 27 - \log_{27} 9$

$= \log_9 3^3 - \log_{27} 3^2$

$= 3\log_9 3 - 2\log_{27} 3 \quad [By\ power\ rule]$

$$= \frac{3}{\log_3 9} - \frac{2}{\log_3 27} \quad [By\ change\ of\ box\ rule]$$

$$= \frac{3}{\log_3 3^2} - \frac{2}{\log_3 3^3} = \frac{3}{2\log_3 3} - \frac{2}{3\log_3 3} \qquad [\because \log_3 3 = 1]$$

$$= \frac{3}{2} - \frac{2}{3} = \frac{9-4}{6} = \frac{5}{6}$$

3. *Solve*: $\log_8 x + \log_4 x + \log_2 x = 11$.

$Given\ \log_8 x + \log_4 x + \log_2 x = 11$

$$\frac{1}{\log_x 8} + \frac{1}{\log_x 4} + \frac{1}{\log_x 2} = 11$$

$$\frac{1}{\log_x 2^3} + \frac{4}{2\log_x 2^2} + \frac{1}{\log_x 2} = 11$$

$$\frac{1}{3\log_x 2} + \frac{4}{2\log_x 2} + \frac{1}{\log_x 2} = 11 \Longrightarrow \frac{1}{\log_x 2}\left(\frac{1}{3} + \frac{1}{2} + 1\right) = 11$$

$$\frac{1}{\log_x 2}\left(\frac{11}{6}\right) = 11 \Longrightarrow \frac{1}{\log_x 2} = 11 \times \frac{6}{11}$$

$$\frac{1}{\log_x 2} = 6 \Longrightarrow \log_2 x = 6 \Longrightarrow 2^6 = x$$

$$\boxed{x = 64}$$

4. *Solve*: $\log_4 2^{8x} = 2^{\log_2 8}$

$$8x\log_4 2 = 2^{\log_2 2^3} \Longrightarrow 8x\log_4 2 = 2^{3(1)}$$

$$\frac{8x}{\log_2 4} = 8 \Longrightarrow \frac{x}{\log_2 4} = 1 \Longrightarrow x = \log_2 4 \qquad \boxed{\log_2 2 = 1}$$

$x = \log_2 2^2 \implies x = 2\log_2 2$

$\boxed{x = 2}$

5. $If\ a^2 + b^2 = 7ab. Show\ that\ \log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a + \log b)$

$Given\ a^2 + b^2 = 7ab$

Adding 2ab both side we get,

$a^2 + b^2 + 2ab = 7ab + 2ab$

$(a+b)^2 = 9ab \implies \frac{(a+b)^2}{9} = ab$

$\frac{(a+b)^2}{3^2} = ab \implies \frac{a+b}{3} = \sqrt{ab} \implies \frac{a+b}{3} = (ab)^{\frac{1}{2}}$

$\log\left(\frac{a+b}{3}\right) = \log(ab)^{\frac{1}{2}} \implies \log\left(\frac{a+b}{3}\right) = \frac{1}{2}\log(ab)$

$\log\left(\frac{a+b}{3}\right) = \frac{1}{2}[\log a + \log b]$ *Hence proved.*

6. $Prove\ \log\frac{a^2}{bc} + \log\frac{b^2}{ca} + \log\frac{c^2}{ab} = 0.$

$L.H.S = log\frac{a^2}{bc} + log\frac{b^2}{ca} + log\frac{c^2}{ab}$

$= \log\left(\frac{a^2}{bc} \times \frac{b^2}{ca} \times \frac{c^2}{ab}\right) = \log\left(\frac{a^2b^2c^2}{a^2b^2c^2}\right) = \log 1$

$= 0 = RHS$ Hence proved.

7. $Prove\ that\ \log 2 + 16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80} = 1.$

$LHS = \log 2 + 16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80}$

$\boxed{(a^m)^n = a^{mn}}$

$= \log 2 + \log\left(\frac{16}{15}\right)^{16} + \log\left(\frac{25}{24}\right)^{12} + \log\left(\frac{81}{80}\right)^7$

$\boxed{\frac{a^m}{a^n} = a^{m-n}}$

$= \log 2 \times \frac{(2^4)^{16}}{(3\times 5)^{16}} \times \frac{(5^2)^{12}}{(2^3\times 3)^{12}} \times \frac{(3^4)^7}{(2^4\times 5)^7}$

$= \log_2 1 \times \frac{2^{64}}{3^{16}\times 5^{16}} \times \frac{5^{24}}{2^{16}\times 3^{12}} \times \frac{3^{28}}{2^{28}\times 5^7}$

$= \log\frac{2^{1+64}\times 5^{24}\times 3^{28}}{3^{16+12}\times 5^{16+7}\times 2^{36+28}} = \log\frac{2^{65}\times 5^{24}\times \cancel{3^{28}}}{\cancel{3^{28}}\times 5^{23}\times 2^{64}}$

$= \log_2 65 - 64 \times 5^{24-23} = \log_2 1 \times 5^1$

$= \log_{10} 10 = 1 = RHS$. *Hence proved.*

8. *Prove* $\boldsymbol{\log_a 2\ a \log_b 2\ b \log_c 2\ c = \frac{1}{8}}$

$\boxed{\log_a a = 1}$

$L.H.S = \log_a 2\,a \times \log_b 2\,b \times \log_c 2\,c$

$$= \frac{1}{\log_a a^2} \times \frac{1}{\log_b b^2} \times \frac{1}{\log_c c^2} = \frac{1}{2\log_a a} \times \frac{1}{2\log_b b} \times \frac{1}{2\log_c c}$$

$$= \frac{1}{2(1)} \times \frac{1}{2(1)} \times \frac{1}{2(1)} = \frac{1}{8} = R.H.S \quad .Hence\ proved.$$

9. *Prove* $\boldsymbol{\log a + \log_a 2 + \log_a 3 + \cdots + \log_a n = \frac{n(n+1)}{3} \log a}$

$L.H.S = \log a + \log_a 2 \log_a 3 + \cdots + \log_a n$

$= \log a + \log_a 2 \log_a 3 + \cdots + \mathrm{n} \log a$

$\boxed{\sum n = \frac{n(n+1)}{2}}$

$= \log a\ (1 + 2 + 3 + \cdots n)$

$= \log a \frac{(n)(n+1)}{2} = \frac{(n)(n+1)}{2} \log a \quad = R.H.S$

10. $\boldsymbol{If\ \frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}, then\ prove\ that\ xyz = 1.}$

$$Let\ \frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y} = k$$

$\log x = k(y-z) = ky - kz \ \dots(1)$

$\log y = k(z-x) = kz - kx \ \dots(2)$

$\log z = k(x-y) = kx - ky \ \dots(3)$

Adding (1), (2) *and* (3)

$\log x + \log y + \log z = ky - kz + kz - kx + kx - ky = 0$

$\log xyz = 0$

$\log xyz = \log 1$

$xyz = 1$

Hence proved.

11. $\boldsymbol{Solve: \log_2 x - 3\log_{\frac{1}{2}} x = 6}$

$Given\ \log_2 x - 3\log_{\frac{1}{2}} x = 6$ [*using quotient rule*]

$$\frac{1}{\log_x 2} - \frac{3}{\log_x \frac{1}{2}} = 6$$

$$\frac{1}{\log_x 2} - \frac{3}{\log_x 1 - \log_x 2} = 6 \ [using\ quotient\ rule]$$

$$\frac{1}{\log_x 2} + \frac{3}{\log_x 2} = 6 \Longrightarrow \frac{1}{\log_x 2}(1+3) = 6$$

$$\frac{1}{\log_x 2}(4) = 6 \Longrightarrow \frac{1}{\log_x 2} = \frac{6}{4}$$

$$\log_x 2 = \frac{3}{2} \Longrightarrow 2^{\frac{3}{2}} = x \Longrightarrow (2^3)^{\frac{1}{2}} = x \Longrightarrow (8)^{\frac{1}{2}} = x$$

$$x = \sqrt{8} = \sqrt{2 \times 2 \times 2} \Longrightarrow x = 2\sqrt{2}$$

12. $\boldsymbol{Solve\ \log 5 - x\left(x^2 - 6x + 65\right) = 2.}$

$Given: log_{5\,-\,x}(x^2 - 6x + 65) = 2$

$$(5-x)^2 = x^2 - 6x + 65 \quad [Converting\ into\ exponential\ form]$$

$$25 + x^2 - 10x = x^2 - 6x + 65$$

$$-6x + 65 - 25 + 10x = 0$$

$$4x + 40 = 0 \Longrightarrow 4x = -40$$

$$x = \frac{-40}{4} \Longrightarrow x = -10$$

$$= \frac{3}{2\,log_3 3} - \frac{2}{3\,log_3 3} = \frac{3}{2} - \frac{2}{3} = \frac{9-4}{6} = \frac{5}{6}$$

3. $\boldsymbol{Solve}$**:** $\mathbf{\log_8 x + \log_4 x + \log_2 x = 11}$.

$$log_8 x + log_4 x + log_2 x = 11$$

$$\frac{1}{log_x 8} + \frac{1}{log_x 4} + \frac{1}{log_x 2} = 11 \Rightarrow \frac{1}{\log_x 2^3} + \frac{1}{\log_x 2^2} + \frac{1}{\log_x 2} = 11$$

$$\frac{1}{3\log_x 2} + \frac{1}{2\log_x 2} + \frac{1}{\log_x 2} = 11 \Rightarrow \frac{1}{3a} + \frac{1}{2a} + \frac{1}{a} = 11$$

$$where\ a = \log_x 2$$

$$\frac{2+3+6}{6a} = 11 \Rightarrow \frac{11}{6a} = 11 \Rightarrow \frac{1}{6a} = 1 \Rightarrow a = \frac{1}{6}$$

$$\log_x 2 = \frac{1}{6} \Rightarrow x^{\frac{1}{6}} = 2 \Rightarrow x = 2^6$$

$$\boxed{x = 64}$$

4. $\boldsymbol{Solve}$**:** $\mathbf{\log_4 2^{8x} = 2^{\log_2 8}}$

$$\log_4 2^{8x} = 2^{\log_2 8}$$

$$8x\log_4 2 = 2^{\log_2 2^3} \Rightarrow 8x log_4 2 = 2^3 \qquad \boxed{\log_2 2 = 1}$$

$$\frac{8x}{\log_2 4} = 8 \Rightarrow \frac{x}{\log_2 4} = 1$$

$$x = log_2 4 \Rightarrow x = log_2 2^2$$

$$x = 2\,log_2 2 \Rightarrow x = 2$$

5. $\boldsymbol{If\ a^2 + b^2 = 7ab. Show\ that\ log\left(\frac{a+b}{3}\right) = \frac{1}{2}(log\,a + log\,b)}$

$$a^2 + b^2 = 7ab$$

$$Adding\ 2ab\ on\ both\ side$$

$$a^2 + b^2 + 2ab = 7ab + 2ab \Rightarrow (a+b)^2 = 9ab$$

$$\frac{(a+b)^2}{9} = ab \Rightarrow \frac{(a+b)^2}{3^2} = ab$$

$$\left(\frac{a+b}{3}\right)^2 = ab \Rightarrow \frac{a+b}{3} = \sqrt{ab}$$

$$\frac{a+b}{3} = (ab)^{\frac{1}{2}} \Rightarrow \log\left(\frac{a+b}{3}\right) = \log(ab)^{\frac{1}{2}}$$

$$\log\left(\frac{a+b}{3}\right) = \frac{1}{2}\log(ab) \Rightarrow \log\left(\frac{a+b}{3}\right) = \frac{1}{2}[\log a + \log b]$$

6. $\boldsymbol{Prove\ \log\frac{a^2}{bc}+\log\frac{b^2}{ca}+\log\frac{c^2}{ab}=0.}$

$$L.H.S = \ log\frac{a^2}{bc}+log\frac{b^2}{ca}+log\frac{c^2}{ab}$$

$$= log\left(\frac{a^2}{bc}\times\frac{b^2}{ca}\times\frac{c^2}{ab}\right) \ = log\left(\frac{a^2b^2c^2}{a^2b^2c^2}\right)$$

$$= log\,1 = 0 = R.H.S\ .Hence\ proved.$$

TRIGONOMETRY

EXERCISE: 3.1

Example 3.1 Prove that $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\dfrac{1+\sin\theta}{\cos\theta}$

$$L.H.S=\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\frac{\tan\theta+\sec\theta-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1}$$

$$=\frac{\tan\theta+\sec\theta-(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}{\tan\theta-\sec\theta+1}$$

$$\boxed{\because 1+\tan^2\theta=\sec^2\theta} \quad \boxed{\sec^2\theta-\tan^2\theta=1}$$

$$=\frac{(\tan\theta+\sec\theta)\,[1-(\sec\theta-\tan\theta)]}{\tan\theta-\sec\theta+1}$$

$$=\frac{(\tan\theta+\sec\theta)\cancel{(1-\sec\theta+\tan\theta)}}{\cancel{\tan\theta-\sec\theta+1}}=\tan\theta+\sec\theta$$

$$=\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}=\frac{\sin\theta+1}{\cos\theta}=\frac{1+\sin\theta}{\cos\theta}$$

Example 3.2 Prove that $(\sec A-\operatorname{cos}ecA)(1+\tan A+\cot A)$
$=\tan A\sec A-\cot A\operatorname{cosec}A$

$$L.H.S=(\sec A-\operatorname{cos}ecA)(1+\tan A+\cot A)$$

$$\boxed{\tan\theta=\frac{\sin\theta}{\cos\theta}} \quad \boxed{\sec\theta=\frac{1}{\cos\theta}} \quad \boxed{\cot\theta=\frac{\cos\theta}{\sin\theta}}$$

$$=\left(\frac{1}{\cos A}-\frac{1}{\sin A}\right)\left(1+\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}\right)$$

$$=\frac{1}{\cos A}+\frac{\sin A}{\cos A}\times\frac{1}{\cos A}+\frac{\cancel{\cos A}}{\sin A}\times\frac{1}{\cancel{\cos A}}-\frac{1}{\sin A}+\frac{\cancel{\sin A}}{\cos A}\times-\frac{1}{\cancel{\sin A}}+\frac{\cos A}{\sin A}\times-\frac{1}{\sin A}$$

$$=\cancel{\frac{1}{\cos A}}+\frac{\sin A}{\cos^2A}+\cancel{\frac{1}{\sin A}}-\cancel{\frac{1}{\sin A}}-\cancel{\frac{1}{\cos A}}-\frac{\cos A}{\sin^2A}=\frac{\sin A}{\cos^2A}-\frac{\cos A}{\sin^2A}$$

$$=\frac{\sin A}{\cos A}\times\frac{1}{\cos A}-\frac{\cos A}{\sin A}\times\frac{1}{\sin A}$$

$$\boxed{\operatorname{cosec}\theta=\frac{1}{\sin\theta}}$$

$$=\tan A\times\sec A-\cot A\times\operatorname{cosec}A=\tan A\sec A-\cot A\operatorname{cosec}A$$

Example 3.2 Eliminate θ from $a\cos\theta=b$ and $c\sin\theta=d$ where a,b,c,d are constants.

$a\cos\theta=b$ $\Rightarrow$ $ac\cos\theta=bc$

Multiplying both side by c — *squaring on both sides*

$a^2c^2\cos^2\theta=b^2c^2\ldots(1)$

$c\sin\theta=d$ $\Rightarrow$ $ac\sin\theta=ad$

Multiplying both side by a — *squaring on both sides*

$a^2c^2\sin^2\theta=a^2d^2\ldots(2)$

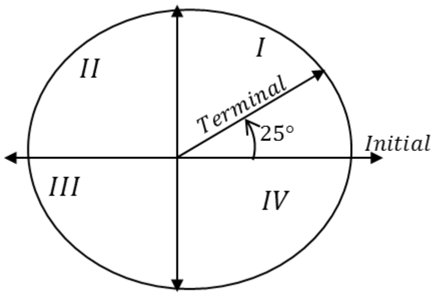
***Adding* (1) *and* (2)**

$$a^2c^2cos^2\theta + a^2c^2sin^2\theta = b^2c^2 + a^2d^2$$

$$a^2c^2(sin^2\theta + cos^2\theta) = b^2c^2 + a^2d^2 \Longrightarrow a^2c^2 = b^2c^2 + a^2d^2$$

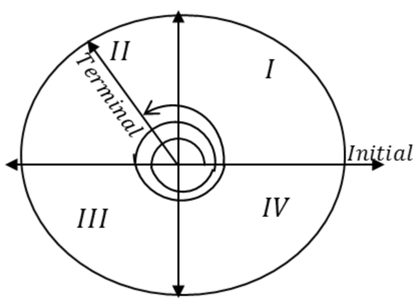
1. *Identify the quadrant in which an angle of each given measure lies* (i) 25° (ii) 825° (iii) −55° (iv) 328° (v) −230°

(ii) 25°



$25°$ *lies in the* 1^{st} *Quadrant*

(ii) 825°



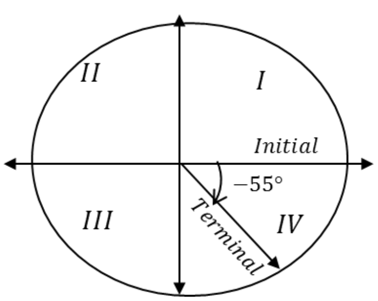
$$360 \overline{)\,825\,}\,(2$$
$$720$$
$$105$$

$825 = 2 \times 360 + 105$

$360° + 360° + 105° = 825°$

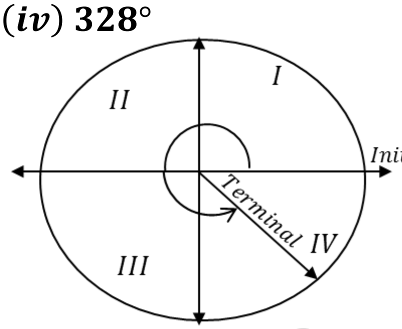
$825°$ *lies in the* 2^{nd} *Quadrant*

(iii) −55°



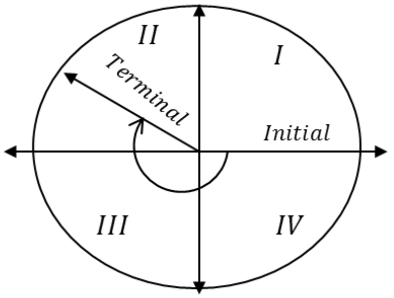
$-55°$ *lies in the* 4^{th} *Quadrant*

(iv) 328°



$328°$ *lies in the* 4^{th} *Quadrant*

$(v)\ -\mathbf{230°}$

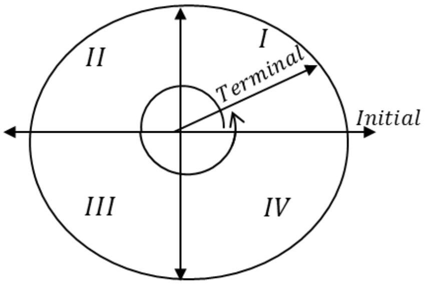


$\boxed{-180° - 50° = -230°}$

$-230°$ *lies in the* 2^{nd} *Quadrant*

(3) Determine the quadrants in which the following degrees lie.
(i) 380° (ii) – 140° (iii) 1100°

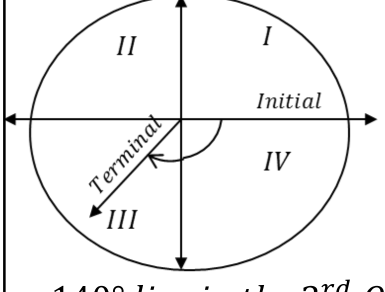
$(\boldsymbol{i})\ \mathbf{380°}$



$\boxed{360° + 20°}$

$380°$ *lies in the* 1^{st} *Quadrant*

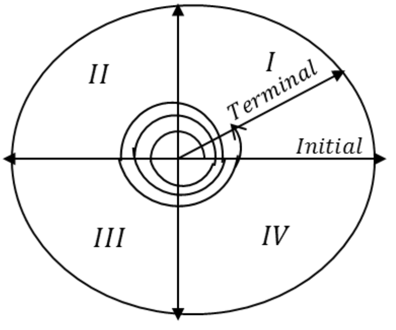
$(\boldsymbol{ii})\ -\mathbf{140°}$



$\boxed{-90° - 50° = -140°}$

$-140°$ *lies in the* 3^{rd} *Quadrant*

$(\boldsymbol{iii})\ \mathbf{1100°}$



$$360 \overline{)\,1100\,}(3$$
$$\frac{1080}{20}$$

$1100 = 3 \times 360 + 20$

$360 + 360° + 360° + 20° = 1100$

$1100°$ *lies in the* 1^{st} *Quadrant*

2. *For each given angle, find a coterminal angle with measure of such that* $0 \leq \theta < 360°$

$(i)\ 395°$ $(ii)\ 525°$ $(iii)\ 1150°$ $(iv)\ -270°$ $(v)\ -450°$

$(i)\ 395°$

$395° = 360° + 35°$

$\boxed{\theta = 35°}$

$(ii)\ 525°$

$525° = 360° + 165°$

$\boxed{\theta = 165°}$

$$360 \,\big)\, \begin{array}{r} 525 \\ 360 \\ \hline 165 \end{array} \,\big(\, 1$$

$525° = 1 \times 360° + 165°$

$(iii)\ 1150°$

$1150° = 3 \times 360° + 70°$

$\boxed{\theta = 70°}$

$$360 \,\big)\, \begin{array}{r} 1150 \\ 1080 \\ \hline 70 \end{array} \,\big(\, 3$$

$1150° = 3 \times 360° + 70°$

$(iv)\ -270°$

$-270° = -(360° - 90°)$

$= -360° + 90°$

$\boxed{\theta = 90°}$

$(v)\ -450°$

$-450° = -(720° - 270°)$

$= -720° + 270°$

$\boxed{\theta = 270°}$

3) *If* $a\cos\theta - b\sin\theta = c$, *show that* $a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

$a\cos\theta - b\sin\theta = c$

squaring on both sides

$(a\cos\theta - b\sin\theta)^2 = c^2$

$(a\cos\theta)^2 + (b\sin\theta)^2 - 2(a\cos\theta)(b\sin\theta) = c^2$

$a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta = c^2$

$a^2\cos^2\theta + b^2\sin^2\theta - c^2 = 2ab\sin\theta\cos\theta$

$(a\sin\theta + b\cos\theta)^2 = (a\sin\theta)^2 + (b\cos\theta)^2 + 2(a\sin\theta)(b\cos\theta)$

$= a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta$

$= a^2\sin^2\theta + b^2\cos^2\theta + a^2\cos^2\theta + b^2\sin^2\theta - c^2$

$= a^2\sin^2\theta + a^2\cos^2\theta + b^2\cos^2\theta + b^2\sin^2\theta - c^2$

$= a^2(\sin^2\theta + \cos^2\theta) + b^2(\cos^2\theta + \sin^2\theta) - c^2$

$(a\sin\theta + b\cos\theta)^2 = a^2(1) + b^2(1) - c^2$

$(a\sin\theta + b\cos\theta)^2 = a^2 + b^2 - c^2 \Rightarrow a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

4. $\boldsymbol{If\ \sin\theta + \cos\theta = m\ show\ that\ \cos^6\theta + \sin^6\theta = \frac{4-3(m^2-1)^2}{4}}$ $\boldsymbol{where\ m^2 \le 2}$

$\sin\theta + \cos\theta = m$ $\Longrightarrow (\sin\theta + \cos\theta)^2 = m^2$

squaring on both sides

$\boxed{a^3+b^3 = (a+b)^3 - 3ab(a+b)}$

$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = m^2$

$1 + 2\sin\theta\cos\theta = m^2 \Longrightarrow 2\sin\theta\cos\theta = m^2 - 1$

$$\boxed{\sin\theta\cos\theta = \frac{m^2-1}{2}}$$

$L.H.S = \cos^6\theta + \sin^6\theta = (\cos^2\theta)^3 + (\sin^2\theta)^3$

$= (\cos^2\theta + \sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)$

$= 1^3 - 3\cos^2\theta\sin^2\theta(1) = 1^3 - 3\sin^2\theta\cos^2\theta$

$= 1 - 3(\sin\theta\cos\theta)^2 \quad = 1 - 3\left(\frac{m^2-1}{2}\right)^2$

$= 1 - \frac{3(m^2-1)^2}{4} = \frac{4-3(m^2-1)^2}{4}$

5. $\boldsymbol{If\ \frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = 1,}$ $\boldsymbol{Prove\ that\ (i)\ \sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta}$

$\boldsymbol{(ii)\ \frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} = 1}$

$\frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = 1 \quad \Longrightarrow \quad \frac{\cos^4\alpha\sin^2\beta + \sin^4\alpha\cos^2\beta}{\cos^2\beta\sin^2\beta} = 1$

$\cos^4\alpha\sin^2\beta + \sin^4\alpha\cos^2\beta = \cos^2\beta\sin^2\beta$

$(\cos^2\alpha)^2\sin^2\beta + \sin^4\alpha\cos^2\beta = \cos^2\beta\sin^2\beta$

$(1-\sin^2\alpha)^2\sin^2\beta + \sin^4\alpha(1-\sin^2\beta) = (1-\sin^2\beta)\sin^2\beta$

$[1^2 + (\sin^2\alpha)^2 - 2(1)(\sin^2\alpha)]\sin^2\beta + \sin^4\alpha - \sin^4\alpha\sin^2\beta = \sin^2\beta - \sin^4\beta$

$(1 + \sin^4\alpha - 2\sin^2\alpha)\sin^2\beta + \sin^4\alpha - \sin^4\alpha\sin^2\beta = \sin^2\beta - \sin^4\beta$

$\cancel{\sin^2\beta} + \cancel{\sin^4\alpha\sin^2\beta} - 2\sin^2\alpha\sin^2\beta + \sin^4\alpha - \cancel{\sin^4\alpha\sin^2\beta} = \cancel{\sin^2\beta} - \sin^4\beta$

$\sin^4\alpha - 2\sin^2\alpha\sin^2\beta = -\sin^4\beta \Longrightarrow \sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta$

$\frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = 1 \quad \Longrightarrow \quad \frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} = 1$

Interchange α and β

6. $\boldsymbol{If\ y = \frac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}\ then\ prove\ that\ \frac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha} = y}$

$$y = \frac{2sin\alpha}{1 + cos\alpha + sin\alpha}$$

$$y = \frac{2sin\alpha}{1 + sin\alpha + cos\alpha}$$

$$y = \frac{2sin\alpha\,(1 + sin\alpha - cos\alpha)}{(\underbrace{1 + sin\alpha}_{a} + \underbrace{cos\alpha}_{b})\,(\underbrace{1 + sin\alpha}_{a} - \underbrace{cos\alpha}_{b})}$$

$$y = \frac{2sin\alpha\,(1 + sin\alpha - cos\alpha)}{(1 + sin\alpha)^2 - cos^2\alpha}$$

$$y = \frac{2sin\alpha\,(1 + sin\alpha - cos\alpha)}{1^2 + 2(1)sin\alpha + sin^2\alpha - cos^2\alpha} \Rightarrow y = \frac{2sin\alpha\,(1 + sin\alpha - cos\alpha)}{1 + 2sin\alpha + sin^2\alpha - cos^2\alpha}$$

$$y = \frac{2sin\alpha\,(1 + sin\alpha - cos\alpha)}{1 - cos^2\alpha + sin^2\alpha + 2sin\alpha}$$

$$y = \frac{2sin\alpha\,(1 + sin\alpha - cos\alpha)}{sin^2\alpha + sin^2\alpha + 2sin\alpha} \Rightarrow y = \frac{2sin\alpha\,(1 + sin\alpha - cos\alpha)}{2sin^2\alpha + 2sin\alpha}$$

$$y = \frac{\cancel{2sin\alpha}\,(1 + sin\alpha - cos\alpha)}{\cancel{2sin\alpha}\,(sin\alpha + 1)} \Rightarrow y = \frac{1 - cos\alpha + sin\alpha}{1 + sin\alpha}$$

7. $\boldsymbol{If\ x = \sum_{n=0}^{\infty} cos^{2n}\theta\,;\, y = \sum_{n=0}^{\infty} sin^{2n}\theta\ and\ z = \sum_{n=0}^{\infty} cos^{2n}\theta sin^{2n}\theta, 0 < \theta < \frac{\pi}{2}}$

$\boldsymbol{then\ show\ that\ xyz = x + y + z}$

$$\boxed{1 + x + x^2 + \cdots = \frac{1}{1 - x}}$$

$$x = \sum_{n=0}^{\infty} cos^{2n}\theta = cos^0\theta + cos^2\theta + cos^4\theta + \cdots$$

$$= 1 + cos^2\theta + (cos^2\theta)^2 + \cdots = \frac{1}{1 - cos^2\theta}$$

$$x = \sum_{n=0}^{\infty} cos^{2n}\theta = \frac{1}{sin^2\theta} \Rightarrow x = \frac{1}{sin^2\theta}$$

$$y = \sum_{n=0}^{\infty} sin^{2n}\theta = sin^0\theta + sin^2\theta + sin^4\theta + \cdots$$

$$= 1 + sin^2\theta + (sin^2\theta)^2 + \cdots = \frac{1}{1 - sin^2\theta}$$

$$\boxed{y = \frac{1}{cos^2\theta}}$$

$$z = \sum_{n=0}^{\infty} cos^{2n}\theta sin^{2n}\theta = cos^0 sin^0\theta + cos^2\theta sin^2\theta + cos^4\theta sin^4\theta + \cdots$$

$$= 1 + cos^2\theta sin^2\theta + cos^4\theta sin^4\theta + \cdots$$

$$= 1 + cos^2\theta sin^2\theta + (cos^2\theta sin^2\theta)^2 + \cdots$$

$$\boxed{z = \frac{1}{1 - sin^2\theta cos^2\theta}}$$

$$x + y + z = \frac{1}{sin^2\theta} + \frac{1}{cos^2\theta} + \frac{1}{1 - sin^2\theta cos^2\theta}$$

$$= \frac{cos^2\theta(1 - sin^2\theta cos^2\theta) + sin^2\theta(1 - sin^2\theta cos^2\theta) + sin^2\theta cos^2\theta}{sin^2\theta\ cos^2\theta(1 - sin^2\theta cos^2\theta)}$$

$$= \frac{cos^2\theta - sin^2\theta cos^4\theta + sin^2\theta - sin^4\theta cos^2\theta + sin^2\theta cos^2\theta}{sin^2\theta\ cos^2\theta(1 - sin^2\theta cos^2\theta)}$$

$$= \frac{cos^2\theta + sin^2\theta - sin^2\theta cos^4\theta - sin^4\theta cos^2\theta + sin^2\theta cos^2\theta}{sin^2\theta\ cos^2\theta(1 - sin^2\theta cos^2\theta)}$$

$$= \frac{1 - sin^2\theta cos^2\theta(cos^2\theta + sin^2\theta) + sin^2\theta cos^2\theta}{sin^2\theta\ cos^2\theta(1 - sin^2\theta cos^2\theta)}$$

$$= \frac{1 - sin^2\theta cos^2\theta\ (1) + sin^2\theta cos^2\theta}{sin^2\theta\ cos^2\theta(1 - sin^2\theta cos^2\theta)} = \frac{1 - sin^2\theta cos^2\theta + sin^2\theta cos^2\theta}{sin^2\theta\ cos^2\theta(1 - sin^2\theta cos^2\theta)}$$

$$= \frac{1}{sin^2\theta\ cos^2\theta(1 - sin^2\theta cos^2\theta)} = \frac{1}{sin^2\theta} \times \frac{1}{cos^2\theta} \times \frac{1}{1 - sin^2\theta cos^2\theta}$$

$$= x \times y \times z = xyz$$

8. $\boldsymbol{If\ tan^2\theta = 1 - k^2\ .show\ that\ sec\theta + tan^3\theta cosec\theta = \left(2 - k^2\right)^{\frac{3}{2}}.Also\ ,}$ $\boldsymbol{find\ the\ values\ of\ k\ which\ this\ result\ holds}$

$tan^2\theta = 1 - k^2$

Adding 1 on both sides $\Longrightarrow\ 1 + tan^2\theta = 1 - k^2 + 1$

$\boxed{\because 1 + \tan^2\theta = \sec^2\theta}$

$Sec^2\theta = 2 - k^2 \ \Longrightarrow\ \sec\theta = (2 - k^2)^{\frac{1}{2}}$

$L.H.S = \sec\theta + tan^3\theta\ cosec\theta$

$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$

$\boxed{sec\ \theta = \frac{1}{\cos\theta}}$

$$= \frac{1}{cos\theta} + \frac{\overset{sin^2\theta}{\cancel{sin^3\theta}}}{cos^3\theta} \times \frac{1}{\cancel{sin\theta}}$$

$$= \frac{1}{cos\theta} + \frac{sin^2\theta}{cos^3\theta} = \frac{cos^2\theta + sin^2\theta}{cos^3\theta} = \frac{1}{cos^3\theta}$$

$$= sec^3\theta = \left[(2 - k^2)^{\frac{1}{2}}\right]^3 = (2 - k^2)^{\frac{3}{2}}$$

$tan^2\theta = 1 - k^2$

$When\ \theta = 0^\circ,\ tan^2 0 = 1 - k^2 \Longrightarrow\ 0^2 = 1 - k^2$

$0 = 1 - k^2 \ \Longrightarrow\ k^2 = 1$

$k = \sqrt{1}\ \Longrightarrow k = \pm 1$

$when\ \theta = 45^\circ, tan^2 45^\circ = 1 - k^2$

$(1)^2 = 1 - k^2 \Longrightarrow\ 1 = 1 - k^2$

$when\ \theta = 60^\circ, tan^2 60^\circ = 1 - k^2$

$\left(\sqrt{3}\right)^2 = 1 - k^2$

$3 = 1 - k^2 \ \Longrightarrow\ k^2 = 1 - 3$

$k^2 = -2 \Longrightarrow\ k = \sqrt{-2}$

$k^2 = 1 - 1 \implies k^2 = 0$

$\boxed{k = 0}$

$When\ \theta = 90^{\circ}, tan\theta\ is\ undefined$

$When\ \theta > 45^{\circ}, k^2\ will\ be\ negative\ and\ k\ will\ be\ imaginary$

$since\ k\ lies\ between - 1\ and\ 1$

9 . If $sec\theta + tan\theta = p$, obtain the values of $sec\theta, tan\theta$ and $sin\theta$ in terms of p

$sec\theta + tan\theta = p \ldots (1)$

$1 + tan^2\theta = sec^2\theta$

$sec^2\theta - tan^2\theta = 1 \implies (sec\theta + tan\theta)(sec\theta - tan\theta) = 1$

$$p(sec\theta - tan\theta) = 1 \implies sec\theta - tan\theta = \frac{1}{p} \ldots (2)$$

$Adding\ (1) and\ (2)$

$$sec\theta + \cancel{tan\theta} = p$$

$$sec\theta - \cancel{tan\theta} = \frac{1}{p}$$

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

$$\boxed{sec\,\theta = \frac{1}{\cos\theta}}$$

$$2sec\theta = p + \frac{1}{p} \implies 2sec\theta = \frac{p^2+1}{p}$$

$$\boxed{sec\theta = \frac{p^2+1}{2p}}$$

Subracting (1) and (2)

$$\cancel{sec\theta} + tan\theta = p$$

$$\underset{(-)}{\cancel{sec\theta}} \ \underset{(+)}{-}\ tan\theta \ \underset{(-)}{=} \frac{1}{p}$$

$$2\ tan\theta = p - \frac{1}{p} \implies 2tan\theta = \frac{p^2-1}{p}$$

$$\boxed{tan\theta = \frac{p^2-1}{2p}}$$

$$\frac{tan\theta}{sec\theta} = \frac{\frac{p^2-1}{\cancel{2p}}}{\frac{p^2+1}{\cancel{2p}}}$$

$$\frac{\frac{sin\theta}{\cancel{cos\theta}}}{\frac{1}{\cancel{cos\theta}}} = \frac{p^2-1}{p^2+1} \implies \boxed{sin\theta = \frac{p^2-1}{p^2+1}}$$

10. If $\cot\theta\,(1+\sin\theta) = 4m$ and $\cot\theta\,(1-\sin\theta) = 4n$, then prove that $(m^2-n^2)^2 = mn$.

$(1+\sin\theta)^2 = 1^2 + \sin^2\theta + 2(1)\sin\theta$
$(1+\sin\theta)^2 = 1 + \sin^2\theta + 2\sin\theta$
$(1-\sin\theta)^2 = 1 + \sin^2\theta - 2\sin\theta$

$$4m = \cot\theta\,(1+\sin\theta) \implies m = \frac{\cot\theta(1+\sin\theta)}{4}$$

square on both sides

$$m^2 = \frac{\cot^2\theta\,(1+\sin\theta)^2}{16}$$

$$4n = \cot\theta\,(1-\sin\theta) \implies n = \frac{\cot\theta(1-\sin\theta)}{4}$$

square on both sides

$$n^2 = \frac{\cot^2\theta\,(1-\sin\theta)^2}{16}$$

$$m^2-n^2 = \frac{\cot^2\theta\,(1+\sin\theta)^2}{16} - \frac{\cot^2\theta\,(1-\sin\theta)^2}{16}$$

$$m^2-n^2 = \frac{\cot^2\theta}{16}[(1+\sin\theta)^2 - (1-\sin\theta)^2]$$

$$= \frac{\cot^2\theta}{16}[1+\sin^2\theta+2\sin\theta-(1+\sin^2\theta-2\sin\theta)]$$

$$m^2-n^2 = \frac{\cot^2\theta}{16}[\cancel{1}+\cancel{\sin^2}\theta+2\sin\theta-\cancel{1}-\cancel{\sin^2}\theta+2\sin\theta]$$

$$m^2-n^2 = \frac{\cot^2\theta}{\cancel{16}_4}\times\cancel{4}\sin\theta$$

$$m^2-n^2 = \frac{\sin\theta\,\cot^2\theta}{4}$$

$$L.H.S = (m^2-n^2)^2$$

$$(m^2-n^2)^2 = \frac{\sin^2\theta\cot^4\theta}{16} \quad \ldots(1)$$

$$R.H.S = mn$$

$$= \frac{\cot\theta(1+\sin\theta)}{4}\times\frac{\cot\theta(1-\sin\theta)}{4} = \frac{\cot^2\theta(1^2-\sin^2\theta)}{16}$$

$$mn = \frac{\cot^2\theta(1^2-\sin^2\theta)}{16} = \frac{\cot^2\theta(1-\sin^2\theta)}{16}$$

$\cot^2\theta\times\sin^2\theta = \cos^2\theta$

$\cot^2\theta\times\sin^2\theta = \frac{\cos^2\theta}{\cancel{\sin^2}\theta}\times\cancel{\sin^2}\theta$

$$= \frac{\cot^2\theta\cos^2\theta}{16} = \frac{\cot^2\theta\times\cot^2\theta\times\sin^2\theta}{16}$$

$$mn = \frac{\cot^4\theta\,\sin^2\theta}{16} \quad \ldots(2)$$

From (1) and (2) $L.H.S = R.H.S$

11. If $cosec\theta - sin\theta = a^3$ and $sec\theta - cos\theta = b^3$, then prove that $a^2b^2(a^2+b^2) = 1$.

Given : $a^3 = cosec\theta - sin\theta$

$$= \frac{1}{sin\theta} - sin\theta = \frac{1 - sin^2\theta}{sin\theta}$$

$$a^3 = \frac{cos^2\theta}{sin\theta} \implies \boxed{a = \left(\frac{cos^2\theta}{sin\theta}\right)^{\frac{1}{3}}}$$

$b^3 = sec\theta - cos\theta$

$$= \frac{1}{cos\theta} - cos\theta = \frac{1 - cos^2\theta}{cos\theta}$$

$$b^3 = \frac{sin^2\theta}{cos\theta} \implies \boxed{b = \left(\frac{sin^2\theta}{cos\theta}\right)^{\frac{1}{3}}}$$

$L.H.S = a^2b^2\,(a^2+b^2)$

$$a^2 = \left[\left(\frac{cos^2\theta}{sin\theta}\right)^{\frac{1}{3}}\right]^2 \implies a^2 = \left(\frac{cos^2\theta}{sin\theta}\right)^{\frac{2}{3}} \implies a^2 = \frac{cos^{\frac{4}{3}}\theta}{sin^{\frac{2}{3}}\theta}$$

$$b^2 = \left[\left(\frac{sin^2\theta}{cos\theta}\right)^{\frac{1}{3}}\right]^2 \implies b^2 = \left(\frac{sin^2\theta}{cos\theta}\right)^{\frac{2}{3}} \implies b^2 = \frac{sin^{\frac{4}{3}}\theta}{cos^{\frac{2}{3}}\theta}$$

$$= \left(\frac{cos^{\frac{4}{3}}\theta}{sin^{\frac{2}{3}}\theta}\right)\left(\frac{sin^{\frac{4}{3}}\theta}{cos^{\frac{2}{3}}\theta}\right)\left[\frac{cos^{\frac{4}{3}}\theta}{sin^{\frac{2}{3}}\theta} + \frac{sin^{\frac{4}{3}}\theta}{cos^{\frac{2}{3}}\theta}\right]$$

$$= \left(\frac{\overset{cos^{\frac{2}{3}}\theta}{\cancel{cos^{\frac{4}{3}}\theta}}}{\cancel{sin^{\frac{2}{3}}\theta}}\right)\left(\frac{\overset{sin^{\frac{2}{3}}\theta}{\cancel{sin^{\frac{4}{3}}\theta}}}{\cancel{cos^{\frac{2}{3}}\theta}}\right)\left(\frac{cos^{\frac{6}{3}}\,\theta + sin^{\frac{6}{3}}\,\theta}{sin^{\frac{2}{3}}\theta\ cos^{\frac{2}{3}}\theta}\right)$$

$$= \cancel{cos^{\frac{2}{3}}\theta} \times \cancel{sin^{\frac{2}{3}}\theta}\left(\frac{cos^2\theta + sin^2\theta}{\cancel{sin^{\frac{2}{3}}\theta}\ \cancel{cos^{\frac{2}{3}}\theta}}\right) = 1$$

12. Eliminate θ from the equations $a\,sec\theta - c\,tan\theta = b$, $b\,sec\theta + d\,tan\theta = c$.

$a\sec\theta - c\,tan\theta - b = 0$

$b\sec\theta + d\,tan\theta - c = 0$

Using cross multiplication method

$$\begin{array}{cccccccc} & sec\theta & & tan\theta & & 1 & & \\ -c & & -b & & a & & -c & \\ d & & -c & & b & & d & \end{array}$$

$$\frac{sec\theta}{c^2 - (-\mathrm{bd})} = \frac{tan\theta}{-b^2-(-ac)} = \frac{1}{\mathrm{ad} - (-bc)}$$

$$\frac{sec\theta}{c^2+bd} = \frac{tan\theta}{-b^2+ac} = \frac{1}{ad+bc}$$

$$\frac{sec\theta}{c^2+bd} = \frac{1}{ad+bc} \ and \quad \frac{tan\theta}{ac-b^2} = \frac{1}{ad+bc}$$

$$sec\theta = \frac{bd+c^2}{ad+bc} \ and \ tan\theta = \frac{ac-b^2}{ad+bc}$$

$$sec^2\theta - tan^2\theta = 1$$

$$\left[\frac{bd+c^2}{ad+bc}\right]^2 - \left[\frac{ac-b^2}{ad+bc}\right]^2 = 1$$

$$\frac{(bd+c^2)^2}{(ad+bc)^2} - \frac{(ac-b^2)^2}{(ad+bc)^2} = 1 \Longrightarrow \frac{(bd+c^2)^2-(ac-b^2)^2}{(ad+bc)^2} = 1$$

$$(bd+c^2)^2 - (ac-b^2)^2 = (ad+bc)^2$$

EXERCISE : 3.2

Example 3.6 Find the length of an arc of a circle of radius 5cm subtending a central angle measuring 15°.

$\theta = 15°,\ \ r = 5cm\ \ \ s = ?$

$\boxed{degree = \frac{\pi}{180} \times radian}$

$\theta = 15\ degree$

$\theta = 15 \times \frac{\pi}{180}\ radians$

$\boxed{s = r\theta}$

$s = 5 \times \frac{\cancel{15}^{3}\pi}{\cancel{180}_{\cancel{36}\ 12}} \implies \boxed{S = \frac{5\pi}{12} cm}$

Example 3.7 If the arcs of same length in two circles subtend central angles 30° and 80°, find the radio of thiir radi Let r_1 and r_2 be the radi of the two given circles

$\theta_1 = 30° \implies \theta_1 = \cancel{30°} \times \frac{\pi}{\cancel{180°}_{6}}$

$\theta_1 = \frac{\pi}{6}$

$\theta_2 = 80° = \theta = \cancel{80°}^{4} \times \frac{\pi}{\cancel{180°}_{9}}$

$\theta_2 = \frac{4\pi}{9}$

$S_1 = S_2 \implies r_1\,\theta_1 = r_2\,\theta_2 \implies \frac{\cancel{\pi}}{\cancel{6}_{2}} r_1 = \frac{\cancel{4\pi}}{\cancel{9}_{3}} r_2$

$\frac{r_1}{2} = \frac{4}{3} r_2 \implies \frac{r_1}{r_2} = \frac{8}{3} \implies r_1 : r_2 = 8 : 3$

1. Express each of the following angles in radian measure: (i) 30° (ii) 135° (iii) − 205° (iv) 150° (v) 330°.

(i) 30° into radians

$\boxed{degree = \frac{\pi}{180} \times radian}$

$30° = 30\ degree$

$= \cancel{30} \times \frac{\pi}{\cancel{180}_{6}} \times radians = \frac{\pi}{6}$

(ii) 135° into radians

$\boxed{degree = \frac{\pi}{180} \times radian}$

$135° = 135\ degree$

$= \cancel{135}^{\cancel{27}\ 3} \times \frac{\pi}{\cancel{180}_{\cancel{36}\ 4}} \times radians = \frac{3\pi}{4}$

$(\boldsymbol{ii})\ \boldsymbol{-205°\ into\ radians}$

$\boxed{degree = \frac{\pi}{180} \times radian}$

$-205° = -205\ degree$

$= -\overset{41}{\cancel{205}} \times \frac{\pi}{\underset{36}{\cancel{180}}} \times radians$

$= -\frac{41\pi}{36}$

$(\boldsymbol{iii})\boldsymbol{150°\ into\ radians}$

$\boxed{degree = \frac{\pi}{180} \times radian}$

$150° = 150 degree$

$= 150 \times \frac{\pi}{180} \times radians$

$= \overset{5}{\cancel{150}} \times \frac{\pi}{\underset{6}{\cancel{180}}} = \frac{5\pi}{6}$

$(\boldsymbol{iv})\ \boldsymbol{330°\ into\ radians}$

$\boxed{degree = \frac{\pi}{180} \times radian}$

$330° = 330\ degree$

$= \overset{11}{\cancel{330}} \times \frac{\pi}{\underset{6}{\cancel{180}}} \times radians$

$= \frac{11\pi}{6}$

$\boldsymbol{2.}$ ***Find the degree measure corresponding to the following radian measures***

$(\boldsymbol{i})\frac{\boldsymbol{\pi}}{\boldsymbol{3}}\ (\boldsymbol{ii})\frac{\boldsymbol{\pi}}{\boldsymbol{9}}\ (\boldsymbol{iii})\frac{\boldsymbol{2\pi}}{\boldsymbol{5}}\ (\boldsymbol{iv})\frac{\boldsymbol{7\pi}}{\boldsymbol{3}}\ (\boldsymbol{v})\frac{\boldsymbol{10\pi}}{\boldsymbol{9}}$

$(\boldsymbol{ii})\frac{\boldsymbol{\pi}}{\boldsymbol{3}}\boldsymbol{into\ degrees}$

$\boxed{radians = \frac{180°}{\pi}}$

$\frac{\pi}{3} = \frac{\pi}{3} radians$

$= \frac{\cancel{\pi}}{\cancel{3}} \times \frac{\overset{60°}{\cancel{180°}}}{\cancel{\pi}} = 60°$

$(\boldsymbol{ii})\frac{\boldsymbol{\pi}}{\boldsymbol{9}}$

$\boxed{\boldsymbol{radians = \frac{180°}{\pi}}}$

$\frac{\pi}{9} = \frac{\pi}{9} radians$

$= \frac{\cancel{\pi}}{\cancel{9}} \times \frac{\cancel{180°}\ 20°}{\cancel{\pi}} = 20°$

$(iii)\ \dfrac{2\pi}{5}$

$$\boxed{radians = \frac{180^\circ}{\pi}}$$

$$\frac{2\pi}{5} = \frac{2\pi}{5} radians$$

$$= \frac{\cancel{2\pi}}{\cancel{5}} \times \frac{\cancel{180^\circ}^{36^\circ}}{\cancel{\pi}} = 2 \times 36^\circ = 72^\circ$$

$(iii)\ \dfrac{7\pi}{3}$

$$\boxed{radians = \frac{180^\circ}{\pi}}$$

$$\frac{7\pi}{3} = \frac{7\pi}{3} radians$$

$$= \frac{\cancel{7\pi}}{\cancel{3}} \times \frac{\cancel{180^\circ}^{60^\circ}}{\cancel{\pi}} = 7 \times 60^\circ = 420^\circ$$

$(iv)\ \dfrac{10\pi}{9}$

$$\boxed{radians = \frac{180^\circ}{\pi}}$$

$$\frac{7\pi}{3} = \frac{7\pi}{3} radians$$

$$= \frac{\cancel{7\pi}}{\cancel{3}} \times \frac{\cancel{180^\circ}^{60^\circ}}{\cancel{\pi}} = 7 \times 60^\circ = 420^\circ$$

3. What must be the radius of a circular running path, around which an athlete must run 5 times in order to describe 1 km?

Let r be the radii of the circle.

$$Circumference = 2\pi r$$

$$Given\ that: 5(2\pi r) = 1km$$

$$10\pi r = 1000m$$

$$r = \frac{1000}{10\pi} = \frac{100}{\pi} m = \frac{100}{\frac{22}{7}} m = 100 \times \frac{7}{22} m = \frac{700}{22} m$$

$$= \frac{350}{11} m = 31.82m$$

4. In a circle of diameter 40cm, a chord is of length 20 cm. Find the length of the minor arc of the chord.

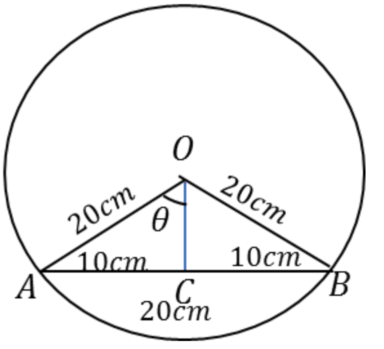
$$\text{Diameter} = 40cm$$

$$radius = 20cm$$

AB is the chord of length 20 cm. Let C be the midpoint of AB

O is the centre of cricle $AC = CB = 10cm$

$In\ \Delta\ OCA, \angle AOC = \theta$



$$sin\theta = \frac{opp}{hyp}$$

$$sin\theta = \frac{AC}{OA} \Longrightarrow \quad sin\theta = \frac{10}{20}$$

$$sin\theta = \frac{1}{2} \quad \Longrightarrow \quad \theta = 30°$$

$$Here\ AOB = 2 \times 30° = 60° = \frac{\pi}{3} radians$$

$$Arc\ length : s = r\theta \ \ (central\ angle)$$

$$= 20 \times \frac{\pi}{3} = \frac{20\pi}{3} cm$$

$$= \frac{20}{3} \times \frac{22}{7} = \frac{440}{21} = 20.95cm$$

5. *Find the degree measure of the angle subtended at the centre of circle of radius 100 cm by an arc of length 22cm.*

$$Given : r = 100cm; s = 22cm$$

$$Arc\ length : s = r\theta \ (\theta\ is\ the\ angle\ at\ the\ centre)$$

$$22 = 100\ \theta$$

$$\theta = \frac{22}{100} radians$$

$$\boxed{0.6 \times 60min = 36'}$$

$$radians = \frac{180°}{\pi}$$

$$\theta = \frac{22}{100} \times \frac{180°}{\pi} \Longrightarrow \theta = \frac{22}{100} \times \frac{180°}{\frac{22}{7}} = \frac{\cancel{22}}{\cancel{100}} \times \cancel{180°} \times \frac{7}{\cancel{22}}$$

$$= \frac{126}{10} = 12.6° = 12°36'$$

6. *What is the length of the arc intercepted by a central angle ofmeasure 41° in a circle of radius 10ft?*

$$\theta = 41° \quad , r = 10ft \ \ , s = ?$$

$$\boxed{degree = \frac{\pi}{180} \times radian}$$

$$\theta = 41\ degree$$

$$\theta = 41 \times \frac{\pi}{180}\ radians$$

$$\boxed{S = r\theta}$$

$$S = 10 \times \frac{41 \times \pi}{180} = \frac{41\pi}{18} ft = \frac{41}{\cancel{18}_{9}} \times \frac{\overset{11}{\cancel{22}}}{7} = \frac{451}{63} = 7.16ft$$

7. *If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.*

Let r_1 and r_2 be the radii of the given circles

$$\theta_1 = 60° \implies \theta_1 = \cancel{60} \times \frac{\pi}{\cancel{180}_3} \qquad \boxed{degree = \frac{\pi}{180} \times radian}$$

$$\theta_1 = \frac{\pi}{3} radians$$

$$\theta_2 = 75° \implies \theta_2 = \overset{\overset{5}{\cancel{15}}}{\cancel{75}} \times \frac{\pi}{\cancel{180}\ \underset{\cancel{36}}{12}}$$

$$\theta_2 = \frac{5\pi}{12} radians$$

Given : $S_1 = S_2$

$$r_1\theta_1 = r_2\theta_2 \implies r_1 \times \frac{\cancel{\pi}}{\cancel{3}} = r_2 \times \frac{5\cancel{\pi}}{\cancel{12}\ 4}$$

$$r_1 = r_2 \times \frac{5}{4} \implies \frac{r_1}{r_2} = \frac{5}{4} \implies r_1 : r_2 = 5 : 4$$

8. *The perimeter of a certain sector of a cricle is equal to the length of the arc of a semi – circle having the same radius. Express the angle of the sector in degrees, minutes and seconds.*

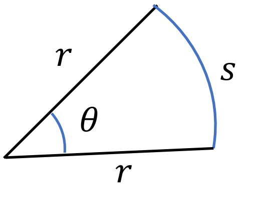
Given that perimeter of the sector = length of arc of semicircle

$$r + r + s = \pi r \implies r + r + r\theta = \pi r$$

$$2r + r\theta = \pi r \implies \cancel{r}(2 + \theta) = \pi\cancel{r}$$

$$2 + \theta = \pi$$

$$\theta = \pi - 2\ radians$$



$$= \frac{22}{7} - 2 = \frac{22 - 14}{7}$$

$$\boxed{radians = \frac{180°}{\pi}}$$

$$\theta = \frac{8}{7}\ radians \implies \theta = \frac{8}{7} \times \frac{180°}{\pi}$$

$$\theta = \frac{8}{7} \times \frac{180°}{\pi}$$

$$= \frac{8}{7} \times \frac{180°}{\frac{22}{7}} = \frac{\overset{4}{\cancel{8}}}{\cancel{7}} \times 180° \times \frac{\cancel{7}}{\cancel{22}\ 11}$$

$$= \frac{180° \times 4}{11} = \frac{720°}{11} = 65.4545°$$

$$= 65°\,27'16''$$

$$\boxed{\begin{aligned} 0.4545° \times 60 &= 27.27' \\ 0.27' &= 0.27' \times 60 \\ &= 16.2'' \approx 16'' \end{aligned}}$$

11)720(65.45
66
60
55
50
44
60
55
5

9. ***An airplane propeller rotates 1000 times per minute. Find the number of degrees that a point the edge of the propeller will rotate in 1 second.***

$No.\,of\ rotations\ per\ minute = 1000$

$$No.\,of\ rotations\ per\ second = \frac{\cancel{1000}^{50}}{{}_{3}\cancel{60}} = \frac{50}{3}\ rotations\ = \frac{50}{\cancel{3}} \times \cancel{360°}^{12°}$$

$$= 6000°$$

10. ***A train is moving on a circular track of 1500m radius at the rate of 66km\hr. What angle will be turn in 20 seconds?***

$Given: r = 1500m, Speed = 66km\backslash hr$

$$\boxed{\begin{array}{c} 1km = 1000m \\ 1hr = 60\ mins \end{array}}$$

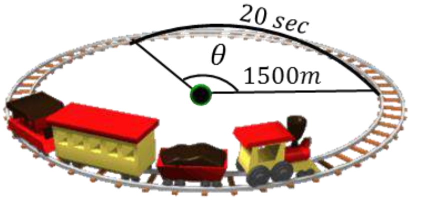
$Time = 20sec\,.To\ find\ \theta = ?$

$Speed = 66km\backslash hr$

$Speed = 66 \times 1000\ m\backslash hr$

$$Speed = 66000\ m\backslash hr \implies Speed = \frac{66000}{60} m\backslash min$$

$$Speed = \frac{66000}{60 \times 60} m\backslash \sec = \frac{\cancel{\cancel{66000}}^{\cancel{110}\ 55}}{\cancel{60}_{3} \times \cancel{60}} m\backslash \sec$$



$$Speed = \frac{55}{3} m\backslash \sec$$

$$\boxed{\begin{array}{c} speed = \dfrac{Distance}{Time} \\ Distance = speed \times Time \end{array}}$$

$$Distance = \frac{55}{3} m\backslash \cancel{sec} \times 20\cancel{sec}$$

$$Distance = \frac{55}{3} \times 20m \implies S = \frac{1100}{3} m$$

$$\boxed{S = r\theta}$$

$$\frac{11\cancel{00}}{3} m = \cancel{15}00m \times \theta \implies \frac{11}{3} = 15 \times \theta$$

$$\frac{11}{3} \times \frac{1}{15} = \theta \implies \theta = \frac{11}{45}$$

$$\theta = \frac{11}{45} \times \frac{180°}{\pi} = \frac{11}{45} \times \frac{180°}{\frac{22}{7}}$$

$$= \frac{\cancel{11}}{\cancel{45}} \times \cancel{180°}^{\cancel{4}\ 2} \times \frac{7}{\cancel{22}\ \cancel{2}}$$

$$\boxed{\theta = 14°}$$

11. ***A ciruclar metallic plate of radius* 8cm *and thickness* 6mm *is melted and moulded into a pie*(*a sector of the circle with thickness*) *of radius* 16cm *and thickness* 4mm. *Find the angle of the sector.***

Given:

$r_1 = radius\ of\ circular\ coin$

$m = radius\ of\ sector$

$Volume\ of\ circular\ plate = volume\ of\ sector$

$$\pi {r_1}^2 \times thickness \quad = \frac{r_2^2}{2}\theta \times thickness$$

$$\pi(8)^2 \times \frac{6}{10} \quad = \frac{(16)^2}{2} \times \frac{4}{10}\theta$$

$$64\pi \times 6 \ = 128\,\theta \times 4$$

$$\theta = \frac{64\pi \times 6}{128\ \times 4} \ = \frac{3}{4}\pi\ radians \quad = 135°$$

$EXERCISE : 3.3$

Example: 3.9 If $\sin\theta = \frac{3}{5}$ and the angle θ is in the second quadrant, then find the values of other five trigonometric functions.

$$Given: sin\theta = \frac{3}{5}$$

$$sin^2\theta + cos^2\theta = 1 \implies cos^2\theta = 1 - sin^2\theta$$

$$\cos\theta = \pm\sqrt{1 - sin^2\theta}$$

$$\cos\theta = \pm\sqrt{1 - \frac{9}{25}} \implies \cos\theta = \pm\sqrt{\frac{25-9}{25}}$$

$$\cos\theta = \pm\sqrt{\frac{16}{25}} \implies \cos\theta = \pm\frac{4}{5}$$

$$\cos\theta = -\frac{4}{5} \quad \theta \text{ lies in the second quardrant.}$$

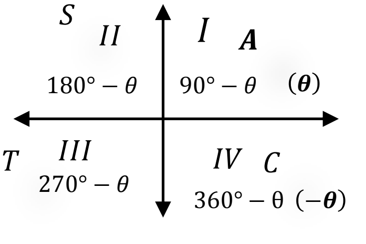
$$\sin\theta = \frac{3}{5} \implies cosec\,\theta = \frac{5}{3}$$

$$\cos\theta = -\frac{4}{5} \implies \sec\theta = -\frac{5}{4} \; [\text{since } \theta \text{ lies in the second quardrant}].$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \implies \tan\theta = \frac{\frac{3}{\cancel{5}}}{\frac{-4}{\cancel{5}}} \implies \tan\theta = -\frac{3}{4}$$

$$\cot\theta = -\frac{4}{3}$$

Example : 3.10. Find the values of (i) $\sin(-45°)$, (ii) $\cos(-45°)$, (iii) $\cot(-45°)$.



(i)
$$sin(-45°) = -sin\,45° \qquad \boxed{sin(-\theta) = -sin\theta}$$
$$= -\frac{1}{\sqrt{2}}$$

(ii)
$$cos(-45°) = cos\,45° \qquad \boxed{cos(-\theta) = cos\theta}$$
$$= \frac{1}{\sqrt{2}}$$

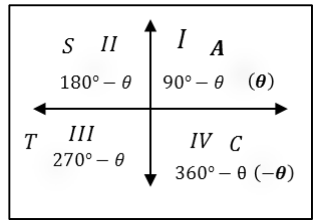
(iv)
$$cot(-45°) = -cot\,45° \qquad \boxed{cot(-\theta) = -cot\theta}$$
$$= -1$$

Example: 3.11 Find the value of (i) $\sin 150°$, (ii) $\cos 135°$, (iii) $\tan 120°$

$(\boldsymbol{i})\quad sin150° = sin(90° + 60°)$ $\quad \boxed{sin(90° + \theta) = cos\theta}$

$$= \cos 60°$$

$$= \frac{1}{2}$$

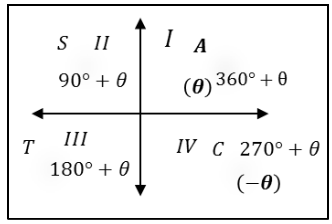


(or)

$$sin\,150° = sin(180° - 30°)$$

$$= sin\,30° \quad sin(180° - \theta) = sin\theta$$

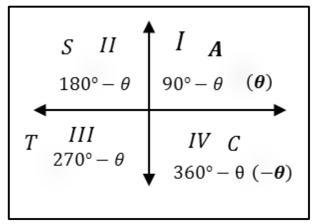
$$= \frac{1}{2}$$



$(\boldsymbol{iii})\quad cos\,135° = \cos(180° - 45°)$ $\quad \boxed{cos(180° - \theta) = -cos\theta}$

$$= -\cos(45°)$$

$$= -\frac{1}{\sqrt{2}}$$

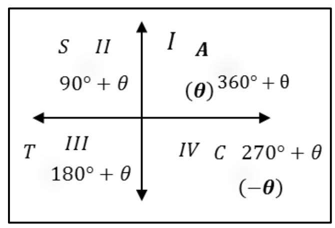


(or)

$cos\,135° = cos(90° + 45°)$ $\quad \boxed{cos(90° + \theta) = -sin\theta}$

$$= -sin(45°)$$

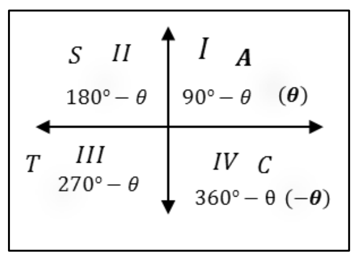
$$= -\frac{1}{\sqrt{2}}$$



$(\boldsymbol{iii})\tan 120° = tan(180° - 60°)$

$= -tan\,60°$ $\quad \boxed{tan(180° - \theta) = -tan\theta}$

$$= -\sqrt{3}$$



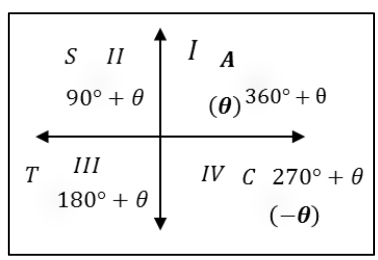
(or)

$tan\,120° = tan(90° + 30°)$ $\quad \boxed{tan(90° + \theta) = -cot\theta}$

$$= -cot\,30°$$

$$= -\sqrt{3}$$

$\boldsymbol{Example}$: 3.12 ***Find the value of*** $(\boldsymbol{i})\ \boldsymbol{sin\,765°}$, $(\boldsymbol{ii})\ \boldsymbol{cosec(-1410°)}$, $(\boldsymbol{iii})\ \boldsymbol{cot\left(\frac{-15\pi}{4}\right)}$.

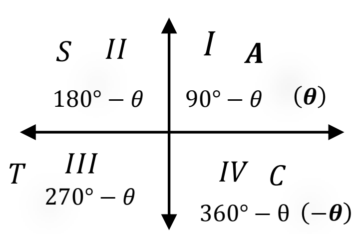


$(i)\ sin\,765° = sin(2 \times 360° + 45°)$

$= sin\,45°$ $\quad \boxed{sin(360° + \theta) = sin\theta}$

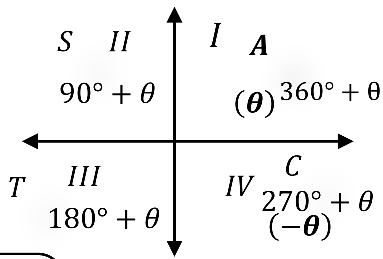
$$= \frac{1}{\sqrt{2}}$$

360) 765(2
720
45
$765 = 2 \times 360 + 45$



(ii) cosec(−1410°)

$$\text{cosec}(-1410^\circ) = -\text{cosec}(1410^\circ)$$
$$= -\text{cosec}(3\times 360^\circ + 330^\circ)$$
$$= -\text{cosec}\,330^\circ$$
$$= -\text{cosec}(360^\circ - 30^\circ)$$
$$= -\{-\text{cosec}\,30^\circ\}$$
$$= \text{cosec}\,30^\circ = 2$$

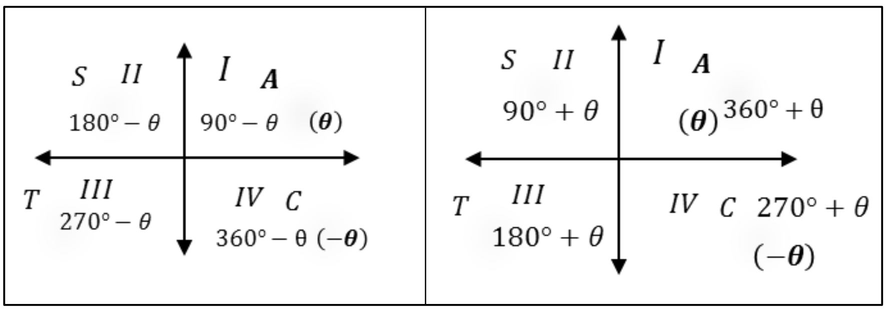


$$360\,\overline{)1410}\,(3$$
$$\frac{1080}{330}$$

$\boxed{1410 = 3\times 360 + 330}$

$\boxed{\text{cosec}(360^\circ + \theta) = \text{cosec}\theta}$

$\boxed{cosec(360^\circ - \theta) = -cosec\theta}$



(iii) $\boldsymbol{cot\left(\frac{-15\pi}{4}\right)}$

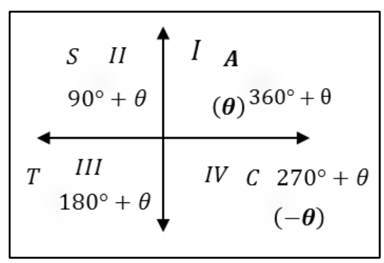
$\boxed{cot(-\theta) = -cot\theta}$

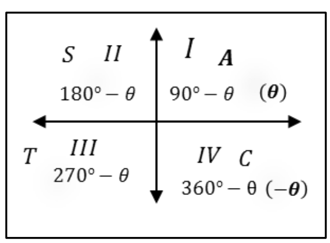
$$\frac{15\pi}{4} = \frac{15\times 180^\circ}{4} = 15\times 45^\circ = 675^\circ$$

$$\cot\left(\frac{-15\pi}{4}\right) = -\cot\left(\frac{15\pi}{4}\right) = -\cot 675^\circ$$

$\boxed{cot(360^\circ - \theta) = -cot\theta}$

$$= -\cot(360^\circ + 315^\circ) = -\cot(315^\circ)$$
$$= -\cot(360^\circ - 45^\circ) = -\{-\cot 45^\circ\}$$
$$= \cot 45^\circ = 1$$





***Example*: 3.13 *Prove* :** $\boldsymbol{tan\,315^\circ\ cot(-405^\circ) + cot\,495^\circ\, tan(-585^\circ) = 2.}$

$$L.H.S = tan\,315^\circ\ cot(-405^\circ) + cot\,495^\circ\, tan(-585^\circ)$$

$\boxed{cot(-\theta) = -cot\theta}$

$\boxed{tan(-\theta) = -tan\theta}$

$$= tan\,315^\circ \times -cot(405^\circ) + cot\,495^\circ \times -tan(585^\circ)$$
$$= tan(360^\circ - 45^\circ) \times -cot(360^\circ + 45^\circ) + cot(360^\circ + 135^\circ) \times -tan(360^\circ + 225^\circ)$$
$$= -tan\,45^\circ \times -cot\,45^\circ + cot\,135^\circ \times -tan 225^\circ$$
$$= tan\,45^\circ \times cot\,45^\circ + cot(180^\circ - 45^\circ) \times -tan(270^\circ - 45^\circ)$$
$$= tan\,45^\circ \times cot\,45^\circ - cot\,45^\circ \times -cot 45^\circ$$
$$= tan\,45^\circ \times cot\,45^\circ + cot\,45^\circ \times cot 45^\circ$$
$$= 1\times 1 + 1\times 1 = 2$$

$tan(360^\circ + \theta) = tan\theta$
$tan(360^\circ - \theta) = -tan\theta$
$cot(360^\circ + \theta) = cot\theta$

***Example*: 3.14. *Determine whether the following functions are even, odd or neither*. (*i*) $\sin^2 x - 2\cos^2 x - \cos x$, (*ii*) $\sin(\cos(x))$, (*iii*) $\cos(\sin(x))$, (*iv*) $\sin x + \cos x$.**

$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$

(*i*) $f(x) = \sin^2 x - 2\cos^2 x - \cos x$

$f(x) = (\sin x)^2 - 2(\cos x)^2 - \cos x$

$f(-x) = [\sin(-x)]^2 - 2[\cos(-x)]^2 - \cos(-x)$

$= (-\sin x)^2 - 2(\cos x)^2 - \cos x$

$f(-x) = \sin^2 x - 2\cos^2 x - \cos x$

$f(-x) = f(x) \Longrightarrow f(x)$ *is even*

(*ii*) $f(x) = \sin[\cos(x)]$

$f(-x) = \sin[\cos(-x)] = \sin(\cos x)$

$f(-x) = f(x) \Longrightarrow f(x)$ *is even*

(*iii*) $f(x) = \cos(\sin(x))$

$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$

$f(-x) = \cos[\sin(-x)]$

$= \cos[-\sin x] = \cos[\sin x]$

$f(-x) = f(x) \Longrightarrow f(x)$ *is even.*

(*iv*) $f(x) = \sin x + \cos x$

$f(-x) = \sin(-x) + \cos(-x)$

$f(-x) = -\sin x + \cos x$

$f(-x) \neq f(x)$

$f(-x) \neq -f(x)$

$f(x) = \sin x + \cos x$ *is neither even nor odd.*

1. *Find the values of* (*i*) $\sin(480°)$ (*ii*) $\sin(-1110°)$ (*iii*) $\cos(300°)$, (*iv*) $\tan(1050°)$, (*v*) $\cot(660°)$ (*vi*) $\tan\left[\frac{19\pi}{3}\right]$, (*vii*) $\sin\left[\frac{-11\pi}{3}\right]$.

(*i*) $\sin(480°)$

$= \sin(360° + 120°) = \sin 120°$

$= \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}$

$\sin(360° + \theta) = \sin\theta$
$\sin(180° - \theta) = \sin\theta$

(*ii*) $\sin(-1110°)$

$\sin(-\theta) = -\sin\theta$

$= -\sin(1110°) = -\sin(3 \times 360° + 30°)$

$= -\sin 30° = -\frac{1}{2}$

$360 \,)\, 1110 \,(\, 3$
$\quad 1080$
$\quad\overline{\;\;30\;\;}$

$1110 = 3 \times 360 + 30$
$\sin(360° + \theta) = \sin\theta$

$(iii)\ \cos 300°$

$\boxed{\cos(360° - \theta) = \cos\theta}$

$$\cos 300° = \cos(360° - 60°) = \cos 60° = \frac{1}{2}$$

$(iv)\ \tan(1050°)$

$$360\,)\,1050\,(2 \quad 720 \quad 330$$

$1050 = 3 \times 360 + 30$

$\boxed{\tan(360° + \theta) = \tan\theta}$

$\boxed{\tan(360° - \theta) = -\tan\theta}$

$$\tan(1050°) = \tan(2 \times 360° + 330°) = \tan 330° = \tan(360° - 30°) = -\tan 30° = -\frac{1}{\sqrt{3}}$$

$(v)\ \cot(660°)$

$\cot(360° + \theta) = \cot\theta$

$\cot(360° - \theta) = -\cot\theta$

$$= \cot(660°) = \cot(360° + 300°) = \cot 300° = \cot(360° - 60°) = -\cot 60° = -\frac{1}{\sqrt{3}}$$

$(vi)\ \tan\left(\frac{19\pi}{3}\right)$

$$360\,)\,1140\,(3 \quad 1080 \quad 60$$

$1140 = 3 \times 360 + 60$

$$\frac{19\pi}{3} = \frac{19 \times 180°}{3} = \frac{19 \times \cancel{180°}^{60°}}{\cancel{3}} = 19 \times 60° = 1140°$$

$$= \tan 1140° = \tan(3 \times 360° + 60°) = \tan 60° = \sqrt{3}$$

$(vii)\ \sin\left(\frac{-11\pi}{3}\right)$

$\boxed{\sin(-\theta) = -\sin\theta}$

$$\frac{11\pi}{3} = \frac{11 \times 180°}{3} = \frac{11 \times \cancel{180°}^{60°}}{\cancel{3}} = 11 \times 60° = 660°$$

$$\sin\left(-\frac{11\pi}{3}\right) = -\sin\left(\frac{11\pi}{3}\right) = -\sin 660°$$

$$= -\{\sin(360° + 300°)\}$$

$$= -\sin 300° = -\{\sin(360° - 60°)\}$$

$$= -\{-\sin 60°\} = \sin 60° = \frac{\sqrt{3}}{2}$$

$$360\,)\,660\,(1 \quad 360 \quad 300$$

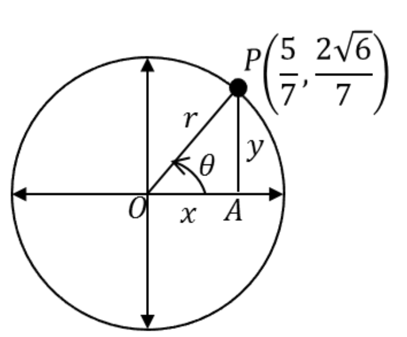
$660 = 360 + 300$

$\boxed{\sin(360° + \theta) = \sin\theta}$ $\boxed{\sin(360° - \theta) = -\sin\theta}$

2. $\left(\frac{5}{7}, \frac{2\sqrt{6}}{7}\right)$ ***is a point on the terminal side of an angle θ in standard position. Determine trigonometric function values of angle θ.***

$$x = \frac{5}{7},\ y = \frac{2\sqrt{6}}{7}$$

$$r=\sqrt{x^2+y^2} \implies r=\sqrt{\left(\frac{5}{7}\right)^2+\left(\frac{2\sqrt{6}}{7}\right)^2}$$



$$=\sqrt{\frac{25}{49}+\frac{4(6)}{49}}=\sqrt{\frac{25}{49}+\frac{24}{49}}=\sqrt{\frac{25+24}{49}}=\sqrt{\frac{49}{49}}$$

$$\boxed{r=1}$$

$$\therefore \cos\theta=\frac{x}{r} \implies \cos\theta=\frac{5}{7}$$

$$\boxed{\sec\theta=\frac{7}{5}}$$

$$\sin\theta=\frac{y}{r}$$

$$\sin\theta=\frac{2\sqrt{6}}{7} \implies \operatorname{cosec}\theta=\frac{7}{2\sqrt{6}}$$

$$\tan\theta=\frac{y}{x} \implies \tan\theta=\frac{\frac{2\sqrt{6}}{\cancel{7}}}{\frac{5}{\cancel{7}}}$$

$$\tan\theta=\frac{2\sqrt{6}}{5} \implies \cot\theta=\frac{5}{2\sqrt{6}}$$

3. *Find the value of other five trigonometric function for the following*:

(i) $\boldsymbol{\cos\theta=-\frac{1}{2}}$, ***θ lies in the III quardrant.***

(ii) $\boldsymbol{\cos\theta=\frac{2}{3}}$, ***θ lies in the I quardrant.***

(iii) $\boldsymbol{\sin\theta=-\frac{2}{3}}$, ***θ lies in the IV quardrant.***

(iv) $\boldsymbol{\tan\theta=-2}$, ***θ lies in the II quardrant.***

(v) $\boldsymbol{\sec\theta=\frac{13}{5}}$ ***θ lies in the IV quardrant.***

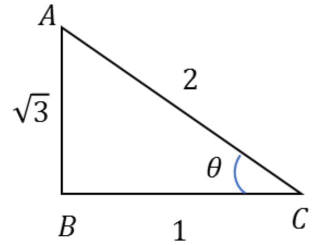
(i) $\boldsymbol{\cos\theta=-\frac{1}{2}}$, ***θ lies in the III quardrant.***

$$\cos\theta=-\frac{1}{2}\quad\frac{adj}{hyp}$$

$$AC^2=AB^2+BC^2$$
$$AB^2=AC^2-BC^2$$
$$AB=\sqrt{AC^2-BC^2}$$

$$AB=\sqrt{AC^2-BC^2} \implies AB=\sqrt{2^2-1^2}$$

$AB = \sqrt{4-1} \implies AB = \sqrt{3}$



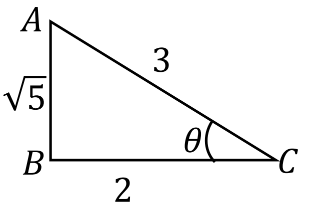
$\boldsymbol{\theta}$ ***lies in the III quadrant, only*** $\boldsymbol{\tan\theta}$ ***and*** $\boldsymbol{\cot\theta}$ ***are positive***:

$$\sin\theta = \frac{opp}{hyp} = -\frac{\sqrt{3}}{2};\ \tan\theta = \frac{opp}{adj} = \sqrt{3}$$

$$cosec\theta = \frac{1}{\sin\theta} = \frac{-2}{\sqrt{3}};\ \sec\theta = \frac{1}{\cos\theta} = -2;\ cos\theta = -\frac{1}{2}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sqrt{3}}$$

$\boldsymbol{(ii)\ \cos\theta = \frac{2}{3}, \theta}$ ***lies in the I quardrant.***



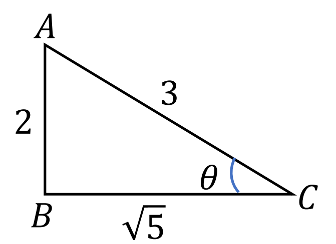
$$\cos\theta = \frac{2}{3}\ \ \frac{adj}{hyp}$$

$$AB = \sqrt{3^2 - 2^2} = \sqrt{9-4} = \sqrt{5}$$

θ *lies in the I quadrant all the trigonometric rations are positive.*

$$\sin\theta = \frac{\sqrt{5}}{3},\ \tan\theta = \frac{\sqrt{5}}{2},\ cosec\theta = \frac{3}{\sqrt{5}}$$

$$\sec\theta = \frac{3}{2},\ \cot\theta = \frac{2}{\sqrt{5}}$$



$\boldsymbol{(iii)\ \sin\theta = -\frac{2}{3}, \theta}$ ***lies in the IV quardrant.***

$$\sin\theta = -\frac{2}{3}\ \ \frac{opp}{hyp}$$

$$BC = \sqrt{3^2 - 2^2} = \sqrt{9-4}$$

$$\boxed{BC = \sqrt{5}}$$

θ *lies in the IV quadrant, only* $\cos\theta$ *and* $\sec\theta$ *are positive.*

$$\therefore \cos\theta = \frac{\sqrt{5}}{3};\ \tan\theta = \frac{-2}{\sqrt{5}};\ \sec\theta = \frac{3}{\sqrt{5}};$$

$$cosec\theta = \frac{-3}{2};\ \cot\theta = \frac{-\sqrt{5}}{2}$$

$\boldsymbol{(iv)\ \tan\theta = -2, \theta}$ ***lies in the II quardrant.***

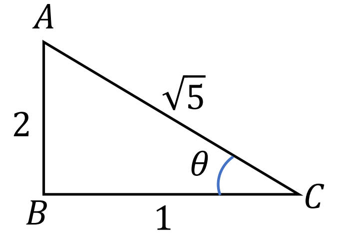
$$\tan\theta = -\frac{2}{1}\ \ \frac{opp}{adj}$$

$$AC = \sqrt{2^2 + 1^2} = \sqrt{5}$$

θ *lies in the II quadrant, only* $\sin\theta$ *and* $cosec\theta$ *are positive.*

$$\sin\theta = \frac{2}{\sqrt{5}},\quad \cos\theta = \frac{-1}{\sqrt{5}},\quad cosec\theta = \frac{\sqrt{5}}{2},$$

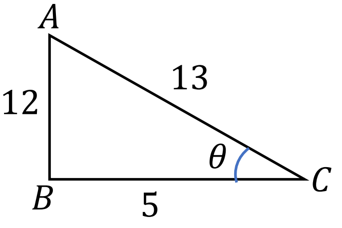
$$\sec\theta = -\sqrt{5},\quad \cot\theta = \frac{-1}{2}$$



$(v)\ \sec\theta = \frac{13}{5}$, θ *lies in the IV quardrant*

$$\sec\theta = \frac{13}{5} \implies \cos\theta = \frac{5}{13}$$

$$AB = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$$



θ lies in the IV quadrant, only $\cos\theta$ and $\sec\theta$ are positive.

$$\sin\theta = -\frac{12}{13},\ \cos\theta = \frac{5}{13},\ \tan\theta = -\frac{12}{5},$$

$$cosec\theta = -\frac{13}{12},\quad \cot\theta = -\frac{5}{12}.$$

4. *Prove that* $\frac{\cot(180° + \theta)\sin(90° - \theta)\cos(-\theta)}{\sin(270° + \theta)\tan(-\theta)\,cosec\theta\ (360° + \theta)} = \cos^2\theta\cot\theta$

$$L.H.S = \frac{\cot(180° + \theta)\sin(90° - \theta)\cos(-\theta)}{\sin(270° + \theta)\tan(-\theta)\,cosec\theta\ (360° + \theta)}$$

$$= \frac{\cot\theta \times \cos\theta \times \cos\theta}{-\cos\theta \times -\tan\theta \times cosec\theta}$$

$$= \frac{\frac{\cos\theta}{\sin\theta} \times \cancel{\cos\theta} \times \cos\theta}{\cancel{\cos\theta} \times \frac{\cancel{\sin\theta}}{\cos\theta} \times \frac{1}{\cancel{\sin\theta}}} = \frac{\frac{\cos\theta}{\sin\theta} \times \cos\theta}{\frac{1}{\cos\theta}}$$

$$= \frac{\cos^2\theta}{\sin\theta} \times \frac{\cos\theta}{1} \quad = \cos^2\theta \times \frac{\cos\theta}{\sin\theta}$$

$$= \cos^2\theta\cot\theta$$

$\cot(180° + \theta) = \cot\theta$
$\sin(90° - \theta) = \cos\theta$
$\cos(-\theta) = \cos\theta$
$\sin(270° + \theta) = -\cos\theta$
$\tan(-\theta) = -\tan\theta$
$cosec(360° + \theta) = cosec\theta$

5. *Find all the angles between* $0°$ *and* $360°$ *which satisfy the equation* $\sin^2\theta = \frac{3}{4}$.

$$\sin^2\theta = \frac{3}{4}$$

$$\sin\theta = \sqrt{\frac{3}{4}} \implies \sin\theta = \pm\frac{\sqrt{3}}{2}$$

$$\sin(270° - 30°) = -\cos 30°$$

$$\sin 240° = -\frac{\sqrt{3}}{2}$$

$$\boldsymbol{\sin\theta = \frac{\sqrt{3}}{2}}$$

$$\sin\theta = \sin 60^\circ$$

$$\theta = 60^\circ$$

$$\sin\theta = \sin(180^\circ - 60^\circ)$$

$$\sin\theta = \sin 120^\circ$$

$$\theta = 120^\circ$$

$$\boldsymbol{\sin\theta = -\frac{\sqrt{3}}{2}}$$

$$\sin\theta = \sin(270^\circ - 30^\circ)$$

$$\sin\theta = \sin 240^\circ$$

$$\theta = 240^\circ$$

$$\sin\theta = \sin(360^\circ - 60^\circ)$$

$$\sin\theta = \sin 300^\circ \implies \theta = 300^\circ$$

$$\therefore \theta = 60^\circ, 120^\circ, 240^\circ, 300^\circ$$

6. Show that $\boldsymbol{\sin^2\frac{\pi}{18} + \sin^2\frac{\pi}{9} + \sin^2\frac{7\pi}{18} + \sin^2\frac{4\pi}{9} = 2.}$

$$L.H.S = \sin^2\frac{\pi}{18} + \sin^2\frac{\pi}{9} + \sin^2\frac{7\pi}{18} + \sin^2\frac{4\pi}{9}$$

$$= \sin^2\left(\frac{\cancel{\pi}}{\cancel{18}} \times \frac{\overset{10^\circ}{\cancel{180^\circ}}}{\cancel{\pi}}\right) + \sin^2\left(\frac{\cancel{\pi}}{\cancel{9}} \times \frac{\overset{20^\circ}{\cancel{180^\circ}}}{\cancel{\pi}}\right) + \sin^2\left(\frac{7\cancel{\pi}}{\cancel{18}} \times \frac{\overset{10^\circ}{\cancel{180^\circ}}}{\cancel{\pi}}\right) + \sin^2\left(\frac{4\cancel{\pi}}{\cancel{9}} \times \frac{\overset{20^\circ}{\cancel{180^\circ}}}{\cancel{\pi}}\right)$$

$$= \sin^2 10^\circ + \sin^2 20^\circ + \sin^2 70^\circ + \sin^2 80^\circ$$

$$= [\sin(90 - 80^\circ)]^2 + [\sin(90 - 70)]^2 + \sin^2 70^\circ + \sin^2 80^\circ$$

$$= \cos^2 80^\circ + \cos^2 70^\circ + \sin^2 70^\circ + \sin^2 80^\circ$$

$$= (\cos^2 80^\circ + \sin^2 80^\circ) + (\cos^2 70^\circ + \sin^2 70^\circ)$$

$$= 1 + 1 = 2 = R.H.S$$

EXERCISE : 3.4

COMPOUND ANGLES

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

***Example* : 3.5 *Find the value of* i) cos 15°**

$$\boxed{\cos(A-B) = \cos A \cos B + \sin A \sin B}$$

$$\cos 15° = \cos(45° - 30°)$$

$$= \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

***ii) tan* 165°**

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan 120° = \tan(180° - 60°) = -\tan 60° = -\sqrt{3}$$

$$\tan 165° = \tan(120° + 45°)$$

$$= \frac{\tan 120° + \tan 45°}{1 - \tan 120° \tan 45°}$$

$$= \frac{-\sqrt{3} + 1}{1 - (-\sqrt{3})(1)} = \frac{-\sqrt{3}+1}{1+\sqrt{3}} = \frac{1-\sqrt{3}}{1+\sqrt{3}}$$

***Example* 3.16 *If* $\boldsymbol{\sin x = \frac{4}{5}}$ *(in I quadrant) and* $\boldsymbol{\cos y = \frac{-12}{13}}$ *(in II quadrant), then find* (i) $\boldsymbol{\sin(x-y)}$, (ii) $\boldsymbol{\cos(x-y)}$.**

$$Given: \sin x = \frac{4}{5}$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cos^2 x = 1 - \sin^2 x \implies \cos x = \sqrt{1 - \sin^2 x}$$

$$\cos x = \sqrt{1 - \left(\frac{4}{5}\right)^2} \implies \cos x = \sqrt{1 - \frac{16}{25}}$$

$$cosx = \sqrt{\frac{25-16}{25}} \Rightarrow cosx = \sqrt{\frac{9}{25}}$$

$$cosx = \pm\frac{3}{5} \quad \text{since } 0 < x < \frac{\pi}{2} \qquad \therefore cosx = \frac{3}{5}$$

$$Given : cosy = -\frac{12}{13} \text{ in the II quadrant}$$

$$\boxed{\begin{array}{l} sin^2 y + cos^2 y = 1 \\ sin^2 y = 1 - cos^2 y \\ sin\, y = \sqrt{1 - cos^2 y} \end{array}}$$

$$\sin y = \sqrt{1-\cos^2 y}$$

$$\sin y = \sqrt{1-\left(-\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}}$$

$$= \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}}$$

$$sin\, y = \pm\frac{5}{13}$$

$$sin\, y = \frac{5}{13} \; (since\ y\ lies\ in\ the\ II\ quadrant)$$

$$(\boldsymbol{i}) \;\; \boldsymbol{sin(x-y) = sinx\, cosy - cosx\, siny}$$

$$= \frac{4}{5}\left(\frac{-12}{13}\right) - \frac{3}{5}\left(\frac{5}{13}\right) = -\frac{48}{65} - \frac{15}{65} = \frac{-48-15}{65}$$

$$= -\frac{63}{65}$$

$$(\boldsymbol{ii})\; \boldsymbol{cos(x-y) = cosx\, cosy + sinx\, siny}$$

$$= \frac{3}{5}\left(\frac{-12}{13}\right) + \frac{4}{5}\left(\frac{5}{13}\right) = -\frac{36}{65} + \frac{20}{65} = -\frac{16}{65}$$

Example 3.17 Prove that $\boldsymbol{cos\left(\frac{3\pi}{4}+x\right) - cos\left(\frac{3\pi}{4}-x\right) = -\sqrt{2}\, sin\, x}$

$$L.H.S = cos\left(\frac{3\pi}{4}+x\right) - cos\left(\frac{3\pi}{4}-x\right)$$

$$\boxed{\frac{3\pi}{4} = \frac{3\cancel{\pi}}{\cancel{4}} \times \frac{\cancel{180°}^{45°}}{\cancel{\pi}} = 135°}$$

$$= cos\frac{3\pi}{4}cosx - sin\frac{3\pi}{4}sinx - \left\{cos\frac{3\pi}{4}cosx + sin\frac{3\pi}{4}sinx\right\}$$

$$= \cancel{cos\frac{3\pi}{4}cosx} - sin\frac{3\pi}{4}sinx - \cancel{cos\frac{3\pi}{4}cosx} - sin\frac{3\pi}{4}sinx$$

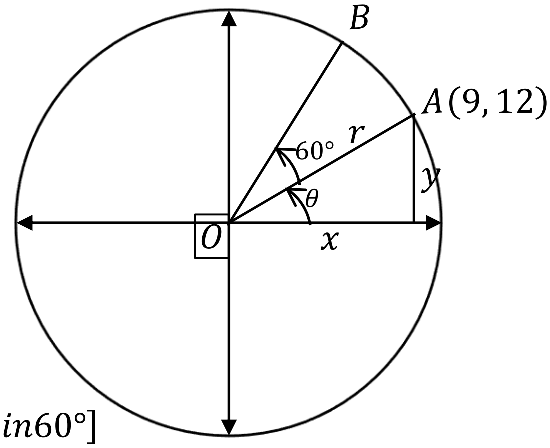
$$= -2sin\frac{3\pi}{4}sinx = -2sin135°sinx$$

$$\boxed{\begin{array}{l} sin135° = sin(180° - 45°) \\ sin135° = sin45° \end{array}}$$

$$= -2sin45°sinx$$

$$= -2\left(\frac{1}{\sqrt{2}}\right) sinx \quad = -\sqrt{2} \times \cancel{\sqrt{2}} \times \frac{1}{\cancel{\sqrt{2}}} \times sinx \; = -\sqrt{2}\, sinx$$

Example 3.18 Point $A(9,12)$ rotates around the origin O in a plane through $60°$ in the anticlock wise direction to a new position B. Find the coordinates of the point B,



$Let \;\; A(9,12) = A\,(r\,cos\theta, r\,sin\theta)$

$Then \;\; r\,cos\theta = 9 \; and \; r\,sin\theta = 12.$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{9^2 + 12^2} = \sqrt{81 + 144} = \sqrt{225}$$

$$\boxed{r = 15}$$

$Hence, the\ point\ A\ is\ (15cos\theta, 15sin\theta).$

$Now, the\ point\ B\ is$

$[15\cos(\theta + 60°), 15\sin(\theta + 60°)]$

$$\boxed{\begin{array}{l} 15\,cos\theta = 9 \\ 15\,sin\theta = 12. \end{array}}$$

$$15\cos(\theta + 60°) = 15[cos\theta\,cos60° - sin\theta sin60°]$$

$$= 15[cos\theta\,cos60° - sin\theta sin60°]$$

$$= 15cos\theta\,cos60° - 15sin\theta sin60°$$

$$= 9 \times \frac{1}{2} - 12 \times \frac{\sqrt{3}}{2} = \frac{9}{2} - \frac{12\sqrt{3}}{2}$$

$$= \frac{3}{2}(3 - 4\sqrt{3})$$

$$15\,sin(\theta + 60°) = 15\,(sin\,\theta\,cos60° + cos\theta\,sin60°)$$

$$= 15\,sin\,\theta\,cos60° + 15cos\theta\,sin60°$$

$$= 12 \times \frac{1}{2} + 9 \times \frac{\sqrt{3}}{2}$$

$$= \frac{12}{2} + \frac{9\sqrt{3}}{2} \quad = \frac{3}{2}\left(4 + 3\sqrt{3}\right)$$

$$B = \left[\frac{3}{2}\left(3 - 4\sqrt{3}\right), \frac{3}{2}\left(4 + 3\sqrt{3}\right)\right]$$

Example3.19 A ripple rank demonstrates the effect of two water waves being added together. The two waves are described by $h = 8\cos t$ and $h = 6\sin t$, where $t \in (0, 2\pi)$ is in secounds and h the height in millimeters above still water. Find the maximum height of the resultant wave and the value of t at a which it occurs.

Let H be the height of the resulatant wave at time t. Then H is given by

$Given : h = 8cost \ldots (1)\ and\ h = 6sint \ \ldots (2)$

$$H = 8\cos t + 6\sin t$$

$$\boxed{\cos A \cos B + \sin A \sin B = \cos(A - B)}$$

$$Let\ 8\cos t + 6\sin t = \cos t\,(k\cos\alpha) + \sin t\,(k\sin\alpha)$$
$$= k(\cos t\cos\alpha + \sin t\sin\alpha)$$
$$8\cos t + 6\sin t = k\,cos(t-\alpha)$$
$$k = \sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100}$$
$$Hence,\ k = 10$$
$$\boldsymbol{solve\ (1)\ and\ (2)}$$
$$6\ sint = 8\ cost$$
$$\frac{sint}{cost} = \frac{8}{6} \Longrightarrow \tan t = \frac{4}{3}$$
$$8\cos t + 6\sin t = k\,cos(t-\alpha)$$
$$H = k\,cos(t-\alpha)$$
$$H = 10\,cos(t-\alpha)$$
$$when\ t = \alpha$$
$$H = 10\,cos(\alpha-\alpha)$$
$$H = 10\,cos0 \Longrightarrow H = 10$$
$$Thus, the\ maximum\ of\ H = 10mm.\ The\ maximum\ occurs\ when\ t = \alpha,$$
$$Where\ \tan\alpha = \frac{3}{4}.$$

Example 3.20 Expand (i) $\boldsymbol{sin(A+B+C)}$ (ii) $\boldsymbol{tan(A+B+C)}$

$$(i)\ sin(A+B+C) = sin[A+\underbrace{B+C}_{B}] \qquad \boxed{sin(A+B) = sin\,A\cos B + cos\,A\sin B}$$
$$= sinA\,cos(B+C) + cos\,A\,sin(B+C)$$
$$= sin\,A\,(cosB\ cosC - sinBsinC) + cosA\,(sinB\ cosC + cosB\ sinC)$$
$$= sin\,A\,cosB\ cosC - sin\,A\,sinBsinC + cos\,A\,sinB\ cosC + cos\,A\,cosB\ sinC$$
$$= sin\,A\,cosB\ cosC + cos\,A\,sinB\ cosC + cos\,A\,cosB\ sinC - sinA\ sinB\ sinC$$

$$\boldsymbol{(ii)\ tan(A+B+C) = tan[A+\underbrace{B+C}_{B}]} \qquad \boxed{tan(A+B) = \frac{tan\,A + tan\,B}{1 - tan\,A\,tan\,B}}$$
$$= \frac{tanA + tan(B+C)}{1 - tanA\ tan(B+C)}$$
$$= \frac{tanA + \dfrac{tanB + tanC}{1 - tanB\ tan\,C}}{1 - tanA\left(\dfrac{tanB + tanC}{1 - tanB\ tan\,C}\right)} = \frac{\dfrac{tanA(1 - tanB\ tan\,C) + tanB + tanC}{\cancel{1 - tanB\ tan\,C}}}{\dfrac{1 - tanB\ tan\,C - tanA(tanB + tanC)}{\cancel{1 - tanB\ tan\,C}}}$$
$$= \frac{tanA - tanA\ tanB\ tanC + tanB + tanC}{1 - tanB\ tan\,C - tanA(tanB + tanC)}$$

$$= \frac{tanA + tanB + tanC - tanA\,tanB\,tanC}{1 - tanB\,tan\,C - tanA\,tanB - tanA\,tanC}$$

$\mathbf{1.}$ $\boldsymbol{If\ \sin x = \frac{15}{17}\ and \cos y = \frac{12}{13}, 0 < x < \frac{\pi}{2}, 0 < y < \frac{\pi}{2}, find\ the\ value\ of}$ $\boldsymbol{(i)\ \sin(x+y)\ (ii)\cos(x-y)\ \ (iii)\ \ tan(x+y).}$

$0 < x < \frac{\pi}{2}, 0 < y < \frac{\pi}{2}, x\ and\ y\ lies\ in\ the\ I\ quadrant.$

$\therefore$ *All the trigonometric ratios are positive*

$$\sin^2\theta + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \sin^2\theta$$
$$cos\theta = \sqrt{1 - \sin^2\theta}$$

$$\sin x = \frac{15}{17}$$

$$\cos x = \sqrt{1 - sin^2 x}$$

$$\cos x = \sqrt{1 - \left(\frac{15}{17}\right)^2} \Rightarrow \cos x = \sqrt{1 - \frac{225}{289}} \Rightarrow \cos x = \sqrt{\frac{289 - 225}{289}}$$

$$\cos x = \sqrt{\frac{64}{289}} \quad \Rightarrow \quad \cos x = \pm\frac{8}{17}$$

$$\cos x = \frac{8}{17}\ \ \text{since}\ \ 0 < x < \frac{\pi}{2}$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta = 1 - \cos^2\theta$$

$$\cos y = \frac{12}{13} \Rightarrow \sin y = \sqrt{1 - \cos^2 y}$$

$$\sin y = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \sqrt{1 - \frac{144}{169}} = \sqrt{\frac{169 - 144}{169}} = \sqrt{\frac{25}{169}}$$

$$\sin y = \pm\frac{5}{13} \Rightarrow \sin y = \frac{5}{13} \quad since\ 0 < y < \frac{\pi}{2}$$

$$tan\,x = \frac{sin\,x}{cos\,x} = \frac{\frac{15}{\cancel{17}}}{\frac{8}{\cancel{17}}} \Rightarrow \tan x = \frac{15}{8}$$

$$\tan y = \frac{\sin y}{\cos y} = \frac{\frac{5}{\cancel{13}}}{\frac{12}{\cancel{13}}} \Rightarrow tan\,y = \frac{5}{12}$$

$$\boxed{sin(x+y) = sin\,x\,cos\,y + cos\,x\,sin\,y}$$

$$= \frac{15}{17} \times \frac{12}{13} + \frac{8}{17} \times \frac{5}{13} = \frac{180}{221} + \frac{40}{221} = \frac{220}{221}$$

$(\boldsymbol{ii})\ \boldsymbol{cos}(\boldsymbol{x}-\boldsymbol{y})$

$$\boxed{\cos(x-y)=\cos x\cos y+\sin x\sin y}$$

$$=\frac{8}{15}\times\frac{12}{13}+\frac{15}{17}\times\frac{5}{13}=\frac{96}{15\times 13}+\frac{75}{17\times 13}=\frac{96\times 17+75\times 15}{13\times 15\times 17}$$

$$=\frac{1632+1125}{3315}=\frac{2757}{3315}$$

$(\boldsymbol{iii})\ \boldsymbol{tan}(\boldsymbol{x}+\boldsymbol{y})$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

$$=\frac{\frac{15}{8}+\frac{5}{12}}{1-\frac{15}{8}\times\frac{5}{12}}=\frac{\frac{45+10}{24}}{1-\frac{75}{96}}=\frac{\frac{55}{24}}{\frac{96-75}{96}}$$

$$\tan(x+y)=\frac{\frac{55}{24}}{\frac{21}{96}}\Longrightarrow\quad tan(x+y)=\frac{55}{\cancel{24}}\times\frac{\overset{4}{\cancel{96}}}{21}$$

$$\tan(x+y)=\frac{220}{21}$$

2. $\boldsymbol{If\ \ sin\,A=\frac{3}{5}\ and\ \ cos\,B=\frac{9}{41},0<A<\frac{\pi}{2},0<B<\frac{\pi}{2}.Find\ the\ value\ of}$

$(\boldsymbol{i})\ \boldsymbol{sin}(\boldsymbol{A}+\boldsymbol{B})\ (\boldsymbol{ii})\ \boldsymbol{cos}(\boldsymbol{A}-\boldsymbol{B})$

$0<A<\frac{\pi}{2},0<B<\frac{\pi}{2},A,B\ lies\ in\ the\ I\ quadrant.$

$\therefore$ *All the trigonometric ratios are positive*

$$sin\,A=\frac{3}{5}$$

$$\cos A=\sqrt{1-\sin^2 A}=\sqrt{1-\left(\frac{3}{5}\right)^2}$$

$$=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{25-9}{25}}=\sqrt{\frac{16}{25}}$$

$$cos\,A=\frac{4}{5}\quad since\ \ 0<A<\frac{\pi}{2}$$

$$\cos B = \frac{9}{41}$$

$$\sin B = \sqrt{1-\cos^2 B} = \sqrt{1-\left(\frac{9}{41}\right)^2} = \sqrt{1-\frac{81}{1681}} = \sqrt{\frac{1681-81}{1681}}$$

$$\sin B = \sqrt{\frac{1600}{1681}} \implies \sin B = \frac{40}{41}$$

$(\boldsymbol{i})\ \boldsymbol{\sin(A+B) = \sin A \cos B + \cos A \sin B}$

$$= \frac{3}{5}\times\frac{9}{41}+\frac{4}{5}\times\frac{40}{41} = \frac{27}{205}+\frac{160}{205} = \frac{187}{205}$$

$(\boldsymbol{ii})\ \boldsymbol{\cos(A-B) = \cos A \cos B + \sin A \sin B}$

$$= \frac{4}{5}\times\frac{9}{41}+\frac{3}{5}\times\frac{40}{41} = \frac{36}{205}+\frac{120}{205} = \frac{156}{205}$$

3. $\boldsymbol{Find\ \cos(x-y), given\ that \cos x = -\frac{4}{5}\ with\ \pi < x < \frac{3\pi}{2}\ and}$ $\boldsymbol{\sin y = -\frac{24}{25} with\ \pi < y < \frac{3\pi}{2}}$.

$\pi < x < \frac{3\pi}{2}$, *x lies in the III quadrant only.* $\cot x$ *and* $\tan x$ *are positive*

$$\cos x = -\frac{4}{5}$$

$$\sin x = \sqrt{1-\cos^2 x}$$

$$\sin x = \sqrt{1-\left(-\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{25-16}{25}}$$

$$\sin x = \sqrt{\frac{9}{25}} \implies \sin x = \pm\frac{3}{5}$$

$$\sin x = -\frac{3}{5} \quad since\ \pi < x < \frac{3\pi}{2}$$

$\pi < y < \frac{3\pi}{2}$, *y is also lies in the III quardrant.* $\cot y$ *and* $\tan y$ *are positive*

$$\sin y = -\frac{24}{25}$$

$$\cos y = \sqrt{1-\sin^2 y}$$

$$\cos y = \sqrt{1-\left(-\frac{24}{25}\right)^2} = \sqrt{1-\frac{576}{625}} = \sqrt{\frac{625-576}{625}}$$

$$\cos y = \sqrt{\frac{49}{625}} \Longrightarrow \cos y = \pm\frac{7}{25} \quad since\ \pi < y < \frac{3\pi}{2}$$

$$\boxed{cos\ y = -\frac{7}{25}}$$

$$\boxed{\therefore \boldsymbol{cos(x-y) = cos\,x\,cos\,y + sin\,x\,sin\,y}}$$

$$= \left(-\frac{4}{5}\right)\left(-\frac{7}{25}\right) + \left(-\frac{3}{5}\right)\left(-\frac{24}{25}\right) = \frac{28}{125} + \frac{72}{125} = \frac{100}{125} = \frac{4}{5}$$

4. $\boldsymbol{Find\ \ sin(x-y) given\ that\ \ sin\,x = \frac{8}{17} with\ 0 < x < \frac{\pi}{2}\ and\ cos\,y = -\frac{24}{25}\ with\ \pi < y < \frac{3\pi}{2}}$.

$0 < x < \frac{\pi}{2}$, *x lies in the I quardrant*

$\therefore$ *All the trigonometric ratios are positive.*

$$sin\,x = \frac{8}{17}$$

$$cosx = \sqrt{1-sin^2x}$$

$$cosx = \sqrt{1-\left(\frac{8}{17}\right)^2} = \sqrt{1-\frac{64}{289}} = \sqrt{\frac{289-64}{289}}$$

$$cosx = \sqrt{\frac{225}{289}} \Longrightarrow cosx = \pm\frac{15}{17} \quad since\ 0 < x < \frac{\pi}{2}$$

$$cos\,x = \frac{15}{17}$$

$\pi < y < \frac{3\pi}{2}$. *y lies in the III quadrant only* $\tan y$ *and* $\cot y$ *are positive*

$$cos\,y = -\frac{24}{25}$$

$$siny = \sqrt{1-cos^2y}$$

$$\sin y = \sqrt{1-\left(-\frac{24}{25}\right)^2} = \sqrt{1-\frac{576}{625}} = \sqrt{\frac{625-576}{625}}$$

$$\sin y = \sqrt{\frac{49}{625}} \implies \sin y = \pm\frac{7}{25}$$

$$\sin y = -\frac{7}{25} \text{ since } \pi < x < \frac{3\pi}{2}$$

$$\therefore \boldsymbol{\sin(x-y) = \sin x \cos y - \cos x \sin y}$$

$$= \left(\frac{8}{17}\right)\left(-\frac{24}{25}\right) - \left(\frac{15}{17}\right)\left(-\frac{7}{25}\right)$$

$$= -\frac{192}{425} + \frac{105}{425} = -\frac{87}{425}$$

5. Find the value of (i) $\cos 105°$ (ii) $\sin 105°$ (iii) $\tan\frac{7\pi}{12}$

$$\cos 105° = \cos(60° + 45°) \qquad \boxed{\cos(A+B) = \cos A\cos B - \sin A\sin B}$$

$$= \cos 60°\cos 45° - \sin 60°\sin 45°$$

$$= \frac{1}{2}\times\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1-\sqrt{3}}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

***ii*) $\sin 105°$**

$$\boxed{\sin(A+B) = \sin A\cos B + \cos A\sin B}$$

$$\sin 105° = \sin(60° + 45°)$$

$$= \sin 60°\cos 45° + \cos 60°\sin 45°$$

$$= \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} + \frac{1}{2}\times\frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(*iii*) $\tan\frac{7\pi}{12}$

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}}$$

$$\tan\frac{7\pi}{12} = \tan\left(\frac{7\cancel{\pi}}{\cancel{12}_{\cancel{2}}}\times\frac{\overset{15°}{\cancel{\overset{30°}{\cancel{180°}}}}}{\cancel{\pi}}\right)$$

$$\tan\frac{7\pi}{12} = \tan(7\times 15°)$$

$$\tan\frac{7\pi}{12} = \tan 105°$$

$$\tan 105° = \tan(60° + 45°)$$

$$= \frac{\tan 60° + \tan 45°}{1 - \tan 60°\tan 45°} = \frac{\sqrt{3}+1}{1-(\sqrt{3})(1)} = \frac{\sqrt{3}+1}{1-\sqrt{3}}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}} = \frac{(\sqrt{3}+1)^2}{1^2 - (\sqrt{3})^2}$$

$$= \frac{(\sqrt{3}+1)^2}{1^2 - (\sqrt{3})^2} = \frac{\left(\sqrt{3}\right)^2 + 2\left(\sqrt{3}\right)(1) + 1^2}{1-3}$$

$$= \frac{3 + 2\left(\sqrt{3}\right)(1) + 1}{-2} = \frac{4+2\sqrt{3}}{-2} = \cancel{2}\left(\frac{2+\sqrt{3}}{\cancel{-2}}\right)$$

$$tan\frac{7\pi}{12} = -\left(2+\sqrt{3}\right)$$

6. $\boldsymbol{Prove\ that\ (i)\ cos(30° + x) = \frac{\sqrt{3}\ cos\,x - sin\,x}{2}\ (ii)\ cos(\pi + \theta) = -\,cos\,\theta}$
$\boldsymbol{(iii)\ sin(\pi + \theta) = -\,sin\,\theta}$

$\boldsymbol{(i)\ cos(30° + x) = \frac{\sqrt{3}\ cos\,x - sin\,x}{2}}$ $\boxed{\cos(A+B) = \cos A\cos B - \sin A \sin B}$

$$cos(30° + x) = cos\,30° cos\,x - sin\,30°\,sin\,x$$

$$= \frac{\sqrt{3}}{2} \times \cos x - \frac{1}{2}\sin x = \frac{\sqrt{3}\cos x}{2} - \frac{sin\,x}{2}$$

$$= \frac{\sqrt{3}\cos x - \sin x}{2}$$

$\boldsymbol{(ii)\ cos(\pi + \theta) = -\,cos\,\theta}$

$$L.H.S = \cos(\pi+\theta) = cos\,\pi\,cos\,\theta - sin\,\pi\,sin\,\theta$$
$$= -1 \times \cos\theta - 0 \times sin\,\theta$$
$$= -\cos\theta$$

$\boldsymbol{(iii)\ sin(\pi + \theta) = -\,sin\,\theta}$ $\boxed{\sin(A+B) = \sin A\cos B + \cos A\sin B}$

$$sin(\pi + \theta) = sin\,\pi\,cos\,\theta + cos\,\pi\ sin\,\theta$$
$$= (0)\,cos\,\theta + (-1)\,sin\,\theta$$
$$= 0 - sin\,\theta$$
$$sin(\pi + \theta) = -\,sin\,\theta$$

$[\because \sin 180° = 0]$

$[\because \cos 180° = -1]$

7. $\boldsymbol{Find\ a\ quadratic\ equation\ whose\ roots\ are\ sin\,15°\ and\ cos\,15°}$

$$sin\,15° = sin(45° - 30°)$$
$$= sin\,45°\,cos\,30° - cos\,45°\,sin\,30°$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$\boxed{cos(A-B) = cos\,A\,cos\,B + sin\,A\,sin\,B}$

$\boxed{sin(A-B) = sin\,A\,cos\,B - cos\,A\,sin\,B}$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\sin 15° = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \ or \ \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\cos 15° = \cos(45° - 30°)$$

$$= \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos 15° = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \ or \ \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$Sum\ of\ the\ roots = \sin 15° + \cos 15°$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} + \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{3} - \cancel{1} + \sqrt{3} + \cancel{1}}{2\sqrt{2}} = \frac{\sqrt{3}+\sqrt{3}}{2\sqrt{2}} = \frac{2\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}$$

$$Sum\ of\ the\ roots = \frac{\sqrt{6}}{2}$$

$$Product\ of\ the\ roots = \sin 15° \cos 15°$$

$$= \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$

$$= \frac{(\sqrt{3})^2 - 1^2}{8} = \frac{3-1}{8} = \frac{2}{8} = \frac{1}{4}$$

$$Product\ of\ the\ roots = \frac{1}{4}$$

$\therefore$ *Required equation is* $x^2 - (sum\ of\ the\ roots)x + product\ of\ the\ roots = 0$

$$x^2 - \frac{\sqrt{6}}{2}x + \frac{1}{4} = 0$$

multiplying both side by 4

$$4x^2 - 2\sqrt{6}x + 1 = 0$$

8. Expand $\cos(A+B+C)$. Hence prove that $\cos A \cos B \cos C = \sin A \sin B \cos C + \sin B \sin C \cos A + \sin C \sin A \cos B$, if $A+B+C = \frac{\pi}{2}$.

$$\cos(\underbrace{A+B}_{A} + \underset{B}{C}) = \cos(A+B)\cos C - \sin(A+B)\sin C$$

$$= [\cos A \cos B - \sin A \sin B]\cos C - [\sin A \cos B + \cos A \sin B]\sin C$$

$$\cos(A+B+C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C$$
$$- \cos A \sin B \sin C$$

$$A+B+C=\frac{\pi}{2}$$ $\boxed{cos(A+B)=cos\,A\,cos\,B-sin\,A\,sin\,B}$

$$cos(A+B+C)=cos\frac{\pi}{2}$$ $\boxed{sin(A+B)=sin\,A\,cos\,B+cos\,A\,sin\,B}$

$$cos\,A\,cos\,B\,cos\,C-sin\,A\,sin\,B\,cos\,C-sin\,A\,cos\,B\,sin\,C-cos\,A\,sin\,B\,sin\,C=0$$

$$\therefore cos\,A\,cos\,B\,cos\,C=sin\,A\,sin\,B\,cos\,C+sin\,A\,cos\,B\,sin\,C+cos\,A\,sin\,B\,sin\,C$$

9. Prove that (i) $\boldsymbol{sin(45^\circ+\theta)-sin(45^\circ-\theta)=\sqrt{2}\,sin\,\theta}$

(ii) $\boldsymbol{sin(30^\circ+\theta)+cos(60^\circ+\theta)=cos\,\theta}$

(i) $\boldsymbol{sin(45^\circ+\theta)-sin(45^\circ-\theta)=\sqrt{2}\,sin\,\theta}$

$L.H.S=sin(45^\circ+\theta)-sin(45^\circ-\theta)$ $\boxed{sin(A+B)=sin\,A\,cos\,B+cos\,A\,sin\,B}$

$$=sin\,45^\circ\,cos\,\theta+cos\,45^\circ\,sin\,\theta-(sin\,45^\circ\,cos\,\theta-cos\,45^\circ\,sin\,\theta)$$

$$=\frac{1}{\sqrt{2}}cos\,\theta+\frac{1}{\sqrt{2}}sin\,\theta-\left(\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta\right)$$

$$=\cancel{\frac{1}{\sqrt{2}}cos\,\theta}+\frac{1}{\sqrt{2}}sin\,\theta-\cancel{\frac{1}{\sqrt{2}}cos\,\theta}+\frac{1}{\sqrt{2}}sin\,\theta$$

$$=\frac{1}{\sqrt{2}}sin\,\theta+\frac{1}{\sqrt{2}}sin\,\theta=\frac{1}{\sqrt{2}}(sin\,\theta+sin\,\theta)=\frac{1}{\sqrt{2}}\times 2\,sin\,\theta$$

$$=\frac{1}{\sqrt{2}}\times 2\,sin\,\theta=\frac{1}{\cancel{\sqrt{2}}}\times\cancel{\sqrt{2}}\times\sqrt{2}\,sin\,\theta=\sqrt{2}\,sin\theta$$

(ii) $\boldsymbol{sin(30^\circ+\theta)+cos(60^\circ+\theta)=cos\,\theta}$

$L.H.S=sin(30^\circ+\theta)+cos(60^\circ+\theta)$ $\boxed{cos(A+B)=cos\,A\,cos\,B-sin\,A\,sin\,B}$

$$=sin\,30^\circ\,cos\,\theta+cos\,30^\circ\,sin\,\theta+cos\,60^\circ\,cos\,\theta-sin\,60^\circ\,sin\,\theta$$

$$=\frac{1}{2}cos\,\theta+\cancel{\frac{\sqrt{3}}{2}sin\,\theta}+\frac{1}{2}cos\,\theta-\cancel{\frac{\sqrt{3}}{2}sin\,\theta}$$

$$=\frac{1}{2}cos\,\theta+\frac{1}{2}cos\,\theta=\frac{1}{2}(cos\,\theta+cos\,\theta)=\frac{1}{2}\times 2\,cos\,\theta$$

$$=cos\,\theta$$

10. If $\boldsymbol{a\,cos(x+y)=b\,cos(x-y)}$, show that $\boldsymbol{(a+b)\,tan\,x=(a-b)\,cot\,y}$.

$$a\,cos(x+y)=b\,cos(x-y)$$

$$a[cos\,x\,cos\,y-sin\,x\,sin\,y]=b[cos\,x\,cos\,y+sin\,x\,sin\,y]$$

$$a\,cos\,x\,cos\,y-a\,sin\,x\,sin\,y=b\,cos\,x\,cos\,y+b\,sin\,x\,sin\,y$$

$$a\,cos\,x\,cos\,y-b\,cos\,x\,cos\,y=a\,sin\,x\,sin\,y+b\,sin\,x\,sin\,y$$

$$(a-b)\,cos\,x\,cos\,y=(a+b)\,sin\,x\,sin\,y$$

$$(a-b)\frac{cos\,y}{sin\,y}=(a+b)\frac{sin\,x}{cos\,x}\implies(a-b)\,cot\,y=(a+b)\,tan\,x$$

$$(a+b)\,tan\,x=(a-b)\,cot\,y$$

11. Prove that $\sin 105° + \cos 105° = \cos 45°$

$$L.H.S = \sin 105° + \cos 105°$$
$$= \sin(60° + 45°) + \cos(60° + 45°)$$
$$= \sin 60° \cos 45° + \cos 60° \sin 45° + \cos 60° \cos 45° - \sin 60° \sin 45°$$
$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$
$$= \cancel{\frac{\sqrt{3}}{2\sqrt{2}}} + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} - \cancel{\frac{\sqrt{3}}{2\sqrt{2}}} = \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{2}{2\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}} = \cos 45° = R.H.S$$

12. Prove that $\sin 75° - \sin 15° = \cos 105° + \cos 15°$.

$$\boxed{\sin(90° - \theta) = \cos\theta}$$
$$\boxed{\cos(90° + \theta) = -\sin\theta}$$

$$L.H.S = \sin 75° - \sin 15°$$
$$= \sin(90° - 15°) - \sin 15°$$
$$= \cos 15° - \sin 15°$$

$$R.H.S = \cos 105° + \cos 15°$$
$$= \cos(90 + 15°) + \cos 15°$$
$$= -\sin 15° + \cos 15° = \cos 15° - \sin 15°$$

$\therefore L.H.S = R.H.S$

13. Show that $\tan 75° + \cot 75° = 4$

$$\boxed{\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

$$\tan 75° = \tan(45° + 30°)$$
$$\tan 75° = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$$
$$\tan 75° = \frac{1 + \frac{1}{\sqrt{3}}}{1 - (1)\left(\frac{1}{\sqrt{3}}\right)} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$$
$$\tan 75° = \frac{\frac{\sqrt{3} + 1}{\cancel{\sqrt{3}}}}{\frac{\sqrt{3} - 1}{\cancel{\sqrt{3}}}} \Longrightarrow \tan 75° = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$
$$\cot 75° = \frac{1}{\tan 75°} \Longrightarrow \cot 75° = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$L.H.S = \tan 75° + cot75°$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}{(\sqrt{3}-1)\ (\sqrt{3}+1)}$$

$$= \frac{(\sqrt{3})^2 + (1)^2 + 2\sqrt{3} + (\sqrt{3})^2 - 2\sqrt{3} + 1^2}{(\sqrt{3})^2 - 1^2}$$

$$= \frac{3+1+\cancel{2\sqrt{3}} + 3 - \cancel{2\sqrt{3}} + 1}{3 - 1} = \frac{8}{2} = 4$$

14. Prove that $\cos(A+B)\cos C - \cos(B+C)\cos A = \sin B \sin(C-A)$

$$L.H.S = \cos(A+B)\cos C - \cos(B+C)\cos A$$

$$= (\cos A\cos B - \sin A \sin B)\cos C - \cos A(\cos B\cos C - \sin B\sin C)$$

$$= \cancel{\cos A\cos B\cos C} - \sin A\sin B\cos C - \cancel{\cos A\cos B\cos C} + \cos A\sin B\sin C$$

$$= \cos A\sin B\sin C - \sin A\sin B\cos C$$

$$= \sin B(\cos A\sin C - \sin A\cos C) = \sin B(\sin C\cos A - \cos C\sin A)$$

$$= \sin B\sin(C-A) = R.H.S$$

15. Prove that $\sin(n+1)\theta\sin(n-1)\theta + \cos(n+1)\theta\cos(n-1)\theta.$
$= \cos 2\theta, n \in \mathbb{Z}$

$$L.H.S = \sin\underbrace{(n+1)\theta}_{A}\sin\underbrace{(n-1)\theta}_{B} + \cos\underbrace{(n+1)\theta}_{A}\cos\underbrace{(n-1)\theta}_{B}.$$

$$= \cos[(n+1)\theta - (n-1)\theta] \qquad \boxed{\cos(A-B) = \cos A\cos B + \sin A\sin B}$$

$$= \cos[n\theta + \theta - n\theta + \theta]$$

$$= \cos 2\theta = R.H.S, n \in \mathbb{Z}$$

16. If $x\cos\theta = y\cos\left(\theta + \frac{2\pi}{3}\right) = z\cos\left(\theta + \frac{4\pi}{3}\right)$, find the value of $xy + yz + zx$

$$\boxed{\begin{aligned} cos120° &= cos(180° - 60°) \\ &= -cos60° = \frac{-1}{2}\end{aligned}}$$

$$x\cos\theta = y\cos\left(\theta + \frac{2\pi}{3}\right) = z\cos\left(\theta + \frac{4\pi}{3}\right) = k$$

$$\boxed{\begin{aligned} sin120° &= \sin(180° - 60°) \\ &= sin60° = \frac{\sqrt{3}}{2}\end{aligned}}$$

$$x\cos\theta = k, y\cos\left(\theta + \frac{2\pi}{3}\right) = k,\ z\cos\left(\theta + \frac{4\pi}{3}\right) = k$$

$$\cos\theta = \frac{k}{x},\quad \cos\left(\theta + \frac{2\pi}{3}\right) = \frac{k}{y},\quad \cos\left(\theta + \frac{4\pi}{3}\right) = \frac{k}{Z}$$

$$\frac{k}{x} + \frac{k}{y} + \frac{k}{Z} = \cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right)$$

$$= \cos\theta + \cos(120° + \theta) + \cos(240° + \theta)$$

$= cos\theta + cos120° cos\theta - sin\,120°\, sin\theta + cos240\, cos\theta - sin240°\, sin\theta$

$= cos\theta + \left(\frac{-1}{2}\right)cos\theta - \left(\frac{\sqrt{3}}{2}\right) sin\theta + \left(\frac{-1}{2}\right)cos\theta - \left(\frac{-\sqrt{3}}{2}\right) sin\theta$

$= cos\theta - \frac{1}{2}\, cos\theta - \frac{\sqrt{3}}{2}\, sin\theta - \frac{1}{2}\, cos\theta + \frac{\sqrt{3}}{2}\, sin\theta$

$= cos\theta \;-\; cos\theta - \frac{\sqrt{3}}{2} sin\theta + \frac{\sqrt{3}}{2}\, sin\theta \;= 0$

$\boxed{cos240° = cos(270° - 30°) = -sin30° = -\frac{1}{2}}$

$\frac{k}{x} + \frac{k}{y} + \frac{k}{Z} = 0 \Rightarrow k\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{Z}\right) = 0$

$\frac{1}{x} + \frac{1}{y} + \frac{1}{Z} = 0 \;\Rightarrow\; \frac{yz + xz + xy}{xyz} = 0$

$\boxed{sin240° = \sin(270° - 30°) = -cos30° = -\frac{\sqrt{3}}{2}}$

$yz + xz + xy = 0 \Rightarrow xy + yz + zx = 0$

17. *Prove that* $(i)\ \boldsymbol{sin(A+B)\, sin(A-B) = sin^2 A - sin^2 B}$

$(ii)\ \boldsymbol{cos(A+B)\, cos(A-B) = cos^2 A - sin^2 B = cos^2 B - sin^2 A}$

$(iii)\ \boldsymbol{sin^2(A+B) - sin^2(A-B) = sin\, 2A\, sin\, 2B}$

$(iv)\ \boldsymbol{cos\, 8\,\theta\, cos\, 2\theta = cos^2 5\theta sin^2 30}$

$i)$ *Prove that* $\boldsymbol{sin(A+B)\, sin(A-B) = sin^2 A - sin^2 B}$

$L.H.S = \sin(A+B)\sin(A-B)$ $\quad\boxed{\sin(A+B) = \sin A\cos B + \cos A \sin B}$

$= (\sin A\cos B + \cos A\sin B)\,(\sin A\cos B - \cos A\sin B)$

$= (\sin A\cos B)^2 - (\cos A\sin B)^2$ $\quad\boxed{\cos^2 B = 1 - \sin^2 B}$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$ $\quad\boxed{\cos^2 A = 1 - \sin^2 A}$

$= \sin^2 A(1 - \sin^2 B) - \sin^2 B\,(1 - sin^2 A)$

$= \sin^2 A - \cancel{\sin^2 A\sin^2 B} - \sin^2 B + \cancel{\sin^2 A\sin^2 B}$

$= \sin^2 A - \sin^2 B$ $\quad\boxed{\sin(A-B) = \sin A\cos B - \cos A\sin B}$

$ii)$ *Prove that* $\boldsymbol{cos(A+B)\, cos(A-B) = cos^2 A - sin^2 B}$

$L.H.S = \cos(A+B)\cos(A-B)$ $\quad\boxed{\cos(A+B) = \cos A\cos B - \sin A\sin B}$

$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$= (\cos A\cos B)^2 - (\sin A\sin B)^2$ $\quad\boxed{\cos^2 B = 1 - \sin^2 B}$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$ $\quad\boxed{\sin^2 A = 1 - \cos^2 A}$

$= (1 - \sin^2 B)\cos^2 A - \sin^2 B(1 - \cos^2 A)$

$= \cos^2 A - \cancel{\cos^2 A\sin^2 B} - \sin^2 B + \cancel{\cos^2 A\sin^2 B}$

$= \cos^2 A - \sin^2 B$ $\quad\boxed{\cos(A-B) = \cos A\cos B + \sin A\sin B}$

iii) $\boldsymbol{\cos(A+B)\cos(A-B) = \cos^2 B - \sin^2 A}$

$L.H.S = \cos(A+B)\cos(A-B)$ $\boxed{\cos(A+B) = \cos A\cos B - \sin A\sin B}$

$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$= (\cos A\cos B)^2 - (\sin A\sin B)^2$ $\boxed{\sin^2 B = 1-\cos^2 B}$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$ $\boxed{\cos^2 A = 1-\sin^2 A}$

$= (1-\sin^2 A)\cos^2 B - \sin^2 A(1-\cos^2 B)$

$= \cos^2 B - \cancel{\sin^2 A\cos^2 B} - \sin^2 A + \cancel{\sin^2 A\cos^2 B}$

$= \cos^2 B - \sin^2 A$ $\boxed{\cos(A-B) = \cos A\cos B + \sin A\sin B}$

(***iv***) $\boldsymbol{\sin^2(A+B) - \sin^2(A-B) = \sin 2A\sin 2B}$

$L.H.S = \sin^2(A+B) - \sin^2(A-B)$

$= \sin(A+B+A-B)\ \sin[A+B-(A-\boldsymbol{B})]$

$= \sin(A+\cancel{B}+A-\cancel{B})\ \sin(\cancel{A}+B-\cancel{A}+B)$

$= \sin 2A\sin 2B = R.H.S$ $\boxed{\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B}$

(***iv***) $\boldsymbol{\cos 8\theta\cos 2\theta = \cos^2 5\theta - \sin^2 3\theta}$

$R.H.S = \cos^2 5\theta - \sin^2 3\theta$ $\boxed{\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B}$

$= \cos(5\theta+3\theta)\cos(5\theta-3\theta)$

$= \cos 8\theta\cos 2\theta = L.H.S$

18. ***Show that*** $\boldsymbol{\cos^2 A + \cos^2 B - 2\cos A\cos B\cos(A+B) = \sin^2(A+B)}$

$L.H.S = \cos^2 A + \cos^2 B - 2\cos A\cos B\cos(A+B)$

$= \cos^2 A + 1 - \sin^2 B - 2\cos A\cos B\cos(A+B)$

$= \cos^2 A - \sin^2 B + 1 - 2\cos A\cos B\cos(A+B)$

$\boxed{\because \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)}$

$= \cos(A+B)\cos(A-B) - 2\cos A\cos B\cos(A+B) + 1$

$= \cos(A+B)[\cos(A-B) - 2\cos A\cos B] + 1$

$= \cos(A+B)[\cos A\cos B + \sin A\sin B - 2\cos A\cos B] + 1$

$= \cos(A+B)[\sin A\sin B - \cos A\cos B] + 1$

$= -\cos(A+B)[\cos A\cos B - \sin A\sin B] + 1$

$= -\cos(A+B)\cos(A+B) + 1 = -\cos^2(A+B) + 1$

$= 1 - \cos^2(A+B) = \sin^2(A+B) = R.H.S$

19. ***If*** $\boldsymbol{\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = -\frac{3}{2}}$ ***then***

prove that $\boldsymbol{\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0}$

$\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = -\frac{3}{2}$

$$\cos\alpha\cos\beta+\sin\alpha\sin\beta+\cos\beta\cos\gamma+\sin\beta\sin\gamma+\cos\gamma\cos\alpha+\sin\alpha\sin\gamma=-\frac{3}{2}$$

$$2\left[\cos\alpha\cos\beta+\cos\beta\cos\gamma+\cos\gamma\cos\alpha+\sin\alpha\sin\beta+\sin\beta\sin\gamma+\sin\alpha\sin\gamma\right]=-3$$

$$2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha=-3$$

$$3+2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha=0$$

$$1+1+1+2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha=0$$

$$\cos^2\alpha+\sin^2\alpha+\cos^2\beta+\sin^2\beta+\cos^2\gamma+\sin^2\gamma+2\cos\alpha\cos\beta+2\cos\beta\cos\gamma\;2\cos\gamma\cos\alpha+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha=0$$

$$(\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha)+(\sin^2\alpha+\sin^2\beta+\sin^2\gamma+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha)=0$$

$$(\cos\alpha+\cos\beta+\cos\gamma)^2+(\sin\alpha+\sin\beta+\sin\gamma)^2=0$$

$$(\cos\alpha+\cos\beta+\cos\gamma)^2=0,\quad(\sin\alpha+\sin\beta+\sin\gamma)^2=0$$

$$\cos\alpha+\cos\beta+\cos\gamma=0,\quad\sin\alpha+\sin\beta+\sin\gamma=0$$

20. Show that $(i)\ \tan(45^\circ+A)=\dfrac{1+\tan A}{1-\tan A}$ $(ii)\ \tan(45^\circ-A)=\dfrac{1-\tan A}{1+\tan A}$

$(i)\ \tan(45^\circ+A)=\dfrac{1+\tan A}{1-\tan A}$ $\boxed{\because \tan 45^\circ=1}$

$$L.H.S=\tan(45^\circ+A)$$

$$=\frac{\tan 45^\circ+\tan A}{1-\tan 45^\circ\tan A}=\frac{1+\tan A}{1-\tan A}=R.H.S$$

$(ii)\ \tan(45^\circ-A)=\dfrac{1-\tan A}{1+\tan A}$

$$L.H.S=\tan(45^\circ-A)$$

$$=\frac{\tan 45^\circ-\tan A}{1+\tan 45^\circ\tan A}=\frac{1-\tan A}{1+\tan A}=R.H.S$$

21. Prove that $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A+\cot B}$

$$L.H.S=\cot(A+B)$$

$$=\frac{1}{\tan(A+B)}=\frac{1}{\dfrac{\tan A+\tan B}{1-\tan A\tan B}}=\frac{1-\tan A\tan B}{\tan A+\tan B}$$

$$= \frac{1-\frac{1}{\cot A}\times\frac{1}{\cot B}}{\frac{1}{\cot A}+\frac{1}{\cot B}} = \frac{1-\frac{1}{cotAcotB}}{\frac{cotB+cotA}{cotA\,cotB}} = \frac{\frac{\cot A\cot B-1}{\cancel{cotA\,cotB}}}{\frac{cotB+cotA}{\cancel{cotA\,cotB}}}$$

$$= \frac{\cot A\cot B-1}{\cot A+\cot B} = R.H.S$$

22. $\boldsymbol{If\ tan\,x = \frac{n}{n+1}\ and\ tan\,y = \frac{1}{2n+1}, find\ tan\ (x+y)}$

$$tan(x+y) = \frac{tanx+tany}{1-tanx\,tany}$$

$$= \frac{\frac{n}{n+1}+\frac{1}{2n+1}}{1-\frac{n}{n+1}\frac{1}{2n+1}} = \frac{\frac{n}{n+1}+\frac{1}{2n+1}}{1-\frac{n}{(n+1)(2n+1)}}$$

$$= \frac{\frac{n(2n+1)+n+1}{\cancel{(n+1)(2n+1)}}}{\frac{(n+1)(2n+1)-n}{\cancel{(n+1)(2n+1)}}} = \frac{2n^2+n+n+1}{2n^2+n+2n+1-n} = \frac{2n^2+2n+1}{2n^2+2n+1} = 1$$

23. $\boldsymbol{Prove\ that\ tan\left[\frac{\pi}{4}+\theta\right]tan\left[\frac{3\pi}{4}+\theta\right] = -1}$

$$L.H.S = tan\left[\frac{\pi}{4}+\theta\right]tan\left[\frac{3\pi}{4}+\theta\right]$$

$$= \left(\frac{\overset{1}{tan\frac{\pi}{4}}+\tan\theta}{1-\underset{1}{tan\frac{\pi}{4}}tan\theta}\right)\left(\frac{\overset{(-1)}{tan\frac{3\pi}{4}}+\tan\theta}{1-\underset{(-1)}{tan\frac{3\pi}{4}}tan\theta}\right)$$

$$\frac{3\pi}{4} = \frac{3\times180^\circ}{4} = 3\times45^\circ$$

$$\frac{3\pi}{4} = 135^\circ$$

$$tan135^\circ = tan(180^\circ-45^\circ)$$

$$tan135^\circ = -tan45^\circ$$

$$tan135^\circ = -1$$

$$= \left(\frac{1+\tan\theta}{1-\tan\theta}\right)\left(\frac{-1+\tan\theta}{1+\tan\theta}\right) = \frac{1\cancel{+tan\theta}}{1\cancel{-tan\theta}}\times\frac{-(1\cancel{-tan\theta})}{1\cancel{+tan\theta}}$$

$$= -1$$

24. $\boldsymbol{Find\ the\ values\ of\ tan(\alpha+\beta)\ given\ that\ cot\alpha = \frac{1}{2}\in\left[\pi,\frac{3\pi}{2}\right]\ and}$

$\boldsymbol{sec\,\beta = -\frac{5}{3}, \beta\in\left[\frac{\pi}{2},\pi\right]}$

$$cot\alpha = \frac{1}{2}, \alpha\ \in\left[\pi,\frac{3\pi}{2}\right]$$

$$tan\,\alpha = 2$$

$$also\ sec\beta = -\frac{5}{3}\ , \beta \in \left[\frac{\pi}{2}, \pi\right]$$

$$1 + tan^2\theta = sec^2\theta$$
$$tan^2\theta = sec^2\theta - 1$$
$$tan\theta = \sqrt{sec^2\theta - 1}$$

$$\tan\beta = \sqrt{sec^2\beta - 1} = \sqrt{\left(-\frac{5}{3}\right)^2 - 1}$$

$$= \sqrt{\frac{25}{9} - 1} = \sqrt{\frac{16}{9}} = \pm\frac{4}{3}$$

$$tan\beta = -\frac{4}{3}\ since\ \beta \in \left[\frac{\pi}{2}, \pi\right]$$

$$Hence\ \ tan(\alpha+\beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha\ tan\beta}$$

$$= \frac{2 - \frac{4}{3}}{1 - 2 \times -\frac{4}{3}} = \frac{\frac{6-4}{3}}{1 + \frac{8}{3}} = \frac{\frac{2}{3}}{\frac{3+8}{3}}$$

$$tan(\alpha+\beta) = \frac{2}{11}$$

25. $If\ \theta + \phi = \alpha\ and\ tan\theta = k\tan\phi, then\ prove\ that$
$$\boldsymbol{sin(\theta - \phi) = \frac{k-1}{k+1} sin\,\alpha}$$

$$\theta + \phi = \alpha \implies tan\theta = k\tan\phi,$$

$$k = \frac{tan\theta}{\tan\phi}$$

$$\frac{k-1}{k+1} = \frac{\frac{tan\theta}{\tan\phi} - 1}{\frac{tan\theta}{\tan\phi} + 1} = \frac{\frac{tan\theta - \tan\phi}{\cancel{\tan\phi}}}{\frac{tan\theta + \tan\phi}{\cancel{\tan\phi}}} \implies \frac{k-1}{k+1} = \frac{tan\theta - \tan\phi}{tan\theta + \tan\phi}$$

$$\frac{k-1}{k+1} = \frac{\frac{sin\theta}{cos\theta} - \frac{sin\phi}{cos\phi}}{\frac{sin\theta}{cos\theta} + \frac{sin\phi}{cos\phi}} = \frac{\frac{sin\theta cos\emptyset - sin\emptyset cos\theta}{\cancel{cos\theta\ cos\emptyset}}}{\frac{sin\theta cos\emptyset + sin\emptyset cos\theta}{\cancel{cos\theta\ cos\emptyset}}}$$

$$= \frac{sin\theta\ cos\phi - cos\theta\ sin\phi}{sin\theta cos\phi\ + cos\theta\ sin\phi}$$

$$\frac{k-1}{k+1} = \frac{sin(\theta - \phi)}{sin(\theta + \phi)} \implies \frac{k-1}{k+1} = \frac{sin(\theta - \phi)}{sin\,\alpha}$$

$$sin(\theta - \phi) = \frac{k-1}{k+1} sin\,\alpha$$

$EXERCISE : 3.5$

Example 3.21 ***A football player can kick a football from ground level with an initial velocity of*** $\frac{80ft}{secound}$***. Find the maximum horizontal distance the football travels and at what angle? (Take*** $g = 32$***)***

The formula for horizontal distance R is given by

$$R = \frac{u^2 sin2\alpha}{g}$$

Given : The maximum distance is $R = 200ft$

$$R = \frac{(\overset{20}{\cancel{80}} \times \overset{10}{\cancel{80}})\ sin2\alpha}{\cancel{32}_{\cancel{4}}} \implies 200 = 10 \times 20\ sin2\alpha$$

$$\cancel{200} = \cancel{200}\ sin2\alpha \implies sin2\alpha = 1$$

$$2\alpha = 90° \implies \alpha = 45°$$

Identity 3.10

$$sin2A = \frac{2tanA}{1 + tan^2A}$$

$$sin2A = 2\ sinA\ cosA = \frac{2\ sinA\ cosA}{1} = \frac{\frac{2\ sinA\ cosA}{cos^2A}}{\frac{1}{cos^2A}}$$

$$= \frac{2\ tanA}{sec^2A} = \frac{2\ tanA}{1 + tan^2A}$$

Identity 3.11

$$cos2A = \frac{1 - tan^2A}{1 + tan^2A}$$

$$cos2A = cos^2A - sin^2A = \frac{cos^2A - sin^2A}{1}$$

$$= \frac{\frac{cos^2A - sin^2A}{cos^2A}}{\frac{1}{cos^2A}} = \frac{1 - tan^2A}{1 + tan^2A}$$

Identity 3.12

$$sin3A = 3\ sinA - 4\ sin^3A$$

$$sin3A = \sin(2A + A)$$

$= cos2A\ sinA$

$= 2\ sinA\ cos^2A + (1 - 2\ sin^2A)\ sinA$

$= 2\ sinA(1 - sin^2A) + (1 - 2\ sin^2A)\ sinA$

$= 3\ sinA - 4\ sin^3A$

***Identity* 3.13**

$sin3A = 4\ cos^3A - 3\ cosA.$

$cos3A = \sin(2A + A) = cos2A\ cosA - sin2A\ sinA$

$= (2\ cos^2A - 1)\ cosA - 2\ sinA\ cosA\ sinA$

$= (2\ cos^2A - 1)\ cosA - 2\ cosA\,(1 - cos^2A)$

$= 4\ cos^3A - 3\ cosA$

***Identity* 3.14**

$$tan3A = \frac{3\ tanA - tan^3A}{1 - 3tan^2A}$$

$tan3A = tan(2A + A)$

$$= \frac{tan2A + 3\ tanA}{1 - tan2A\ tanA} = \frac{\frac{2tanA}{1 - tan^2A} + tanA}{1 - \frac{2tanA}{1 - tan^2A}\ tanA} = \frac{3\ tanA - tan^3A}{1 - 3tan^2A}$$

Half – Angle Identities

If we put $2A = \theta$ *or* $A = \frac{\theta}{2}$ *in the double angle indentites*

Let us list out the half angle identites in the following table :

Double angle identity	*Half angle identity*
$sin2A = 2\ sinA\ cosA$	$sin\theta = 2sin\frac{\theta}{2}cos\frac{\theta}{2}$
$cos2A = cos^2A - sin^2A$	$cos\theta = cos^2\frac{\theta}{2} - sin^2\frac{\theta}{2}$
$cos2A = 2cos^2A - 1$	$cos\theta = 2cos^2\frac{\theta}{2} - 1$
$cos2A = 1 - 2\ sin^2A$	$cos\theta = 1 - 2\ sin^2\frac{\theta}{2}$

$tan2A = \frac{2tanA}{1-tan^2A}$	$tanA = \frac{2tan\frac{A}{2}}{1-tan^2\frac{A}{2}}$
$sin2A = \frac{2tanA}{1+tan^2A}$	$sinA = \frac{2tan\frac{A}{2}}{1+tan^2\frac{A}{2}}$
$cos2A = \frac{1-tan^2A}{1+tan^2A}$	$cosA = \frac{1-tan^2\frac{A}{2}}{1+tan^2\frac{A}{2}}$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$sub\ \ B = A$$

$$\sin(A+A) = \sin A\cos A + \cos A\sin A$$

$$\sin(A+A) = \sin A\cos A + \sin A\cos A$$

$$\sin 2A = 2\sin A\cos A \implies \frac{1}{2}\sin 2A = \sin A\cos A$$

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$cos(A+B) = cos\,A\,cos\,B - sin\,A\,sin\,B$$

$$sub\ \ B = A$$

$$\cos(A+A) = \cos A\cos A - \sin A\sin A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$sub\ \ \cos^2 A = 1 - \sin^2 A$$

$$\cos 2A = 1 - \sin^2 A - \sin^2 A$$

$$\cos 2A = 1 - 2\sin^2 A$$

$$\cos A = 1 - 2\sin^2\frac{A}{2} \implies 2\sin^2\frac{A}{2} = 1 - \text{cosA}$$

$$2\sin^2 A = 1 - \cos 2A \implies \sin^2 A = \frac{1-\cos 2A}{2}$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$sub\ \ \sin^2 A = 1 - \cos^2 A$$

$$\cos 2A = \cos^2 A - (1 - \cos^2 A)$$

$$\cos 2A = \cos^2 A - 1 + \cos^2 A$$

$$\cos 2A = 2\cos^2 A - 1$$

$$\cos A = 2\cos^2\frac{A}{2} - 1 \implies 1 + \cos A = 2\cos^2\frac{A}{2}$$

$$\cos 2A = 2\cos^2 A - 1$$

$$1 + \cos 2A = 2\cos^2 A \implies \cos^2 A = \frac{1+\cos 2A}{2}$$

$$\cos 3A = 4cos^3 A - 3cosA$$
$$\cos 3A + 3\cos A = 4cos^3 A$$
$$cos^3 A = \frac{1}{4}(cos3A + 3cosA)$$

$$\sin 3A = 3sinA - 4sin^3 A$$
$$4sin^3 A = 3\sin A - \sin 3A$$
$$sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$sub\ \ B = A$$
$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$
$$\tan 2A = \frac{2\tan A}{1 - tan^2 A}$$

$$\tan(45° + A) = \frac{\tan 45° + \tan A}{1 - \tan 45° \tan A}$$
(1)
$$\tan(45° + A) = \frac{1 + \tan A}{1 - \tan A}$$

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}}$$
$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

22. *Find the value of sin* $\left[22\frac{1}{2}^{\circ}\right]$

$$cos\theta = 1 - 2sin^2\frac{\theta}{2} \implies 2sin^2\frac{\theta}{2} = 1 - cos\theta$$

$$sin^2\frac{\theta}{2} = \frac{1 - cos\theta}{2} \implies sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - cos\theta}{2}}$$

$Take\ \theta = 45°,$

$$sin\frac{45°}{2} = \pm\sqrt{\frac{1 - cos45°}{2}} \quad (since\ 22\tfrac{1}{2}°\ lies\ in\ the\ first\ quadrant)$$

$$sin\ 22\tfrac{1}{2}^{\circ} = \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{2}}{2}}{\frac{2}{1}}}$$

$$sin\ 22\tfrac{1}{2}^{\circ} = \sqrt{\frac{2-\sqrt{2}}{2}\times\frac{1}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} \implies sin\ 22\tfrac{1}{2}^{\circ} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

23. Find the value of $\sin 2\theta$, When $\sin\theta = \frac{12}{13}$, θ lies in the first quadrant.

$$Given: \sin\theta = \frac{12}{13}$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{169-144}{169}}$$

$$\cos\theta = \sqrt{\frac{25}{169}} = \pm\frac{5}{13}$$

$$\cos\theta = \frac{5}{13} \quad since\ \theta\ lies\ in\ the\ first\ quadrant.$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right) = \frac{120}{169}$$

24. Prove that $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$.

$$4\sin A\cos^3 A - 4\cos A\sin^3 A = 4\sin A\cos A\,(\cos^2 A - \sin^2 A)$$

$$= 2(2\sin A\cos A)\cos 2A$$

$$= 2\sin 2A\cos 2A = \sin 2(2A) = \sin 4A$$

25. Prove that $\sin x = 2^{10}\sin\left(\frac{x}{2^{10}}\right)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right)\ldots\ldots\cos\left(\frac{x}{2^{10}}\right)$

$$\boxed{\sin 2A = 2\sin A\cos A}$$

$$\boxed{\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\left(2\sin\frac{x}{4}\cos\frac{x}{4}\right)\cos\frac{x}{2}$$

$$= 2\left(2\sin\frac{x}{2^2}\cos\frac{x}{2^2}\right)\cos\frac{x}{2}$$

$$= 2^2\sin\frac{x}{2^2}\cos\frac{x}{2^2}\cos\frac{x}{2} = 2^2\left(2\sin\frac{x}{2^3}\cos\frac{x}{2^3}\right)\cos\frac{x}{2^2}\cos\frac{x}{2}$$

$$= 2^3\sin\frac{x}{2^3}\cos\frac{x}{2^3}\cos\frac{x}{2^2}\cos\frac{x}{2}$$

$$= 2^3\left(2\sin\frac{x}{2^4}\cos\frac{x}{2^4}\right)\cos\frac{x}{2^3}\cos\frac{x}{2^2}\cos\frac{x}{2}$$

$$= 2^4\sin\frac{x}{2^4}\cos\frac{x}{2^4}\cos\frac{x}{2^3}\cos\frac{x}{2^2}\cos\frac{x}{2}$$

Applying repeatedly the half angle sine formula, we get

$$\sin x = 2^{10}\sin\left(\frac{x}{2^{10}}\right)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right)\ldots\ldots\cos\left(\frac{x}{2^{10}}\right)$$

26. Prove that $\dfrac{\sin\theta + \sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

$$L.H.S = \frac{\sin\theta + \sin 2\theta}{1+\cos\theta+\cos 2\theta}$$

$$= \frac{\sin\theta + 2\sin\theta\cos\theta}{\cos\theta + 2\cos^2\theta} = \frac{\sin\theta\,(\cancel{1+2\cos\theta})}{\cos\theta\,(\cancel{1+2\cos\theta})}$$

$$= \tan\theta$$

27. Prove that $1 - \dfrac{1}{2}\sin 2x = \dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x}$

$$R.H.S = \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} \qquad \boxed{a^3+b^3 = (a+b)(a^2-ab+b^2)}$$

$$= \frac{\cancel{(\sin x + \cos x)}\;(\sin^2 x - \sin x\cos x + \cos^2 x)}{\cancel{\sin x + \cos x}}$$

$$= \sin^2 x + \cos^2 x - \sin x\cos x = 1 - \sin x\cos x$$

$$= 1 - \frac{1}{2}\sin 2x$$

28. Find x **such that** $-\pi \le x \le \pi$ **and** $\cos 2x = \sin x$

$$\cos 2x = \sin x$$

$$1 - 2\sin^2 x = \sin x$$

$$2\sin^2 x + \sin x - 1 = 0$$

$$a = 2,\ b = 1 \text{ and } c = -1$$

$$\sin x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1^2 - 4(2)(-1)}}{2(2)} = \frac{-1 \pm \sqrt{1+8}}{4}$$

$$\sin x = \frac{-1 \pm \sqrt{9}}{4}$$

$$\sin x = \frac{-1 \pm 3}{4} \implies \sin x = \frac{-1+3}{4},\ \sin x = \frac{-1-3}{4}$$

$$\sin x = \frac{2}{4},\ \sin x = \frac{-4}{4}$$

$$\sin x = \frac{1}{2},\ \sin x = -1$$

$$\sin x = \frac{1}{2} \implies x = 30°, 150° \implies x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\boxed{\sin x = -1}$$

$$x = -90° \implies x = -\frac{\pi}{2}$$

$$\therefore x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

29. *Find the values of* $(i)\sin 18°, (ii)\cos 18°, (iii)\sin 72°, (iv)\cos 36°,$ $(v)\sin 54°$

(i) $Let\ \theta = 18°$

$$5\theta = 5 \times 18 \implies 5\theta = 90°$$

$$3\theta + 2\theta = 90° \implies 2\theta = 90° - 3\theta$$

$$\sin 2\theta = \sin(90° - 3\theta) \implies 2\sin\theta\cos\theta = \cos 3\theta$$

$$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$$

$$2\sin\theta\cancel{\cos\theta} = \cancel{\cos}\theta\,(4\cos^2\theta - 3)$$

$$2\sin\theta = 4\cos^2\theta - 3 \implies 2\sin\theta = 4(1 - \sin^2\theta) - 3$$

$$2\sin\theta = 4 - 4\sin^2\theta - 3$$

$$2\sin\theta = 1 - 4\sin^2\theta$$

$$4\sin^2\theta + 2\sin\theta - 1 = 0$$

$$a = 4,\ b = 2,\ c = -1$$

$$\sin\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \implies \sin\theta = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)}$$

$$= \frac{-2 \pm \sqrt{4 + 16}}{8} = \frac{-2 \pm \sqrt{20}}{8}$$

$$\sin\theta = \frac{-2 \pm \sqrt{4 \times 5}}{8} \implies \sin\theta = \frac{-2 \pm 2\sqrt{5}}{8}$$

$$\sin\theta = \frac{\cancel{2}(-1 \pm \sqrt{5})}{\cancel{8}\ 4} \implies \sin\theta = \frac{-1 \pm \sqrt{5}}{4}$$

$$\sin\theta = \frac{-1 + \sqrt{5}}{4} \quad (Since\ 18°\ is\ in\ I\ Quadrant)$$

$$\sin\theta = \frac{-1 + \sqrt{5}}{4} \quad (Where\ \theta = 18°)$$

$$\sin 18° = \frac{\sqrt{5} - 1}{4}$$

(ii) $\boldsymbol{cos18°}$

$$cos18° = \sqrt{1 - sin^2 18°} = \sqrt{1 - \left(\frac{\sqrt{5}-1}{4}\right)^2}$$

$$= \sqrt{1 - \frac{(\sqrt{5}-1)^2}{16}} = \sqrt{\frac{16-(\sqrt{5}-1)^2}{16}}$$

$$= \sqrt{\frac{16-[\sqrt{5}^2 - 2\sqrt{5}(1) + 1^2]}{16}} = \sqrt{\frac{16-[5-2\sqrt{5}+1]}{16}}$$

$$= \sqrt{\frac{16-5+2\sqrt{5}-1}{16}}$$

$$cos18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

(iii) $\boldsymbol{sin72°}$

$$sin72° = sin(90° - 18°)$$

$$= cos18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$sin72° = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

(iv) $\boldsymbol{cos36°}$

$$cos36° = cos\ 2 \times 18°$$

$$= 1 - 2sin^2 18°$$

$$= 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = 1 - \cancel{2}\frac{(\sqrt{5}-1)^2}{\cancel{16}\ 8} = 1 - \frac{(\sqrt{5})^2 - 2\sqrt{5}(1) + 1^2}{8}$$

$$= 1 - \frac{5-2\sqrt{5}+1}{8} = 1 - \frac{6-2\sqrt{5}}{8} = \frac{8-(6-2\sqrt{5})}{8}$$

$$= \frac{8-6+2\sqrt{5}}{8} = \frac{2+2\sqrt{5}}{8} = \frac{\cancel{2}(1+\sqrt{5})}{\cancel{8}\ 4} = \frac{1+\sqrt{5}}{4}$$

$$cos36° = \frac{\sqrt{5}+1}{4}$$

(v) $\boldsymbol{sin54^\circ}$

$$sin54^\circ = sin(90^\circ - 36^\circ)$$
$$= cos36^\circ$$
$$sin54^\circ = \frac{\sqrt{5}+1}{4}$$

$\boldsymbol{30. If\, tan\frac{\theta}{2} = \sqrt{\frac{1-a}{1+a}}\, tan\frac{\phi}{2}, then\ prove\ that\ cos\phi = \frac{cos\theta - a}{1 - acos\theta}}$

$$tan\frac{\theta}{2} = \sqrt{\frac{1-a}{1+a}}\, tan\frac{\emptyset}{2} \Longrightarrow tan\frac{\emptyset}{2} = tan\frac{\theta}{2}\sqrt{\frac{1+a}{1-a}}$$

squaring on both sides

$$tan^2\frac{\emptyset}{2} = \left(\frac{1+a}{1-a}\right) tan^2\frac{\theta}{2}$$

$$cos\emptyset = \frac{1 - tan^2\frac{\phi}{2}}{1 + tan^2\frac{\phi}{2}}$$

$$cos\phi = \frac{1 - \left(\frac{1+a}{1-a}\right)tan^2\frac{\theta}{2}}{1 + \left(\frac{1+a}{1-a}\right)tan^2\frac{\theta}{2}} = \frac{\frac{1 - a - (1+a)tan^2\frac{\theta}{2}}{1\cancel{-}a}}{\frac{1 - a + (1+a)tan^2\frac{\theta}{2}}{1\cancel{-}a}}$$

$$= \frac{1 - a - tan^2\frac{\theta}{2} - a\,tan^2\frac{\theta}{2}}{1 - a + tan^2\frac{\theta}{2} + a\,tan^2\frac{\theta}{2}} = \frac{1 - tan^2\frac{\theta}{2} - a - a\,tan^2\frac{\theta}{2}}{1 + tan^2\frac{\theta}{2} - a + a\,tan^2\frac{\theta}{2}}$$

$$= \frac{1 - tan^2\frac{\theta}{2} - a\left(1 + tan^2\frac{\theta}{2}\right)}{1 + tan^2\frac{\theta}{2} - a\left(1 - tan^2\frac{\theta}{2}\right)} = \frac{\frac{1 - tan^2\frac{\theta}{2} - a\left(1 + tan^2\frac{\theta}{2}\right)}{1 + tan^2\frac{\theta}{2}}}{\frac{1 + tan^2\frac{\theta}{2} - a\left(1 - tan^2\frac{\theta}{2}\right)}{1 + tan^2\frac{\theta}{2}}}$$

$$= \frac{\frac{1-tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}} - a\,\frac{\left(1+tan^2\frac{\theta}{2}\right)}{1+tan^2\frac{\theta}{2}}}{\frac{1+tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}} - a\,\frac{\left(1-tan^2\frac{\theta}{2}\right)}{1+tan^2\frac{\theta}{2}}} = \frac{\left(\frac{1-tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}\right) - a}{1 - a\left(\frac{1-tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}\right)}$$

$$= \frac{cos\theta - a}{1 - acos\theta}$$

31. Find the value of $\sqrt{3}cosec\ 20^\circ - sec\ 20$

$$\boxed{\sin 2A = 2\sin A\cos A}$$

$$\boxed{\frac{1}{2}\sin 2A = \sin A\cos A}$$

$$\sqrt{3}\ cosec\ 20^\circ - sec\ 20^\circ = \frac{\sqrt{3}}{sin\ 20^\circ} - \frac{1}{cos20^\circ}$$

$$= \frac{\sqrt{3}cos20^\circ - sin20^\circ}{sin20^\circ cos20^\circ} = \frac{\frac{\sqrt{3}cos20^\circ - sin20^\circ}{2}}{\frac{sin20^\circ cos20^\circ}{2}}$$

$$= \frac{\frac{\sqrt{3}}{2}cos20^\circ - \frac{1}{2}sin20^\circ}{\frac{1}{2}sin20^\circ\ cos20^\circ} = \frac{\frac{\sqrt{3}}{2}cos20^\circ - \frac{1}{2}sin20^\circ}{\frac{1}{2}sin20^\circ cos20^\circ}$$

$$= \frac{\frac{\sqrt{3}}{2}cos20^\circ - \frac{1}{2}sin20^\circ}{\frac{1}{2}\left[\frac{1}{2}sin2(20^\circ)\right]} = \frac{\frac{\sqrt{3}}{2}cos20^\circ - \frac{1}{2}sin20^\circ}{\frac{1}{4}sin2(20^\circ)}$$

$$= 4\left(\frac{sin60^\circ cos20^\circ\ -\ cos60^\circ sin20^\circ}{sin\ 2\ (20^\circ)}\right) = 4\left(\frac{sin(60^\circ - 20^\circ)}{sin40^\circ}\right)$$

$$= \frac{4\ sin40^\circ}{sin40^\circ} = 4$$

32. Prove that $\cos A\cos 2A\, cos2^2Acos2^3A\ \ldots cos2^{n-1}A = \frac{sin2^nA}{2^n\ sinA}$

$$L.H.S = \cos A\cos 2A\ \ cos2^2Acos2^3A\ldots\ldots\ cos2^{n-1}A$$

$$= \frac{1}{2sinA}\ 2sinA\cos A\cos 2A\ \ cos2^2A\ \ cos2^3A\ldots\ldots\ cos^{2n-1}A$$

$$\boxed{\sin 2A = 2\sin A\cos A}$$

$$= \frac{1}{2sinA} sin2A \cos 2A \quad cos2^2A\,cos2^3A\ldots\ldots.. \, cos^{2n-1}A$$

$$= \frac{1}{2sinA} \times \frac{1}{2} \; sin4A \; cos2^2A cos2^3A\ldots\ldots.. \, cos^{2n-1}A$$

$$= \frac{1}{2^2 sinA} \boxed{sin2^2A\, cos2^2A} cos2^3A\ldots\ldots.. \, cos^{2n-1}A \qquad \boxed{\frac{1}{2}\sin 2A = \sin A \cos A}$$

$$= \frac{1}{2^2 sinA} \times \frac{1}{2} \, sin2(2^2A)\, cos2^3A\ldots\ldots.. \, cos^{2n-1}A$$

$$= \frac{1}{2^3 sinA} \; sin2^3A\, cos2^3A\ldots\ldots.. \, cos^{2n-1}A$$

Continuing the process, we get

$$= \frac{sin2^nA}{2^n\, sinA}$$

(ii) $\boldsymbol{\cos 2A = 1 - 2\, sin^2A}$

$$where \;\; \sin A = \frac{4}{5}$$

$$\cos 2A = 1 - 2\left(\frac{4}{5}\right)^2$$

$$= 1 - 2\left(\frac{16}{25}\right) = 1 - \frac{32}{25}$$

$$= \frac{25 - 32}{25}$$

$$\cos 2A = -\frac{7}{25}$$

(iii) $\boldsymbol{\cos 2A = \dfrac{1 - tan^2A}{1 + tan^2A}}$

$$where \;\; \tan A = \frac{16}{63}$$

$$\cos 2A = \frac{1 - \left(\frac{16}{63}\right)^2}{1 + \left(\frac{16}{63}\right)^2} = \frac{1 - \frac{16^2}{63^2}}{1 + \frac{16^2}{63^2}} = \frac{\frac{63^2 - 16^2}{\cancel{63^2}}}{\frac{63^2 + 16^2}{\cancel{63^2}}}$$

$$= \frac{3969 - 256}{3969 + 256} = \frac{3713}{4225}$$

2. If θ is an acute angle, then find (i) $\sin\left[\frac{\pi}{4}-\frac{\theta}{2}\right]$ when $\sin\theta=\frac{1}{25}$,

(ii) $\cos\left[\frac{\pi}{4}+\frac{\theta}{2}\right]$ when $\sin\theta=\frac{8}{9}$

$$\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\sin\frac{\pi}{4}\cos\frac{\theta}{2}-\cos\frac{\pi}{4}\sin\frac{\theta}{2}$$

$$=\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}-\frac{1}{\sqrt{2}}\sin\frac{\theta}{2}$$

$$\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)$$

squaring on both sides

$$\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\left(\frac{1}{\sqrt{2}}\right)^2\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2$$

$$\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\frac{1}{2}\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\frac{1}{2}\left(1-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)=\frac{1}{2}\left(1-\sin 2\frac{\theta}{2}\right)$$

$$=\frac{1}{2}(1-\sin\theta)\quad where\ \sin\theta=\frac{1}{25}$$

$$\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\frac{1}{2}\left(1-\frac{1}{25}\right)$$

$$=\frac{1}{2}\left(\frac{25-1}{25}\right)=\frac{1}{\cancel{2}}\left(\frac{\overset{12}{\cancel{24}}}{25}\right)$$

$$\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\frac{12}{25}\Rightarrow\sin^2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=\frac{12}{25}$$

$$\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\sqrt{\frac{12}{25}}\Rightarrow\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\frac{2\sqrt{3}}{5}$$

(ii) $\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

$$\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\cos\frac{\pi}{4}\cos\frac{\theta}{2}-\sin\frac{\pi}{4}\sin\frac{\theta}{2}=\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}-\frac{1}{\sqrt{2}}\sin\frac{\theta}{2}$$

$$= \frac{1}{\sqrt{2}}\left(cos\frac{\theta}{2} - sin\frac{\theta}{2}\right)$$

$$cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \left(\frac{1}{\sqrt{2}}\right)^2\left(cos\frac{\theta}{2} - sin\frac{\theta}{2}\right)^2$$

$$= \frac{1}{2}\left[cos^2\frac{\theta}{2} + sin^2\frac{\theta}{2} - 2sin\frac{\theta}{2}cos\frac{\theta}{2}\right]$$

$$cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1}{2}\left[1 - 2sin\frac{\theta}{2}cos\frac{\theta}{2}\right]$$

$$= \frac{1}{2}\left[1 - \sin\cancel{2}\left(\frac{\theta}{\cancel{2}}\right)\right] = \frac{1}{2}(1 - \sin\theta)$$

$$= \frac{1}{2}\left(1 - \frac{8}{9}\right) = \frac{1}{2}\left(\frac{9-8}{9}\right)$$

$$cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1}{2}\left(\frac{1}{9}\right) = \frac{1}{18}$$

$$cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1}{18} \Rightarrow cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \sqrt{\frac{1}{18}}$$

$$cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1}{\sqrt{9 \times 2}} \Rightarrow cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1}{3\sqrt{2}}$$

3. $\boldsymbol{If\, cos\theta = \frac{1}{2}\left[a + \frac{1}{a}\right] Show\ that\ cos3\theta = \frac{1}{2}\left[a^3 + \frac{1}{a^3}\right]}$

$$cos3\theta = 4\,cos^3\theta - 3\cos\theta$$

$$= 4 \times \frac{1}{8}\left(a + \frac{1}{a}\right)^3 - \frac{3}{2}\left(a + \frac{1}{a}\right) = \frac{1}{2}\left(a + \frac{1}{a}\right)^3 - \frac{3}{2}\left(a + \frac{1}{a}\right)$$

$$= \frac{1}{2}\left[\left(a + \frac{1}{a}\right)^3 - 3\left(a + \frac{1}{a}\right)\right]$$

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b)$$

$$= \frac{1}{2}\left[a^3 + \frac{1}{a^3} + 3a\left(\frac{1}{a}\right)\left(a + \frac{1}{a}\right) - 3\left(a + \frac{1}{a}\right)\right]$$

$$= \frac{1}{2}\left[a^3 + \frac{1}{a^3} + 3\left(a + \frac{1}{a}\right) - 3\left(a + \frac{1}{a}\right)\right] = \frac{1}{2}\left[a^3 + \frac{1}{a^3}\right]$$

4. Prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$\cos 5\theta = \cos(3\theta + 2\theta)$ $\boxed{\cos(A+B) = \cos A\cos B - \sin A\sin B}$

$= \cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta$

$= (4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1) - (3\sin\theta - 4\sin^3\theta)(2\sin\theta\cos\theta)$

$= 8\cos^5\theta - 6\cos^3\theta - 4\cos^3\theta + 3\cos\theta - \left(6\cos\theta\sin^2\theta - 8\cos\theta\sin^4\theta\right)$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - \left(6\cos\theta\sin^2\theta - 8\cos\theta(\sin^2\theta)^2\right)$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 6\cos\theta(1-\cos^2\theta) + 8\cos\theta(1-\cos^2\theta)^2$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 6\cos\theta(1-\cos^2\theta) + 8\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 6\cos\theta + 6\cos^3\theta + 8\cos\theta - 16\cos^3\theta + 8\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = R.H.S$

5. Prove that $\sin 4\alpha = 4\tan\alpha \dfrac{1-\tan^2\alpha}{(1+\tan^2\alpha)^2}$

$$R.H.S = 4\tan\alpha \frac{1-\tan^2\alpha}{(1+\tan^2\alpha)^2}$$

$$= \frac{4\tan\alpha}{1+\tan^2\alpha} \times \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = 2 \times \frac{2\tan\alpha}{1+\tan^2\alpha} \times \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

$= 2\sin 2\alpha \cos 2\alpha$

$= \sin 2(2\alpha) = \sin 4\alpha = L.H.S$

6. If $A + B = 45°$ Show that $(1+\tan A)(1+\tan B) = 2$

$A + B = 45°$

$\tan(A+B) = \tan 45°$

$$\frac{\tan A + \tan B}{1 - \tan A\tan B} = 1 \implies \tan A + \tan B = 1 - \tan A\tan B$$

$\tan A + \tan B + \tan A\tan B = 1$

Adding 1 on both side

$1 + \tan A + \tan B + \tan A\tan B = 1 + 1$

$1 + \tan A + \tan B(1 + \tan A) = 2 \implies (1+\tan A)(1+\tan B) = 2$

7. prove that $(1+\tan 1°)(1+\tan 2°)(1+\tan 3°)\ldots\ldots(1+\tan 44°)$ is a multiple of 4.

$(1 + \tan 44°) = 1 + \tan(45° - 1°)$

$$= 1 + \frac{\tan 45° - \tan 1°}{1 + \tan 45°\tan 1°} = 1 + \frac{1 - \tan 1°}{1 + \tan 1°}$$

$$= \frac{1+\tan 1^\circ + 1 - tan1^\circ}{1+tan1^\circ} = \frac{2}{1+tan1^\circ}$$

$$(1+tan\,44^\circ) = \frac{2}{1+tan1^\circ} \quad \Longrightarrow (1+tan\,44^\circ)(1+tan1^\circ) = 2$$

$Similarly\ (1+\tan 2^\circ)(1+\tan 43^\circ) = 2$

$(1+\tan 3^\circ)\ (1+\tan 42^\circ) = 2$

$\cdots\cdots$

$(1+\tan 22^\circ)(1+\tan 23^\circ) = 2$

$= (1+\tan 1^\circ)(1+\tan 2^\circ) \cdots\cdots\cdots (1+\tan 44^\circ)$

$= 2 \times 2 \times \cdots\cdots 22\ times \quad (It\ is\ a\ multiple\ of\ 4.)$

8. $\boldsymbol{prove\ that\ tan\left[\frac{\pi}{4}+\theta\right] - tan\left[\frac{\pi}{4}-\theta\right] = 2\,tan\,2\theta}$

$$L.H.S = tan\left[\frac{\pi}{4}+\theta\right] - tan\left[\frac{\pi}{4}-\theta\right]$$

$$= \frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta} - \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}$$

$$= \frac{1+tan\theta}{1-tan\theta} - \frac{1-tan\theta}{1+tan\theta} = \frac{(1+tan\theta)^2 - (1-tan\theta)^2}{(1-tan\theta)(1+tan\theta)}$$

$$= \frac{1+2tan\theta+tan^2\theta - (1-2tan\theta+tan^2\theta)}{(1-tan\theta)(1+tan\theta)}$$

$$= \frac{1+2tan\theta+tan^2\theta - 1 + 2tan\theta - tan^2\theta}{1^2 - tan^2\theta} = \frac{4\tan\theta}{1-tan^2\theta}$$

$$= 2\left(\frac{2\tan\theta}{1-tan^2\theta}\right) = 2\tan 2\theta$$

9. $\boldsymbol{Show\ that\ \cot\left(7\frac{1}{2}\right)^\circ = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}}$

$\boxed{[\because 2\sin A\cos A = \sin 2A]}$

$\boxed{[\because 2\sin^2 A = 1-\cos 2A]}$

$$L.H.S = \cot\left(7\frac{1}{2}\right)^\circ$$

$$= \frac{\cos\left(7\frac{1}{2}\right)^\circ}{sin\left(7\frac{1}{2}\right)^\circ} = \frac{2\,sin\left(7\frac{1}{2}\right)^\circ \cos\left(7\frac{1}{2}\right)^\circ}{2\sin^2\left(7\frac{1}{2}\right)^\circ} = \frac{sin\,2\left(7\frac{1}{2}\right)^\circ}{1-\cos 2\left(7\frac{1}{2}\right)^\circ}$$

$$= \frac{\sin 15^\circ}{1 - \cos 15^\circ} = \frac{\sin(45^\circ - 30^\circ)}{1 - \cos(45^\circ - 30^\circ)}$$

$$= \frac{\sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ}{1 - [\cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ]}$$

$$= \frac{\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}}{1 - \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}\right)} = \frac{\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}}{\frac{1}{1} - \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}}$$

$$= \frac{\frac{\sqrt{3}-1}{\cancel{2\sqrt{2}}}}{\frac{2\sqrt{2} - (\sqrt{3}+1)}{\cancel{2\sqrt{2}}}} = \frac{\sqrt{3}-1}{2\sqrt{2} - (\sqrt{3}+1)} \times \frac{2\sqrt{2} + (\sqrt{3}+1)}{2\sqrt{2} + (\sqrt{3}+1)}$$

$$= \frac{(\sqrt{3}-1)(2\sqrt{2} + \sqrt{3} + 1)}{(2\sqrt{2})^2 - (\sqrt{3}+1)^2} = \frac{2\sqrt{6} + 3 + \cancel{\sqrt{3}} - 2\sqrt{2} - \cancel{\sqrt{3}} - 1}{4 \times 2 - (3 + 1 + 2\sqrt{3})}$$

$$= \frac{2\sqrt{6} - 2\sqrt{2} + 2}{8 - 3 - 1 - 2\sqrt{3}} = \frac{2\sqrt{6} - 2\sqrt{2} + 2}{8 - 4 - 2\sqrt{3}}$$

$$= \frac{2\sqrt{6} - 2\sqrt{2} + 2}{4 - 2\sqrt{3}} = \frac{\cancel{2}(\sqrt{6} - \sqrt{2} + 1)}{\cancel{2}(2 - \sqrt{3})} = \frac{\sqrt{6} - \sqrt{2} + 1}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$

$$= \frac{(\sqrt{6} - \sqrt{2} + 1)(2 + \sqrt{3})}{2^2 - (\sqrt{3})^2} = \frac{2\sqrt{6} + \sqrt{18} - 2\sqrt{2} - \sqrt{6} + 2 + \sqrt{3}}{4 - 3}$$

$$= \frac{\sqrt{6} + 3\sqrt{2} - 2\sqrt{2} + 2 + \sqrt{3}}{1} = \sqrt{6} + \sqrt{2} + 2 + \sqrt{3}$$

$$\boxed{\begin{aligned} 2 &= \sqrt{2} \times \sqrt{2} \\ &= \sqrt{4} \end{aligned}}$$

$$= \sqrt{6} + \sqrt{2} + \sqrt{4} + \sqrt{3} = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6} = R.H.S$$

10. *Prove that* $(1 + \sec 2\theta)(1 + \sec 4\theta) \ldots\ldots (1 + \sec 2^n\theta) = \tan 2^n\theta \cot \theta$

$$L.H.S = (1 + \sec 2\theta)(1 + \sec 4\theta) \ldots\ldots (1 + \sec 2^n\theta)$$

$$= \left(1 + \frac{1}{\cos 2\theta}\right)\left(1 + \frac{1}{\cos 4\theta}\right) \ldots\ldots\ldots \left(1 + \frac{1}{\cos 2^n\theta}\right)$$

$$= \left(\frac{\cos 2\theta + 1}{\cos 2\theta}\right)\left(\frac{\cos 4\theta + 1}{\cos 4\theta}\right) \dots\dots\dots \left(\frac{\cos 2^n\theta + 1}{\cos 2^n\theta}\right)$$

$$= \frac{(\cos 2\theta + 1)(\cos 4\theta + 1) \dots\dots\dots (\cos 2^n\theta + 1)}{\cos 2\theta \ \cos 4\theta \cdots\cdots\cdots \cos 2^n\theta}$$

$$= \frac{(2\cos^2\theta)(2\cos^2 2\theta) \dots\dots\dots (2\cos^2 2^{n-1}\theta)}{\cos 2\theta \ \cos^2 2\theta \dots\dots\dots \cos 2^n\theta}$$

$$= \frac{2^n \cos\theta}{\cos 2^n\theta}\left[\cos\theta \times \cos 2\theta \times \cos 2^2\theta \cdots\cdots\cdots \times \cos 2^{n-1}\theta\right]$$

$$= \frac{2^n \cos\theta}{\cos 2^n\theta} \times \frac{\sin 2^n A}{2^n \sin A} = \tan 2^n\theta \cot\theta = R.H.S$$

$$\boxed{\because \cos A \cos 2A \cos 2^2 A \dots\dots\dots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}}$$

11. *Prove that* $32(\sqrt{3}) \sin\frac{\pi}{48} \cos\frac{\pi}{48} \cos\frac{\pi}{24} \cos\frac{\pi}{12} \cos\frac{\pi}{6} = 3.$

$$L.H.S = 32(\sqrt{3}) \sin\frac{\pi}{48} \cos\frac{\pi}{48} \cos\frac{\pi}{24} \cos\frac{\pi}{12} \cos\frac{\pi}{6}$$

Multiply divide by 2

$$= \frac{32\sqrt{3}}{2} \times \left(2\sin\frac{\pi}{48}\cos\frac{\pi}{48}\right)\cos\frac{\pi}{24}\cos\frac{\pi}{12}\cos\frac{\pi}{6}$$

$$= \frac{\overset{16}{\cancel{32}}\sqrt{3}}{\cancel{2}} \sin 2\left(\frac{\pi}{48}\right)\cos\frac{\pi}{24}\cos\frac{\pi}{12}\cos\frac{\pi}{6} \qquad [\because 2\sin A\cos A = \sin 2A]$$

$$= \overset{8}{\cancel{16}}\sqrt{3} \times \left(\frac{2}{\cancel{2}}\sin\frac{\pi}{24}\cos\frac{\pi}{24}\right)\cos\frac{\pi}{12}\cos\frac{\pi}{6} = 8\sqrt{3}\sin 2\left(\frac{\pi}{24}\right)\cos\frac{\pi}{12}\cos\frac{\pi}{6}$$

$$= \overset{4}{\cancel{8}}\sqrt{3} \times \left(\frac{2}{\cancel{2}}\sin\frac{\pi}{12}\cos\frac{\pi}{12}\right)\cos\frac{\pi}{6} = 4\sqrt{3}\sin 2\left(\frac{\pi}{12}\right)\cos\frac{\pi}{6}$$

$$= \overset{2}{\cancel{4}}\sqrt{3} \times \left(\frac{2}{\cancel{2}}\sin\frac{\pi}{6}\cos\frac{\pi}{6}\right) = 2\sqrt{3}\sin 2\left(\frac{\pi}{6}\right) = 2\sqrt{3}\sin\frac{\pi}{3}$$

$$= 2\sqrt{3}\sin 60^\circ = \cancel{2}\sqrt{3} \times \frac{\sqrt{3}}{\cancel{2}} = 3 = R.H.S$$

EXERCISE : 3.6

Transformation of a product into a sum or difference

Adding $sin(A+B)$ *and* $sin(A-B)$

$$\sin(A+B) = \sin A\cos B + \cancel{\cos A\sin B}$$

$$+$$

$$\sin(A-B) = \sin A\cos B - \cancel{\cos A\sin B}$$

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B$$

$$\frac{1}{2}[\sin(A+B)+\sin(A-B)] = \sin A\cos B$$

Subtracting $sin(A+B)$ *and* $sin(A-B)$

$$\sin(A+B) = \cancel{\sin A\cos B} + \cos A\sin B$$

$$(-) \qquad\qquad (-) \qquad\qquad (+)$$

$$\sin(A-B) = \cancel{\sin A\cos B} - \cos A\sin B$$

$$\sin(A+B)-\sin(A-B) = 2\cos A\sin B$$

$$\frac{1}{2}[\sin(A+B)-\sin(A-B)] = \cos A\sin B$$

Adding $cos(A+B)$ *and* $cos(A-B)$

$$\cos(A+B) = \cos A\cos B - \cancel{\sin A\sin B}$$

$$+$$

$$\cos(A-B) = \cos A\cos B + \cancel{\sin A\sin B}$$

$$\cos(A+B)+\cos(A-B) = 2\cos A\cos B$$

$$\frac{1}{2}[\cos(A+B)+\cos(A-B)] = \cos A\cos B$$

Subtracting $cos(A+B)$ *and* $cos(A-B)$

$$\cos(A+B) = \cancel{\cos A\cos B} - \sin A\sin B$$

$$(-) \qquad\qquad (-) \qquad\qquad (-)$$

$$\cos(A-B) = \cancel{\cos A\cos B} + \sin A\sin B$$

$$\cos(A+B)-\cos(A-B) = -2\sin A\sin B$$

$$-\cos(A+B)+\cos(A-B)=2\sin A\sin B$$

$$\cos(A-B)-\cos(A+B)=2\sin A\sin B$$

$$\frac{1}{2}[\cos(A-B)-\cos(A+B)]=\sin A\sin B$$

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$

$$where\ A+B=C\ and\ \ A-B=D$$

$$\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)cos\left(\frac{C-D}{2}\right)$$

$$\sin(A+B)-\sin(A-B)=2\cos A\sin B$$

$$\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)sin\left(\frac{C-D}{2}\right)$$

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B$$

$$\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)cos\left(\frac{C-D}{2}\right)$$

$$\cos(A+B)-\cos(A-B)=-2\sin A\sin B$$

$$\cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$\cos(A-B)-\cos(A+B)=2\sin A\sin B$$

$$\cos D-\cos C=2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$A+\cancel{B}=C$$
$$A-\cancel{B}=D$$
$$2A=C+D$$
$$A=\frac{C+D}{2}$$

$$\cancel{A}+B=C$$
$$\underset{(-)}{\cancel{A}}\underset{(+)}{-B}=\underset{(-)}{D}$$
$$2B=C-D$$
$$B=\frac{C-D}{2}$$

Example 3.33 Express each of the following product as a sum or difference

$(i)\ \boldsymbol{sin\,40^{\circ}\ cos\,30^{\circ}}$ $\quad(ii)\ \boldsymbol{cos\,110^{\circ}\ sin\,55^{\circ}}$ $\quad(iii)\ \boldsymbol{sin\frac{x}{2}\ cos\frac{3x}{2}}$

$(i)\ \boldsymbol{sin\,40^{\circ}\ cos\,30^{\circ}}$

$$\boxed{sin(A+B)+sin(A-B)=2\,sin\,A\ cos\,B}$$

$$\frac{1}{2}[sin(A+B)+sin(A-B)]=sin\,A\ cos\,B$$

$$sin\,40^{\circ}\ cos\,30^{\circ}=\frac{1}{2}\ [sin(40^{\circ}+30^{\circ})+sin(40^{\circ}-30^{\circ})]$$

$$=\frac{1}{2}\ [sin\,70^{\circ}+sin\,10^{\circ}]$$

$(ii)\ \boldsymbol{cos\,110^\circ\ sin\,55^\circ}$

$$\boxed{sin(A+B) - sin(A-B) = 2\,cos\,A\ \ sin\,B}$$

$$\frac{1}{2}[sin(A+B) - sin(A-B)] = cos\,A\ \ sin\,B$$

$$cos\,110^\circ\ \ sin\,55^\circ = \frac{1}{2}\ [sin(110^\circ + 55^\circ) - sin(110^\circ - 55^\circ)]$$

$$= \frac{1}{2}\ [sin\,165^\circ - sin\,55^\circ]$$

$(iii)\ \boldsymbol{sin\frac{x}{2}\ cos\frac{3x}{2}}$

$$\boxed{sin(A+B) + sin(A-B) = 2\,sin\,A\ \ cos\,B}$$

$$\frac{1}{2}\ [sin(A+B) + sin(A-B)] = sin\,A\ \ cos\,B$$

$$sin\frac{x}{2}\ cos\frac{3x}{2} = \frac{1}{2}\left(sin\left(\frac{x}{2}+\frac{3x}{2}\right) + sin\left(\frac{x}{2}-\frac{3x}{2}\right)\right)$$

$$= \frac{1}{2}\left(sin\left(\frac{4x}{2}\right) + sin\left(\frac{-2x}{2}\right)\right)$$

$$= \frac{1}{2}\left(sin\,2x + sin(-x)\right) = \frac{1}{2}\left(sin\,2x - \ sin\,x\right)$$

Example3.34 Express each of the following sum or difference as a product

$(\boldsymbol{i})\ \mathbf{sin\,50^\circ} + \boldsymbol{sin\,20^\circ}\ (\boldsymbol{ii})\ \mathbf{cos\,6}\boldsymbol{\theta}\ +\ \mathbf{cos\,2}\boldsymbol{\theta}\ \ (\boldsymbol{iii})\ \boldsymbol{cos\frac{3x}{2} - cos\frac{9x}{2}}$

$(i)\ \mathbf{sin\,50^\circ} + \boldsymbol{sin\,20^\circ}$

$$sinC + sinD = 2\,sin\left(\frac{C+D}{2}\right)cos\left(\frac{C-D}{2}\right)$$

$Take\ \ C\ = 50^\circ\ \ and\ \ D\ = 20^\circ$

$$\sin 50^\circ + sin\,20^\circ = \ 2\,sin\left(\frac{50^\circ + 20^\circ}{2}\right)cos\left(\frac{50^\circ - 20^\circ}{2}\right)$$

$$= 2\,sin\left(\frac{70^\circ}{2}\right)cos\left(\frac{30^\circ}{2}\right) = 2\sin 35^\circ\,cos\,15^\circ$$

$(ii)\ \boldsymbol{\cos 6\theta + \cos 2\theta}$

$$cosC + cosD = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$Take\ C = 6\theta\ and\ D = 2\theta$

$$\cos 6\theta + \cos 2\theta = 2\cos\left(\frac{6\theta+2\theta}{2}\right)\cos\left(\frac{6\theta-2\theta}{2}\right)$$

$$= 2\cos\left(\frac{8\theta}{2}\right)\cos\left(\frac{4\theta}{2}\right)$$

$$= 2\cos 4\theta\cos 2\theta$$

$(iii)\ \boldsymbol{\cos\frac{3x}{2} - \cos\frac{9x}{2}}$

$$\cos D - \cos C = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$Take\ D = \frac{3x}{2}\ and\ C = \frac{9x}{2}$

$$cos\frac{3x}{2} - cos\frac{9x}{2} = 2\ sin\left(\frac{\frac{9x}{2}+\frac{3x}{2}}{2}\right)sin\left(\frac{\frac{9x}{2}-\frac{3x}{2}}{2}\right)$$

$$= 2\ sin\left(\frac{\frac{9x+3x}{2}}{2}\right)sin\left(\frac{\frac{9x-3x}{2}}{2}\right) = 2\ sin\left(\frac{\frac{12x}{2}}{2}\right)sin\left(\frac{\frac{6x}{2}}{2}\right)$$

$$= 2\ sin\left(\frac{6x}{2}\right)sin\left(\frac{3x}{2}\right) = 2\ sin\,3x\,sin\left(\frac{3x}{2}\right)$$

Example 3.35 **Find the value of** $\boldsymbol{\sin 34^\circ + \cos 64^\circ - \cos 4^\circ}$

$$\boxed{\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)}$$

$$\sin 34^\circ + \underbrace{\cos 64^\circ - \cos 4^\circ}$$

$$= \sin 34^\circ - 2\sin\left(\frac{64^\circ+4^\circ}{2}\right)\sin\left(\frac{64^\circ-4^\circ}{2}\right)$$

$$= \sin 34^\circ - 2\sin\left(\frac{68^\circ}{2}\right)\sin\left(\frac{60^\circ}{2}\right) = \sin 34^\circ - 2\sin 34^\circ\sin 30^\circ$$

$$= \sin 34^\circ - 2\sin 34^\circ\left(\frac{1}{2}\right) = \sin 34^\circ - \sin 34^\circ = 0$$

Example 3.36 **Show that** $\boldsymbol{\cos 36^\circ\cos 72^\circ\cos 108^\circ\cos 144^\circ = \frac{1}{16}}$.

$$L.H.S = \cos 36^\circ\cos 72^\circ\cos 108^\circ\cos 144^\circ$$

$$= \cos 36^\circ \cos(90^\circ - 18^\circ)\cos(90^\circ + 18^\circ)\cos(180^\circ - 36^\circ)$$

$$= \cos 36^\circ \times \sin 18^\circ \times -\sin 18^\circ \times -\cos 36^\circ$$

$$= \sin^2 18^\circ \cos^2 36^\circ$$

$$\boxed{sin 18° = \frac{\sqrt{5}-1}{4}}$$

$$\boxed{cos\ 36° = \frac{\sqrt{5}+1}{4}}$$

$$= \left(\frac{\sqrt{5}-1}{4}\right)^2 \left(\frac{\sqrt{5}+1}{4}\right)^2 = \left[\frac{(\sqrt{5}-1)(\sqrt{5}+1)}{16}\right]^2$$

$$= \left[\frac{(\sqrt{5})^2 - 1^2}{16}\right]^2 = \left[\frac{5-1}{16}\right]^2 = \left(\frac{4}{16}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

***Example 3.37**: Simplify*: $\dfrac{\boldsymbol{sin\,75^\circ - sin\,15^\circ}}{\boldsymbol{cos\,75^\circ + cos\,15^\circ}}$

$$\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \frac{2\cos\left(\frac{75^\circ + 15^\circ}{2}\right)\sin\left(\frac{75^\circ - 15^\circ}{2}\right)}{2\cos\left(\frac{75^\circ + 15^\circ}{2}\right)\cos\left(\frac{75^\circ - 15^\circ}{2}\right)}$$

$$= \frac{2\cos\left(\frac{90^\circ}{2}\right)\sin\left(\frac{60^\circ}{2}\right)}{2\cos\left(\frac{90^\circ}{2}\right)\cos\left(\frac{60^\circ}{2}\right)} = \frac{\cancel{2}\,\cancel{\cos 45^\circ}\sin 30^\circ}{\cancel{2}\,\cancel{\cos 45^\circ}\cos 30^\circ}$$

$$= \tan 30^\circ = \frac{1}{\sqrt{3}}$$

***Example 3.38**: Show that* $\boldsymbol{\cos 10^\circ\, cos\, 30^\circ\, cos\, 50^\circ\, cos\, 70^\circ = \frac{3}{16}}$.

$$\cos(60^\circ - 10^\circ)\cos A\cos(60^\circ + 10^\circ) = \frac{1}{4}\cos 3A$$

$$\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \cos 30^\circ\,[\cos 10^\circ \cos 50^\circ \cos 70^\circ]$$

$$= \cos 30^\circ\,[\cos(60^\circ - 10^\circ)\cos 10^\circ \cos(60^\circ + 10^\circ)]$$

$$= \frac{\sqrt{3}}{2}\left[\frac{1}{4}\cos 30^\circ\right] = \frac{\sqrt{3}}{2}\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{16}$$

***Example 3.38**:* $\boldsymbol{cos\,10^\circ\ cos\,30^\circ\ cos\,50^\circ\ cos\,70^\circ} = \frac{3}{16}$

$$L.H.S = cos\,10^\circ\, cos\,30^\circ\, cos\,50^\circ\, cos\,70^\circ$$

$$= cos\,30^\circ\ cos\,10^\circ\, cos\,50^\circ\, cos\,70^\circ$$

$$\boxed{cos(A+B)\,cos(A-B) = \cos^2 A - \sin^2 B}$$

$$= \frac{\sqrt{3}}{2}\, cos\,10^\circ\ cos(60^\circ - 10^\circ)\, cos(60^\circ + 10^\circ)$$

$$= \frac{\sqrt{3}}{2} cos\, 10^{\circ} \left(\cos^2 60^{\circ} - \sin^2 10^{\circ}\right)$$

$$= \frac{\sqrt{3}}{2} cos\, 10^{\circ} \left(\left(\frac{1}{2}\right)^2 - \sin^2 10^{\circ}\right) = \frac{\sqrt{3}}{2} cos\, 10^{\circ} \left(\frac{1}{4} - \sin^2 10^{\circ}\right)$$

$$= \frac{\sqrt{3}}{2} cos\, 10^{\circ} \left(\frac{1 - 4\sin^2 10^{\circ}}{4}\right) = \frac{\sqrt{3}}{2} cos\, 10^{\circ} \left(\frac{1 - 4(1 - \cos^2 10^{\circ})}{4}\right)$$

$$= \frac{\sqrt{3}}{2} cos\, 10^{\circ} \left(\frac{1 - 4 + 4\cos^2 10^{\circ}}{4}\right) = \frac{\sqrt{3}}{2} cos\, 10^{\circ} \left(\frac{4\cos^2 10^{\circ} - 3}{4}\right)$$

$$= \frac{\sqrt{3}}{2} \left(\frac{4\cos^3 10^{\circ} - 3\, cos\, 10^{\circ}}{4}\right) = \frac{\sqrt{3}}{8} \left(4\cos^3 10^{\circ} - 3\, cos\, 10^{\circ}\right)$$

$$\boxed{\cos 3A = 4\cos^3 A - 3\, cos\, A}$$

$$= \frac{\sqrt{3}}{8}\; cos\, 3(10^{\circ}) = \frac{\sqrt{3}}{8}\; cos\, 30^{\circ}$$

$$= \frac{\sqrt{3}}{8} \times \frac{\sqrt{3}}{2} = \frac{3}{16}$$

1. *Express each of the following as a sum or difference.*

(i) $\boldsymbol{sin\, 35^{\circ}\, cos\, 28^{\circ}}$ (ii) $\mathbf{\sin 4x \cos 2x}$ (iii) $\mathbf{2 \sin 10\theta \cos 2\theta}$

(iv) $\mathbf{\cos 5\theta \cos 2\theta}$ (v) $\mathbf{\sin 5\theta \sin 4\theta}$

$$\boxed{sin(A+B) + sin(A-B) = 2\, sin\, A\, cos\, B}$$

$$\boxed{\frac{1}{2}[sin(A+B) + sin(A-B)] = sin\, A\, cos\, B}$$

(i) $\boldsymbol{sin\, 35^{\circ}\, cos\, 28^{\circ}}$

$$\sin 35^{\circ} \cos 28^{\circ} = \frac{1}{2}\left(sin\left(35^{\circ} + 28^{\circ}\right) + sin\left(35^{\circ} - 28^{\circ}\right)\right)$$

$$= \frac{1}{2}\left(\sin 63^{\circ} + \sin 7^{\circ}\right)$$

(ii) $\boldsymbol{sin\, 4x\, cos\, 2x}$

$$= \frac{1}{2}\left(sin(4x + 2x) + sin(4x - 2x)\right)$$

$$= \frac{1}{2}\left(\sin 6x + \sin 2x\right)$$

$$\boxed{\boldsymbol{sin(A+B) + sin(A-B) = 2\, sin\, A\, cos\, B}}$$

$$\boxed{\boldsymbol{\frac{1}{2}[sin(A+B) + sin(A-B)] = sin\, A\, cos\, B}}$$

(iii) $\boldsymbol{2\, sin\, 10\theta\, cos\, 2\theta}$

$$= sin(10\theta + 2\theta) + sin(10\theta - 2\theta)$$

$$= \sin 12\theta + \sin 8\theta$$

$(iv)\ \boldsymbol{\cos 5\theta \cos 2\theta}$

$$= \frac{1}{2}\Big(\cos(5\theta + 2\theta) + \cos(5\theta - 2\theta)\Big)$$

$$= \frac{1}{2}\Big(\cos 7\theta + \cos 3\theta\Big)$$

$$\boxed{\boldsymbol{\cos(A+B) + \cos(A-B) = 2\cos A \cos B}}$$

$$\boxed{\boldsymbol{\frac{1}{2}[\cos(A+B) + \cos(A-B)] = \cos A \cos B}}$$

$(v)\ \boldsymbol{\sin 5\theta \sin 4\theta}$

$$= \frac{1}{2}\Big(\cos(5\theta - 4\theta) - \cos(5\theta + 4\theta)\Big)$$

$$= \frac{1}{2}\Big(\cos\theta - \cos 9\theta\Big)$$

$$\boxed{\boldsymbol{\frac{1}{2}[\cos(A-B) - \cos(A+B)] = \sin A \sin B}}$$

2. Express each of the following as a product.

$(i)\ \boldsymbol{\sin 75° - \sin 35°}$ $\quad (ii)\ \boldsymbol{\cos 65° + \cos 15°}$

$(iii)\ \boldsymbol{\sin 50° + \sin 40°}$ $\quad (iv)\ \boldsymbol{\cos 35° - \cos 75°}$

$(i)\ \boldsymbol{\sin 75^\circ - \sin 35^\circ}$

$$= 2\cos\left(\frac{75^\circ + 35^\circ}{2}\right)\sin\left(\frac{75^\circ - 35^\circ}{2}\right)$$

$$= 2\cos\left(\frac{110^\circ}{2}\right)\sin\left(\frac{40^\circ}{2}\right)$$

$$\boxed{\boldsymbol{\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)}}$$

$$= 2\cos 55^\circ \sin 20^\circ$$

$$\boxed{\boldsymbol{\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)}}$$

$(ii)\ \boldsymbol{\cos 65° + \cos 15°}$

$$= 2\cos\left(\frac{65^\circ + 15^\circ}{2}\right)\cos\left(\frac{65^\circ - 15^\circ}{2}\right)$$

$$= 2\cos\left(\frac{80^\circ}{2}\right)\cos\left(\frac{50^\circ}{2}\right) = 2\cos 40^\circ \cos 25^\circ$$

$(iii)\ \boldsymbol{\sin 50^\circ + \sin 40^\circ}$

$$= 2\sin\left(\frac{50^\circ + 40^\circ}{2}\right)\cos\left(\frac{50^\circ - 40^\circ}{2}\right)$$

$$= 2\sin\left(\frac{90^\circ}{2}\right)\cos\left(\frac{10^\circ}{2}\right)$$

$$\boxed{\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)}$$

$$= 2\sin 45^\circ \cos 5^\circ$$

$$\boxed{\cos D - \cos C = 2\sin\left(\frac{D+C}{2}\right)\cos\left(\frac{D-C}{2}\right)}$$

$(iv)\ \cos 35^\circ - \cos 75^\circ$

$$= 2\sin\left(\frac{35^\circ+75^\circ}{2}\right)\cos\left(\frac{35^\circ-75^\circ}{2}\right) = 2\sin\left(\frac{110^\circ}{2}\right)\cos\left(-\frac{40^\circ}{2}\right)$$

$$= 2\sin 55^\circ \cos 20^\circ$$

$$\boxed{\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)}$$

3. Show that $\sin 12^\circ \sin 48^\circ \sin 54^\circ = \mathbf{\frac{1}{8}}$

$$\boxed{\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]}$$

$$= \frac{1}{2}\left[\cos(12^\circ - 48^\circ) - \cos(12^\circ + 48^\circ)\right]\sin 54^\circ$$

$$= \frac{1}{2}\left[\cos(-36^\circ) - \cos(60^\circ)\right]\sin 54^\circ$$

$$\boxed{\cos 36^\circ = \frac{\sqrt{5}+1}{4}} \quad \boxed{\sin 54^\circ = \frac{\sqrt{5}+1}{4}}$$

$$= \frac{1}{2}\left[\cos 36^\circ - \frac{1}{2}\right]\sin 54^\circ = \frac{1}{2}\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right)\left(\frac{\sqrt{5}+1}{4}\right)$$

$$\boxed{\cos(-\theta) = \cos\theta}$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}+1-2}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{1}{2}\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right)$$

$$= \frac{1}{32}\left[(\sqrt{5})^2 - 1^2\right] = \frac{1}{32}(5-1) = \frac{4}{32} = \frac{1}{8} = R.H.S$$

Hence proved.

4. Show that $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{128}$

$$L.H.S = \cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$$

$$= \cos 12^\circ \cos 24^\circ \cos 36^\circ \cos 48^\circ \cos 60^\circ \cos 72^\circ \cos 84^\circ$$

$$\boxed{\begin{aligned}\cos 36^\circ &= \frac{\sqrt{5}+1}{4}\\ \sin 18^\circ &= \frac{\sqrt{5}-1}{4}\end{aligned}} \quad \boxed{\begin{aligned}\cos 72^\circ &= \cos(90^\circ - 18^\circ)\\ &= \sin 18^\circ\\ &= \frac{\sqrt{5}-1}{4}\end{aligned}} \quad \boxed{\begin{aligned}\frac{\pi}{15}\times\frac{180^\circ}{\pi} &= \frac{180^\circ}{15}\\ &= 12^\circ\end{aligned}}$$

$$= \cos 12^\circ \cos 24^\circ\left(\frac{\sqrt{5}+1}{4}\right)\cos 48^\circ\left(\frac{1}{2}\right)\left(\frac{\sqrt{5}-1}{4}\right)\cos 84^\circ$$

$$= \left[\frac{(\sqrt{5})^2 - 1^2}{32}\right]\cos 48^\circ \cos 12^\circ \cos 84^\circ \cos 24^\circ$$

$$\boxed{\begin{aligned}\frac{1}{2}[\cos(A+B) + \cos(A-B)]\\ = \cos A \cos B\end{aligned}}$$

$$= \left(\frac{4}{32}\right)\frac{1}{2}\left\{\cos(48^\circ+12^\circ) + \cos(48^\circ-12^\circ)\right\}$$

$$\frac{1}{2}\left\{\cos(84^\circ+24^\circ) + \cos(84^\circ-24^\circ)\right\}$$

$$= \left(\frac{1}{8}\right)\frac{1}{4}\left\{\cos 60^\circ + \cos 36^\circ\right\}\left\{\cos 108^\circ + \cos 60^\circ\right\}$$

$$\cos 36^\circ = \frac{\sqrt{5}+1}{4}$$

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4}$$

$$= \frac{1}{32}\left\{\frac{1}{2} + \cos 36^\circ\right\}\left\{\cos 108^\circ + \frac{1}{2}\right\}$$

$$\cos 108^\circ = \cos(90^\circ + 18^\circ)$$
$$= -\sin 18^\circ$$
$$= -\left(\frac{\sqrt{5}-1}{4}\right) = \frac{1-\sqrt{5}}{4}$$

$$= \frac{1}{32}\left\{\frac{1}{2} + \frac{\sqrt{5}+1}{4}\right\}\left\{\frac{1-\sqrt{5}}{4} + \frac{1}{2}\right\}$$

$$= \frac{1}{32}\left(\frac{2+\sqrt{5}+1}{4}\right)\left(\frac{1-\sqrt{5}+2}{4}\right)$$

$$= \frac{1}{32}\left(\frac{3+\sqrt{5}}{4}\right)\left(\frac{3-\sqrt{5}}{4}\right) = \frac{1}{32}\left[\frac{3^2-(\sqrt{5})^2}{16}\right] = \frac{1}{32}\left[\frac{4}{16}\right] = \frac{1}{128}$$

5. *Show that* $\dfrac{\sin 8x \cos x - \sin 6x \cos 3x}{\cos 2x \cos x - \sin 3x \sin 4x} = \tan 2x$

$$L.H.S = \frac{\sin 8x \cos x - \sin 6x \cos 3x}{\cos 2x \cos x - \sin 3x \sin 4x}$$

$$= \frac{\frac{1}{2}[\sin(8x+x) + \sin(8x-x)] - \frac{1}{2}[\sin(6x+3x) + \sin(6x-3x)]}{\frac{1}{2}[\cos(2x+x) + \cos(2x-x)] - \frac{1}{2}[\cos(4x-3x) + \cos(4x+3x)]}$$

$$= \frac{\frac{1}{2}[\sin 9x + \sin 7x] - \frac{1}{2}[\sin 9x + \sin 3x]}{\frac{1}{2}[\cos 3x + \cos x] - \frac{1}{2}[\cos x - \cos 7x]}$$

$$= \frac{\cancel{\frac{1}{2}}[\cancel{\sin 9x} + \sin 7x - \cancel{\sin 9x} - \sin 3x]}{\cancel{\frac{1}{2}}[\cos 3x + \cancel{\cos x} - \cancel{\cos x} + \cos 7x]} = \frac{\sin 7x - \sin 3x}{\cos 3x + \cos 7x}$$

$$= \frac{2\cos\left(\frac{7x+3x}{2}\right)\sin\left(\frac{7x-3x}{2}\right)}{2\cos\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right)}$$

$$\boxed{\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)}$$

$$\boxed{\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)}$$

$$= \frac{2\cos\left(\frac{10x}{2}\right)\sin\left(\frac{4x}{2}\right)}{2\cos\left(\frac{10x}{2}\right)\cos\left(\frac{4x}{2}\right)} = \frac{\cancel{2\cos 5x}\sin 2x}{\cancel{2\cos 5x}\cos 2x} = \tan 2x$$

6. *Show that* $\boldsymbol{\frac{(\cos\theta - \cos 3\theta)(\sin 8\theta + \sin 2\theta)}{(\sin 5\theta - \sin\theta)(\cos 4\theta - \cos 6\theta)} = 1}$

$$L.H.S = \frac{(\cos\theta - \cos 3\theta)(\sin 8\theta + \sin 2\theta)}{(\sin 5\theta - \sin\theta)(\cos 4\theta - \cos 6\theta)}$$

$$= \frac{2\sin\left(\frac{\theta+3\theta}{2}\right)\sin\left(\frac{3\theta-\theta}{2}\right)\times 2\sin\left(\frac{8\theta+2\theta}{2}\right)\cos\left(\frac{8\theta-2\theta}{2}\right)}{2\cos\left(\frac{5\theta+\theta}{2}\right)\sin\left(\frac{5\theta-\theta}{2}\right)\times 2\sin\left(\frac{4\theta+6\theta}{2}\right)\sin\left(\frac{6\theta-4\theta}{2}\right)}$$

$$= \frac{4\times\sin\left(\frac{4\theta}{2}\right)\sin\left(\frac{2\theta}{2}\right)\times\sin\left(\frac{10\theta}{2}\right)\cos\left(\frac{6\theta}{2}\right)}{4\times\cos\left(\frac{6\theta}{2}\right)\sin\left(\frac{4\theta}{2}\right)\times\sin\left(\frac{10\theta}{2}\right)\sin\left(\frac{2\theta}{2}\right)}$$

$$= \frac{\cancel{\sin 2\theta}\,\cancel{\sin\theta}\times\cancel{\sin 5\theta}\,\cancel{\cos 3\theta}}{\cancel{\cos 3\theta}\,\cancel{\sin 2\theta}\times\cancel{\sin 5\theta}\,\cancel{\sin\theta}} = 1 \quad = R.H.S \quad \textit{Hence proved.}$$

7. *Prove that* $\boldsymbol{\sin x + \sin 2x + \sin 3x = \sin 2x\,(1 + 2\cos x)}$

$$L.H.S = (\sin 3x + \sin x) + \sin 2x$$

$$\boxed{\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)}$$

$$= 2\sin\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) + \sin 2x$$

$$= 2\sin\left(\frac{4x}{2}\right)\cos\left(\frac{2x}{2}\right) + \sin 2x = 2\sin 2x\cos x + \sin 2x$$

$$= \sin 2x\,(2\cos x + 1) = \sin 2x\,(1 + 2\cos x)$$

$$= R.H.S \quad \textit{Hence proved.}$$

8. *Prove that* $\boldsymbol{\frac{\sin 4x + \sin 2x}{\cos 4x + \cos 2x} = \tan 3x}$

$$L.H.S = \frac{\sin 4x + \sin 2x}{\cos 4x + \cos 2x}$$

$$= \frac{2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)}{2\cos\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)} = \frac{2\sin\left(\frac{6x}{2}\right)\cos\left(\frac{2x}{2}\right)}{2\cos\left(\frac{6x}{2}\right)\cos\left(\frac{2x}{2}\right)}$$

$$= \frac{\cancel{2}\sin 3x \cancel{\cos x}}{\cancel{2}\cos 3x \cancel{\cos x}} = \tan 3x$$

9. *Prove that* $\mathbf{1 + \cos 2x + \cos 4x + \cos 6x = 4\cos x \cos 2x \cos 3x}$

$$\boxed{\begin{array}{l} \cos 2A = 2\cos^2 A - 1 \\ 1 + \cos 2A = 2\cos^2 A \\ \because \cos(-\theta) = \cos\theta \end{array}}$$

$$L.H.S = 1 + \cos 2x + \cos 4x + \cos 6x$$

$$= 2\cos^2 x + 2\cos\left(\frac{4x+6x}{2}\right)\cos\left(\frac{4x-6x}{2}\right)$$

$$= 2\cos^2 x + 2\cos\left(\frac{10x}{2}\right)\cos\left(\frac{-2x}{2}\right)$$

$$= 2\cos^2 x + 2\cos 5x \cos(-x) = 2\cos^2 x + 2\cos 5x \cos x$$

$$= 2\cos x(\cos x + \cos 5x)$$

$$= 2\cos x \left\{2\cos\left(\frac{x+5x}{2}\right)\cos\left(\frac{x-5x}{2}\right)\right\}$$

$$= 2\cos x\, 2\cos\left(\frac{6x}{2}\right)\cos\left(\frac{-4x}{2}\right) = 2\cos x\, 2\cos 3x \cos(-2x)$$

$$= 4\cos x \cos 2x \cos 3x = R.H.S \quad \textit{Hence proved.}$$

10. *Prove that* $\mathbf{\sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\sin\frac{11\theta}{2} = \sin 2\theta \sin 5\theta}$

$$\boxed{\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]}$$

$$L.H.S = \sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\sin\frac{11\theta}{2}$$

$$= \frac{1}{2}\left\{\cos\left(\frac{\theta}{2} - \frac{7\theta}{2}\right) - \cos\left(\frac{\theta}{2} + \frac{7\theta}{2}\right)\right\} + \frac{1}{2}\left\{\cos\left(\frac{3\theta}{2} - \frac{11\theta}{2}\right) - \cos\left(\frac{3\theta}{2} + \frac{11\theta}{2}\right)\right\}$$

$$= \frac{1}{2}\left[\cos\left(\frac{\theta - 7\theta}{2}\right) - \cos\left(\frac{\theta + 7\theta}{2}\right)\right] + \frac{1}{2}\left[\cos\left(\frac{3\theta - 11\theta}{2}\right) - \cos\left(\frac{3\theta + 11\theta}{2}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{-6\theta}{2}\right) - \cos\left(\frac{8\theta}{2}\right)\right] + \frac{1}{2}\left[\cos\left(\frac{-8\theta}{2}\right) - \cos\left(\frac{14\theta}{2}\right)\right]$$

$$= \frac{1}{2}\left[cos(-3\theta) - \cos 4\theta + \cos(-4\theta) - \cos 7\theta\right]$$

$$\boxed{\because \cos(-\theta) = cos\theta}$$

$$= \frac{1}{2}\left[\cos 3\theta - \cos 4\theta + \cos 4\theta - \cos 7\theta\right]$$

$$= \frac{1}{2}\left[\cos 3\theta - \cos 7\theta\right]$$

$$\boxed{\cos D - \cos C = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)}$$

$$= \frac{1}{2}\left\{2\sin\left(\frac{3\theta+7\theta}{2}\right)\sin\left(\frac{7\theta-3\theta}{2}\right)\right\} = \sin\left(\frac{10\theta}{2}\right)\sin\left(\frac{4\theta}{2}\right)$$

$= sin\, 5\theta\, sin\, 2\theta = R.H.S$ *Hence proved.*

11. *Prove that* $\boldsymbol{cos(30^\circ - A)\,cos(30^\circ + A) + cos(45^\circ - A)\,cos(45^\circ + A)}$

$$\boldsymbol{= \cos 2A + \frac{1}{4}.}$$

$$L.H.S = cos(30^\circ - A)\,cos(30^\circ + A) + cos(45^\circ - A)\,cos(45^\circ + A)$$

$$= \cos^2 30^\circ - \sin^2 A + \cos^2 45^\circ - \sin^2 A$$

$$\boxed{\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B}$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 - \sin^2 A + \left(\frac{1}{\sqrt{2}}\right)^2 - \sin^2 A$$

$$= \frac{3}{4} + \frac{1}{2} - 2\sin^2 A = \frac{3}{4} + \frac{1}{2} - (1 - \cos 2A)$$

$$\boxed{2\sin^2 A = 1 - \cos 2A}$$

$$= \frac{3}{4} + \frac{1}{2} - 1 + \cos 2A = \frac{3+2-4}{4} + \cos 2A$$

$$= \frac{1}{4} + \cos 2A = \cos 2A + \frac{1}{4} = R.H.S$$

Hence proved.

12. *Prove that* $$\boldsymbol{\frac{\sin x + sin\, 3x + sin\, 5x + sin\, 7x}{\cos x + cos\, 3x + cos\, 5x + cos\, 7x} = \tan 4x}$$

$$L.H.S = \frac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x}$$

$$= \frac{2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right) + 2\sin\left(\frac{7x+5x}{2}\right)\cos\left(\frac{7x-5x}{2}\right)}{2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) + 2\cos\left(\frac{7x+5x}{2}\right)\cos\left(\frac{7x-5x}{2}\right)}$$

$$= \frac{2\sin\left(\frac{4x}{2}\right)\cos\left(\frac{-2x}{2}\right) + 2\sin\left(\frac{12x}{2}\right)\cos\left(\frac{2x}{2}\right)}{2\cos\left(\frac{4x}{2}\right)\cos\left(\frac{2x}{2}\right) + 2\cos\left(\frac{12x}{2}\right)\cos\left(\frac{2x}{2}\right)}$$

$$= \frac{2\left[\sin 2x \cos x + \sin 6x \cos x\right]}{2\left[\cos 2x \cos x + \cos 6x \cos x\right]}$$

$$= \frac{\cos x\,(\sin 2x + \sin 6x)}{\cos x\,(\cos 2x + \cos 6x)} = \frac{\sin 2x + \sin 6x}{\cos 2x + \cos 6x}$$

$$= \frac{2\sin\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right)}{2\cos\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right)} = \frac{2\sin\left(\frac{8x}{2}\right)\cos\left(\frac{4x}{2}\right)}{2\cos\left(\frac{8x}{2}\right)\cos\left(\frac{4x}{2}\right)}$$

$$= \frac{\sin 4x \cos 2x}{\cos 4x \cos 2x} = \tan 4x = R.H.S \quad \textit{Hence proved.}$$

13. ***Prove that*** $\boldsymbol{\frac{\sin(4A-2B)+\sin(4B-2A)}{\cos(4A-2B)+\cos(4B-2A)} = \tan(A+B)}$

$$L.H.S = \frac{\sin(4A-2B)+\sin(4B-2A)}{\cos(4A-2B)+\cos(4B-2A)}$$

$$= \frac{2\sin\left(\frac{4A-2B+4B-2A}{2}\right)\cos\left(\frac{4A-2B-(4B-2A)}{2}\right)}{2\cos\left(\frac{4A-2B+4B-2A}{2}\right)\cos\left(\frac{4A-2B-(4B-2A)}{2}\right)}$$

$$= \frac{\cancel{2}\sin\left(\frac{4A-2B+4B-2A}{2}\right)\cos\left(\frac{4A-2B-4B+2A}{2}\right)}{\cancel{2}\cos\left(\frac{4A-2B+4B-2A}{2}\right)\cos\left(\frac{4A-2B-4B+2A}{2}\right)}$$

$$= \frac{\sin\left(\frac{2A+2B}{2}\right)\cancel{\cos\left(\frac{6A-6B}{2}\right)}}{\cos\left(\frac{2A+2B}{2}\right)\cancel{\cos\left(\frac{6A-6B}{2}\right)}} = \frac{\sin\left(\frac{\cancel{2}(A+B)}{\cancel{2}}\right)}{\cos\left(\frac{\cancel{2}(A+B)}{\cancel{2}}\right)}$$

$$= \frac{\sin(A+B)}{\cos(A+B)} = \tan(A+B) \quad \textit{Hence proved.}$$

14. Show that $\cot(A+15^\circ)-\tan(A-15^\circ)=\dfrac{4\cos 2A}{1+2\sin 2A}$

$$L.H.S=\cot(A+15^\circ)-\tan(A-15^\circ)$$

$$=\frac{\cos(A+15^\circ)}{\sin(A+15^\circ)}-\frac{\sin(A-15^\circ)}{\cos(A-15^\circ)}$$

$$=\frac{\cos(A+15^\circ)\cos(A-15^\circ)-\sin(A-15^\circ)\sin(A+15^\circ)}{\sin(A+15^\circ)\cos(A-15^\circ)}$$

$$=\frac{\cos^2 A-\sin^2 15^\circ-\left[\sin^2 A-\sin^2 15^\circ\right]}{\frac{1}{2}\left[\sin\left(A+\cancel{15^\circ}+A-\cancel{15^\circ}\right)+\sin\left(\cancel{A}+15^\circ-\cancel{A}+15^\circ\right)\right]}$$

$$=\frac{2\left[\cos^2 A-\cancel{\sin^2 15^\circ}-\sin^2 A+\cancel{\sin^2 15^\circ}\right]}{\sin 2A+\sin 30^\circ}$$

$$=\frac{2\,(\cos^2 A-\sin^2 A)}{\sin 2A+\frac{1}{2}}=\frac{2\,(\cos^2 A-\sin^2 A)}{\frac{2\sin 2A+1}{2}}$$

$$=2(\cos^2 A-\sin^2 A)\times\frac{2}{2\sin 2A+1}=\frac{4\,(\cos^2 A-\sin^2 A)}{2\sin 2A+1}$$

$$=\frac{4\cos 2A}{1+2\sin 2A}=R.H.S\quad Hence\ proved$$

EXERCISE : 3.7

Example* 3.39: *If* $A+B+C=\pi$ *, Prove that

$$(i)\ \cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$L.H.S=\cos A+\cos B+\cos C$

$$=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C$$

$$\boxed{\begin{aligned}A+B+C&=\pi\\A+B&=\pi-C\\\frac{A+B}{2}&=\frac{\pi-C}{2}\\\frac{A+B}{2}&=\frac{\pi}{2}-\frac{C}{2}\end{aligned}}$$

$$=2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\left(\frac{A}{2}-\frac{B}{2}\right)+\cos C$$

$$=2\sin\frac{C}{2}\cos\left(\frac{A}{2}-\frac{B}{2}\right)+1-2\sin^2\frac{C}{2}$$

$$=1+2\sin\frac{C}{2}\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\sin\frac{C}{2}\right]$$

$$=1+2\sin\frac{C}{2}\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\right]$$

$$=1+2\sin\frac{C}{2}\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\cos\left(\frac{A}{2}+\frac{B}{2}\right)\right]=1+2\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)$$

$$=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\qquad\boxed{\cos(A-B)-\cos(A+B)=2\sin A\sin B}$$

Example* 3.39: *If* $A+B+C=\pi$ *, Prove that $(ii)\ \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}$

Let $u=\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$$\boxed{\begin{aligned}&\cos(A-B)-\cos(A+B)=2\sin A\sin B\\&\frac{1}{2}[\cos(A-B)-\cos(A+B)]=\sin A\sin B\end{aligned}}$$

$$u=\frac{1}{2}\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right]\sin\frac{C}{2}$$

$$=-\frac{1}{2}\left[\cos\left(\frac{A+B}{2}\right)-\cos\left(\frac{A-B}{2}\right)\right]\sin\frac{C}{2}$$

$$=-\frac{1}{2}\left[\cos\left(\frac{A+B}{2}\right)-\cos\left(\frac{A-B}{2}\right)\right]\cos\frac{A+B}{2}$$

$$-2u=\left[\cos\left(\frac{A+B}{2}\right)-\cos\left(\frac{A-B}{2}\right)\right]\cos\frac{A+B}{2}$$

$$\cos^2\frac{A+B}{2}-\cos\frac{A-B}{2}\cos\frac{A+B}{2}+2u=0$$

which is in quadratic in $\cos\frac{A+B}{2}$

since $\cos\frac{A+B}{2}$ is real number, the above equation has a solution.

Thus, the discriminant $b^2-4ac\geq 0$, which gives

$$\cos^2\frac{A+B}{2}-8u\geq 0 \Rightarrow u\leq\frac{1}{8}\cos^2\frac{A+B}{2}\leq\frac{1}{8}$$

Hence, $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{8}$

Example 3.40: **Prove that** $\boldsymbol{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}}$

$$\boldsymbol{=1+4\sin\left(\frac{\pi-A}{2}\right)\sin\left(\frac{\pi-B}{2}\right)\sin\left(\frac{\pi-C}{2}\right), if\ A+B+C=\pi}$$

$$L.H.S=\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}$$

$$=\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)+\cos\left(\frac{\pi}{2}-\frac{B}{2}\right)+\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$=\left[2\cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\cos\left(\frac{B-A}{4}\right)\right]+\left[1-2\sin^2\left(\frac{\pi}{4}-\frac{C}{4}\right)\right]$$

$$=1+2\sin\left(\frac{\pi}{4}-\frac{C}{4}\right)\cos\left(\frac{B-A}{4}\right)-2\sin^2\left(\frac{\pi}{4}-\frac{C}{4}\right)$$

$$=1+2\sin\left(\frac{\pi}{4}-\frac{C}{4}\right)\left[\cos\left(\frac{B-A}{4}\right)-\sin\left(\frac{\pi}{4}-\frac{C}{4}\right)\right]$$

$$=1+2\sin\left(\frac{\pi}{4}-\frac{C}{4}\right)\left[\cos\left(\frac{B-A}{4}\right)-\sin\left(\frac{A+B}{4}\right)\right]$$

$$=1+2\sin\left(\frac{\pi}{4}-\frac{C}{4}\right)\left[\cos\left(\frac{B-A}{4}\right)-\cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\right]$$

$$=1+2\sin\left(\frac{\pi}{4}-\frac{C}{4}\right)\left[2\sin\left(\frac{\frac{B-A}{4}+\frac{\pi}{2}-\frac{A+B}{4}}{2}\right)-\sin\left(\frac{\frac{\pi}{2}-\frac{A+B}{4}-\frac{B-A}{4}}{2}\right)\right]$$

$$=1+4\sin\left(\frac{\pi-A}{2}\right)\sin\left(\frac{\pi-B}{2}\right)\sin\left(\frac{\pi-C}{2}\right)$$

Example 3.41: If $A+B+C=\pi$, prove that $\cos^2 A+\cos^2 B+\cos^2 C = 1-2\cos A\cos B\cos C$

$$L.H.S = \cos^2 A+\cos^2 B+\cos^2 C$$

$$\boxed{\cos^2 A = \frac{1+\cos 2A}{2}}$$

$$= \frac{1+\cos 2A}{2}+\frac{1+\cos 2B}{2}+\frac{1+\cos 2C}{2}$$

$$\boxed{A+B+C=\pi}$$

$$\boxed{A+B=\pi-C}$$

$$= \frac{3}{2}+\frac{1}{2}[\cos 2A+\cos 2B+\cos 2C]$$

$$= \frac{4}{2}\left(2\cos(A+B)\cos(A-B)\right) + \cos 2C$$

$$= 2\left(2\cos(\pi-C)\cos(A-B)\right) + 2\cos^2 C-1$$

$$= 2\left(2\cos(-C)\cos(A-B)\right) + 2\cos^2 C-1$$

$$= 2\left(2\ -\cos C\ \ \cos(A-B)\right) + 2\cos^2 C-1$$

$$= 2\left(2(-\cos C)\cos(A-B)\right)+ 2\cos^2 C-1$$

$$= -1+ 2\times -2\cos C\left(\cos(A-B)-\cos C\right)$$

$$= 1- 2\cos C\left(\cos(A-B) - \cos(\pi-(A+B))\right)$$

$$= 1-2\cos C\left(\cos(A-B) + \cos(A+B)\right)$$

$$= 1-2\cos C\ \cos A\cos B \qquad \cos(A-B)+\cos(A+B) = 2\cos A\cos B$$

$$= 1-2\cos A\cos B\ \cos C = R.H.S$$

(i) $A+B+C=180^\circ$, prove that $\sin 2A+\sin 2B+\sin 2C = 4\sin A\sin B\sin C$

$$L.H.S = \sin 2A+\sin 2B+\sin 2C$$

$$\boxed{\begin{array}{l} A+B+C=180^\circ \\ A+B=180^\circ-C \end{array}}$$

$$\sin C+\sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$= 2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)+ \sin 2C$$

$$\boxed{\begin{array}{l} A+B+C=180^\circ \\ C=180^\circ-(A+B) \end{array}}$$

$$= 2\sin(A+B)\ \cos(A-B)+\sin 2C$$

$$\boxed{\sin 2A = 2\sin A\cos A}$$

$$= 2\sin(180^\circ-C)\cos(A-B)+\sin 2C$$

$$= 2\sin C\cos(A-B)+2\sin C\cos C$$

$$= 2\sin C\ [\cos(A-B)+\cos C]$$

$$= 2\sin C\ [\cos(A-B)+\cos(180^\circ - A+B)\ ]$$

$= 2\sin C\ [\cos(A-B) - \cos(A+B)\ ]$

$= 2\sin C\ (2\sin A\sin B) = 4\ \sin A\sin B\sin C$

(ii) $If\ A+B+C = 180°$, prove that $\cos A + \cos B - \cos C$

$= -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

$$\boxed{\begin{gathered} A+B+C = 180° \\ A+B = 180° - C \\ \frac{A+B}{2} = \frac{180°-C}{2} \\ \frac{A+B}{2} = \frac{180°}{2} - \frac{C}{2} \\ \frac{A+B}{2} = 90° - \frac{C}{2} \end{gathered}}$$

$L.H.S = \cos A + \cos B - \cos C$

$$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \cos C$$

$$= 2\ \cos\left(90° - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) - \cos C$$

$$\boxed{\begin{gathered} A+B+C = 180° \\ C = 180° - (A+B) \\ \frac{C}{2} = \frac{180°}{2} - \left(\frac{A+B}{2}\right) \\ \frac{C}{2} = 90° - \left(\frac{A+B}{2}\right) \end{gathered}}$$

$$= 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - \left(1 - 2\,sin^2\frac{C}{2}\right)$$

$$= 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - 1 + 2\,sin^2\frac{C}{2}$$

$$= -1 + 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + 2\,sin^2\frac{C}{2}$$

$$= -1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \sin\frac{C}{2}\right]$$

$$= -1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \sin\left(90° - \frac{A+B}{2}\right)\right]$$

$$= -1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right]$$

$$= -1 + 2\sin\frac{C}{2}\left[2\cos\frac{A}{2}\cos\frac{B}{2}\right] = -1 + 4\ \cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$iii)$ $If A+B+C = 180°$, prove that $sin^2A + sin^2B + sin^2C$

$= 2 + 2\,cos\,A\,cos\,B\,cos\,C$

$\boxed{A+B+C = 180°}$

$\boxed{A+B = 180° - C}$

$L.H.S = \sin^2 A + \sin^2 B + \sin^2 C$

$= 1 - \underbrace{\cos^2 A + \sin^2 B} + \sin^2 C$

$= 1 - (\cos^2 A - \sin^2 B) + \sin^2 C$

$\boxed{\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B}$

$= 1 - \cos(A+B)\cos(A-B) + \sin^2 C$

$$= 1 - \cos(180^\circ - C)\cos(A-B) + \sin^2 C = 1 + \cos C \ \cos(A-B) + \sin^2 C$$

$$= 1 + \cos C\cos(A-B) + 1 - \cos^2 C \ = 2 + \underbrace{\cos C\cos(A-B) - \cos^2 C}$$

$$= 2 + \cos C\,[\cos(A-B) - \cos C] = 2 + \cos C\,[\cos(A-B) - \cos C]$$

$$= 2 + \cos C\,[\cos(A-B) - \cos(180^\circ - (A+B))] \qquad \boxed{A+B+C = 180^\circ}$$

$$= 2 + \cos C\,[\cos(A-B) + \cos(A+B)] \qquad \boxed{C = 180^\circ - (A+B)}$$

$$= 2 + \cos C\,(2\cos A\cos B) = 2 + 2\cos A\cos B\cos C$$

***iv*) *If* $A+B+C = 180^\circ$, prove that $\sin^2 A + \sin^2 B - \sin^2 C$**

$$= \mathbf{2\sin A\sin B\cos C}$$

$$L.H.S = \sin^2 A + \sin^2 B - \sin^2 C$$

$$\boxed{A+B+C = 180^\circ} \qquad \boxed{A+B = 180^\circ - C}$$

$$= 1 - \underbrace{\cos^2 A + \sin^2 B} - \sin^2 C$$

$$= 1 - (\cos^2 A - \sin^2 B) - \sin^2 C$$

$$\boxed{\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B}$$

$$= 1 - \cos(A+B)\cos(A-B) - \sin^2 C$$

$$= 1 - \cos(180^\circ - C)\cos(A-B) - \sin^2 C$$

$$= 1 + \cos C \ \cos(A-B) - \sin^2 C \quad = 1 + \cos C\cos(A-B) - (1 - \cos^2 C)$$

$$= 1 + \underbrace{\cos C\cos(A-B) - 1 + \cos^2 C}$$

$$= \cos C\cos(A-B) + \cos^2 C = \cos C\,[\cos(A-B) + \cos C]$$

$$= \cos C\,[\cos(A-B) + \cos C] \qquad \boxed{A+B+C = 180^\circ}$$

$$= \cos C\,[\cos(A-B) + \cos(180^\circ - (A+B)] \qquad \boxed{C = 180^\circ - (A+B)}$$

$$= \cos C\,[\cos(A-B) - \cos(A+B)]$$

$$= \cos C\,(2\sin A\sin B) \ = 2\sin A\sin B\cos C$$

(v) If $A+B+C = 180^\circ$, prove that

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1.$$

$$A+B+C = 180^\circ \ \Longrightarrow \frac{A}{2} + \frac{B}{2} = \frac{180^\circ}{2} - \frac{C}{2}$$

$$\tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\frac{180^\circ}{2} - \frac{C}{2}\right) \ \Longrightarrow \ \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \cot\frac{C}{2}$$

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{1}{\tan\frac{C}{2}} \implies \left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\tan\frac{C}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$$

$$\tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$$

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$$

(vi) If $A+B+C=180^\circ$, Prove that $\sin A+\sin B+\sin C$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\boxed{\begin{aligned}\sin 2A &= 2\sin A\cos A\\ \sin A &= 2\sin\frac{A}{2}\cos\frac{A}{2}\end{aligned}}$$

$L.H.S=\sin A\sin B\sin C$

$$=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\sin C$$

$$=2\sin\left(90^\circ-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+\sin C$$

$$=2\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right)+2\sin\frac{C}{2}\cos\frac{C}{2}=2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\sin\frac{C}{2}\right]$$

$$=2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\sin\left(90^\circ-\frac{A+B}{2}\right)\right]$$

$$=2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right]=2\cos\frac{C}{2}\left[2\cos\frac{A}{2}\cos\frac{B}{2}\right]$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\boxed{\begin{aligned}\cos(A-B)+\cos(A+B)&=2\cos A\cos B\\ \cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)&=2\cos\frac{A}{2}\cos\frac{B}{2}\end{aligned}}$$

vii) If $A+B+C=180^\circ$, prove that
$\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)=4\sin A\sin B\cos C$

$L.H.S=\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)$

$$=\sin(180^\circ-A-A)+\sin(180^\circ-B-B)+\sin(180^\circ-C-C)$$

$$=\sin(180^\circ-2A)+\sin(180^\circ-2B)+\sin(180^\circ-2C)$$

$$=\sin 2A+\sin 2B+\sin 2C$$

$$\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\boxed{\begin{aligned}A+B+C&=180^\circ\\ B+C&=180^\circ-A\\ C+A&=180^\circ-B\\ A+B&=180^\circ-C\end{aligned}}$$

$$= 2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) + \sin 2C$$

$$\boxed{\begin{aligned} A+B+C &= 180^\circ \\ A+B &= 180^\circ - C \end{aligned}}$$

$$= 2\sin(A+B)\cos(A-B) + \sin 2C$$

$$= 2\sin(180^\circ - C)\cos(A-B) + \sin 2C$$

$$\boxed{\begin{aligned} A+B+C &= 180^\circ \\ C &= 180^\circ - (A+B) \end{aligned}}$$

$$= 2\sin C\cos(A-B) + 2\sin C\cos C$$

$$= 2\sin C\,[\cos(A-B) + \cos C]$$

$$= 2\sin C\,[\cos(A-B) + \cos(180^\circ - \text{A+B})\,]$$

$$= 2\sin C\,[\cos(A-B) - \cos(\text{A+B})\,]$$

$$= 2\sin C\,(2\sin A\sin B) = 4\sin A\sin B\sin C$$

2) If $A+B+C=2s$, then prove that $\sin(s-A)\,sin(s-B) + \sin s\,sin(s-C) = sin\,A\,sin\,B$

$L.H.S = \sin(s-A)\sin(s-B) + \sin s\sin(s-C)$

$$\boxed{\begin{aligned} \cos(A-B) - \cos(A+B) &= 2\sin A\sin B \\ \frac{1}{2}[\cos(A-B) - \cos(A+B)] &= \sin A\sin B \end{aligned}}$$

$$\boxed{A+B+C = 2s}$$

$$= \frac{1}{2}\left[\cos(s-A-s+B) - \cos(s-A+s-B)\right] + \frac{1}{2}\left[\cos(s-s+C) - \cos(s+s-C)\right]$$

$$= \frac{1}{2}\left[\cos(-A+B) - \cos(2s-A-B)\right] + \frac{1}{2}\left[\cos C - \cos(2s-C)\right]$$

$$= \frac{1}{2}\left[\cos(B-A) \quad - \cos(\cancel{A} + \cancel{B} + C - \cancel{A} - \cancel{B})\right] + \frac{1}{2}\left[\cos C - \cos(A+B+\cancel{C}-\cancel{C})\right]$$

$$= \frac{1}{2}\left[\cos(B-A) - \cos C\right] + \frac{1}{2}\left[\cos C - \cos(A+B)\right]$$

$$= \frac{1}{2}\left[\cos(B-A) - \cancel{\cos C} + \cancel{\cos C} - \cos(A+B)\right]$$

$$= \frac{1}{2}\left[\cos[-(A-B)] - \cos(A+B)\right]$$

$$= \frac{1}{2}\left[\cos(A - B) - \cos(A + B) \right] = \frac{1}{\cancel{2}}(\cancel{2} \sin A \sin B) = \sin A \sin B$$

3) If $x + y + z = xyz$, then prove that

$$\frac{2x}{1 - x^2} + \frac{2y}{1 - y^2} + \frac{2z}{1 - z^2} = \frac{2x}{1 - x^2} \frac{2y}{1 - y^2} \frac{2z}{1 - z^2}$$

Let $x = \tan A$, $y = \tan B$, $z = \tan C$

Take: $A + B + C = 180°$

$$2A + 2B + 2C = 360° \Longrightarrow 2A + 2B = 360° - 2C$$

Taking tan on both sides

$$\tan 2A + \tan 2B = -\tan 2C\,(1 - \tan 2A \tan 2B)$$

$$\tan 2A + \tan 2B = -\tan 2C + \tan 2A \tan 2B \tan 2C$$

$$\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$$

$$\frac{2\tan A}{1 - \tan^2 A} + \frac{2\tan B}{1 - \tan^2 B} + \frac{2\tan C}{1 - \tan^2 C} = \left(\frac{2\tan A}{1 - \tan^2 A}\right)\left(\frac{2\tan B}{1 - \tan^2 B}\right)\left(\frac{2\tan C}{1 - \tan^2 C}\right)$$

$$\frac{2x}{1 - x^2} + \frac{2y}{1 - y^2} + \frac{2z}{1 - z^2} = \left(\frac{2\tan A}{1 - \tan^2 A}\right)\left(\frac{2\tan B}{1 - \tan^2 B}\right)\left(\frac{2\tan C}{1 - \tan^2 C}\right)$$

$$\frac{2x}{1 - x^2} + \frac{2y}{1 - y^2} + \frac{2z}{1 - z^2} = \left(\frac{2x}{1 - x^2}\right)\left(\frac{2y}{1 - y^2}\right)\left(\frac{2z}{1 - z^2}\right)$$

4)(i) If $A + B + C = \frac{\pi}{2}$, prove that $\sin 2A + \sin 2B + \sin 2C = 4\cos A \cos B \cos C$

$$L.H.S = \sin 2A + \sin 2B + \sin 2C$$

$$= 2\sin\left(\frac{2A + 2B}{2}\right)\cos\left(\frac{2A - 2B}{2}\right) + \sin 2C$$

$$\boxed{A + B + C = \frac{\pi}{2}}$$

$$\boxed{A + B = \frac{\pi}{2} - C}$$

$$\boxed{C = \frac{\pi}{2} - (A + B)}$$

$$= 2\sin(A + B)\cos(A - B) + \sin 2C$$

$$= 2\sin\left(\frac{\pi}{2} - C\right)\cos(A - B) + \sin 2C$$

$$= 2\cos C \cos(A - B) + 2\sin C \cos C$$

$$= 2\cos C\left[\cos(A - B) + \sin C\right]$$

$$= 2\cos C\left[\cos(A - B) + \sin\left(\frac{\pi}{2} - (A + B)\right)\right]$$

$$= 2\cos C\left[\cos(A - B) + \cos(A + B)\right]$$

$$= 2\cos C\left[2\cos A\cos B\right] = 4\cos A\cos B\cos C$$

4)(ii) If $A+B+C=\frac{\pi}{2}$, prove that $\cos 2A+\cos 2B+\cos 2C = 1+4\sin A\sin B\cos C$

$$L.H.S = \cos 2A + \cos 2B + \cos 2C$$

$$= 2\cos\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) + \cos 2C$$

$$\boxed{A+B+C=\frac{\pi}{2}}$$

$$\boxed{A+B=\frac{\pi}{2}-C}$$

$$\boxed{\cos 2A = 1-2\sin^2 A}$$

$$\boxed{C=\frac{\pi}{2}-(A+B)}$$

$$= 2\cos(A+B)\cos(A-B) + \cos 2C$$

$$= 2\cos\left(\frac{\pi}{2}-C\right)\cos(A-B) + \cos 2C$$

$$= 2\sin C\cos(A-B) + 1 - 2\sin^2 C$$

$$= \underbrace{2\sin C\ \cos(A-B) - 2\sin^2 C}\ + 1$$

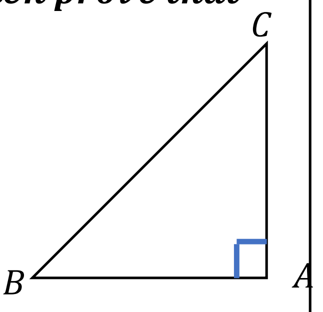
$$= 2\sin C\left[\cos(A-B) - \sin C\right] + 1$$

$$= 2\sin C\left[\cos(A-B) - \sin\left(\frac{\pi}{2}-(A+B)\right)\right] + 1$$

$$= 2\sin C\left[(2\sin A\sin B) - \cos(A+B)\right] + 1$$

$$= 1 + 2\sin C\cos(A-B) = 1 + 4\sin A\sin B\sin C$$

5)(i) If ΔABC, is a right triangle and if $\angle A = \frac{\pi}{2}$, then prove that $\cos^2 B + \cos^2 C = 1$



$$\angle A = \frac{\pi}{2} \quad then\ \ \angle B + \angle C = \frac{\pi}{2}$$

$$L.H.S = \cos^2 B + \cos^2 C$$

$$= \cos^2 B + 1 - \sin^2 C = 1 + \underbrace{\cos^2 B - \sin^2 C}$$

$$= 1 + \cos(B+C)\sin(B-C)$$

$$= 1 + \cos 90^\circ \sin(B-C) = 1 + (0)\sin(B-C) = 1$$

5)(ii) If ΔABC, is a right triangle and if $\angle A = \frac{\pi}{2}$, then prove that $\sin^2 B + \sin^2 C = 1$

$$L.H.S = \sin^2 B + \sin^2 C$$

$$= \sin^2 B + 1 - \cos^2 C = 1 + \sin^2 B - \cos^2 C$$

$$= 1 - (\cos^2 C - \sin^2 B) = 1 - \cos(C+B)\cos(C-B)$$

$$= 1 - \cos 90^\circ\cos(C-B) = 1 - (0)\cos(C-B) = 1$$

5)(iii) If ΔABC, is a right triangle and if $\angle A = \frac{\pi}{2}$, then prove that

$$\cos B - \cos C = -1 + 2\sqrt{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

Let us take : $\cos A + \cos B - \cos C$ *since* $A = 90^\circ$, $\cos A = 0$

$$= -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = -1 + 4\cos\frac{90^\circ}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$= -1 + 4\cos 45^\circ\cos\frac{B}{2}\sin\frac{C}{2} = -1 + 4\times\frac{1}{\sqrt{2}}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$= -1 + 2\times 2\times\frac{1}{\sqrt{2}}\cos\frac{B}{2}\sin\frac{C}{2} = -1 + 2\times\sqrt{2}\times\cancel{\sqrt{2}}\times\frac{1}{\cancel{\sqrt{2}}}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$= -1 + 2\sqrt{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$EXERCISE : 3.8$

Trigonometrical Equations

An equation involving trigonometrical function is called a trigonometrical equation.

$$Example : cos\theta = \frac{1}{2}, \tan\theta = 0, cos^2\theta - 2sin\theta = \frac{1}{2}$$

are some examples for trigonometrical equations.

A solution of a trigonometrical equation is the value of the unknown angle that satisfies the equation. A trigonometrical equation may have infinite number of solutions. The solution in which the absolute value of the angle is the least is called principal solution.

General solutions of sin θ = 0 $\theta = n\pi, n \epsilon Z$

General solutions of cos θ = 0 $\theta = (2n+1)\frac{\pi}{2}, n \epsilon Z$

General solutions of tan θ = 0 $\theta = n\pi, n \epsilon Z$

When a trigonometrical equation is solved, among all solutions the solution which is in intervals

For sine : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\Longrightarrow$ For tangent: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\Longrightarrow$ For cosine : [0,π]

General solutions of sin θ = sin α $\theta = n\pi + (-1)^n\alpha, n \epsilon Z$

General solutions of cos θ = cos α $\theta = 2n\pi \pm \alpha, n \epsilon Z$

General solutions of tan θ = tan α $\theta = n\pi + \alpha, n \epsilon Z$

Example 3.42 : Finding the principal value for $\boldsymbol{sin\theta = \frac{1}{2}}$

$$sin\theta = \frac{1}{2}$$

$$\boxed{sin30° = \frac{1}{2}}$$

$Since\ sin\theta\ is\ positive\ , \theta\ lies\ in\ the\ \boxed{first}$

$Principal\ value\ of\ must\ be\ in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ i.e. $\boxed{first}\ or\ second\ quadrant$

The principal value is in the $first$ quadrant.

$$sin\theta = \frac{1}{2} \Longrightarrow sin\theta = sin30° \Longrightarrow \theta = 30° \Longrightarrow \theta = \frac{\pi}{6}$$

Example 3.42: Find the principal value of : $\boldsymbol{(ii)\ sin\theta = -\frac{\sqrt{3}}{2}}$

$$sin\theta = -\frac{\sqrt{3}}{2}$$

$Since\ sinx\ is\ negative\ , x\ lies\ in\ the\ \ third\ or \boxed{fourth} quadrant$

$Principal\ value\ of\ must\ be\ in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ i.e. $first\ or \boxed{fourth}\ quadrant$

The principal value is in the $fourth$ quadrant. $\boxed{sin(-\theta) = -sin\theta}$

$$sin\theta = -\frac{\sqrt{3}}{2} \Rightarrow sin(-60°) = -sin60° \Rightarrow sin\ (-60°) = -\frac{\sqrt{3}}{2} \Rightarrow \theta = -60°$$

$\therefore$ The principal value of $\theta = -\frac{\pi}{6}$

Example 3.42: Find the principal value of : $(iii)\ cosec\theta = -2$

$$cosec\theta = -2 \Rightarrow sin\theta = -\frac{1}{2}$$

Since $sin\theta$ is negative, θ lies in the third or $\boxed{fourth}$ quadrant

Principal value of must be in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ i.e. *first or* $\boxed{fourth}$ *quadrant*

The principal value is in the $fourth$ quadrant. $\boxed{sin(-\theta) = -sin\theta}$

$$sin\theta = -\frac{1}{2} \Rightarrow sin(-30°) = -sin30° \Rightarrow sin(-30°) = -\frac{1}{2} \Rightarrow \theta = -30°$$

$\therefore$ The principal value of $\theta = -\frac{\pi}{6}$

Example 3.42: Find the principal value of the following : $(iv)\ cos\theta = \frac{1}{2}$

$$cos\theta = \frac{1}{2}$$

Since $cos\theta$ is positive, θ lies in the $\boxed{first}$ *or fourth*

Principal value of must be in $[0, \pi]$ i.e. $\boxed{first}$ *or second quadrant*

The principal value is in the $first$ quadrant.

$$cos\theta = \frac{1}{2} \Rightarrow cos\theta = cos60° \Rightarrow \theta = 60° \Rightarrow \theta = \frac{\pi}{3}$$

$\therefore$ The principal value of θ is $\frac{\pi}{3}$

Example 3.44: (i) *Find the General solution of $sec\theta = -\sqrt{2}$*

Finding the principal value for $cos\theta = -\frac{1}{\sqrt{2}}$

Since $cos\theta$ is negative, θ lies in the $\boxed{second}$ *or Third*

Principal value of must be in $[0, \pi]$ i.e. *first or* $\boxed{second}$ *quadrant*

The principal value is in the $second$ quadrant. $\boxed{cos(180° - \theta) = -cos\theta}$

$$cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow cos\theta = cos(180° - 45°)$$

$$cos\theta = \cos 130°$$

$$\cos\theta = \cos\alpha$$

$$\alpha = 135° \Rightarrow \alpha = \frac{3\pi}{4}$$

$$\cancel{135°}^{\cancel{27}\ \cancel{9}\ 3} \times \frac{\pi}{\cancel{180°}_{\cancel{36}\ \cancel{12}\ 4}} = \frac{3\pi}{4}$$

$\boxed{cos(180° - 45°) = -cos45°}$

General solutions of $\cos\theta = \cos\alpha$ $\theta = 2n\pi \pm \alpha, n \in Z$

$$\theta = 2n\pi \pm \frac{3\pi}{4}, n \in Z$$

$$\boxed{cos(180° - 45°) = -\frac{1}{\sqrt{2}}}$$

Example 3.44: (ii) ***Find the general solution of the equations:*** $\boldsymbol{\tan\theta = \sqrt{3}}$

Finding the principal value for $tan\theta = \sqrt{3}$

Since $tan\theta$ *is positive*, θ *lies in the* $\boxed{first}$ *or third quadrant*

Principal value of must be in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ i.e. $\boxed{first}$ *or fourth quadrant*

The principal value is in the *fourth* quadrant. $\boxed{tan(-\theta) = -tan\theta}$

$tan\theta = \sqrt{3} \Rightarrow tan\theta = tan(60°) \Rightarrow \tan\theta = \tan\alpha \Rightarrow \alpha = 60° \Rightarrow \boxed{\alpha = \frac{\pi}{3}}$

General solutions of tan θ = tan α $\Rightarrow \theta = n\pi + \alpha, n \in Z$

$$General\ solution\ is\ \theta = n\,\pi + \frac{\pi}{3}$$

Example 3.45: Solve $\mathbf{3\cos^2\theta = \sin^2\theta}$

$\boxed{\sin^2\theta + \cos^2\theta = 1}$

$\boxed{\sin^2\theta = 1 - \cos^2\theta}$

$\boxed{\cos^2 A = \frac{1 + \cos 2A}{2}}$

$3\cos^2\theta = \sin^2\theta$

$3\cos^2\theta = 1 - \cos^2\theta$

$3\cos^2\theta + \cos^2\theta = 1 \Rightarrow 4\cos^2\theta = 1$

$\cos^2\theta = \frac{1}{4} \Rightarrow \frac{1 + \cos 2\theta}{\cancel{2}} = \frac{1}{\cancel{4}_{\,2}} \Rightarrow 1 + \cos 2\theta = \frac{1}{2}$

$\cos 2\theta = \frac{1}{2} - 1 \Rightarrow \cos 2\theta = \frac{-1}{2} \Rightarrow cos\,2\theta = cos\frac{2\pi}{3}$

$$\theta = 2\theta,\ \alpha = \frac{2\pi}{3}$$

General solutions of cos θ = cos α

$$\theta = 2n\pi \pm \alpha, n \in Z$$

$$2\theta = 2n\pi \pm \frac{2\pi}{3}, n \in Z \Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in Z$$

$\div 2$

Example 3.46 ***Solve*** **:** $\boldsymbol{\sin x + \sin 5x = \sin 3x}$

$\sin x + \sin 5x = \sin 3x$

$\sin x + \sin 5x - \sin 3x = 0$

$$\boxed{\sin C + \sin D = 2\sin\left(\frac{C + D}{2}\right)cos\left(\frac{C - D}{2}\right)}$$

$2\sin\left(\frac{x + 5x}{2}\right)cos\left(\frac{x - 5x}{2}\right) - sin3x = 0$

$2\sin\left(\frac{6x}{2}\right)cos\left(-\frac{4x}{2}\right) - sin3x = 0 \Rightarrow 2\sin 3x\,cos(-2x) - sin3x = 0$

$2\sin 3x\,cos2x - sin3x = 0 \Rightarrow \sin 3x\,(2cos2x - 1) = 0$

$\sin 3x = 0,\ 2cos2x - 1 = 0 \Rightarrow \sin 3x = 0,\ cos2x = \frac{1}{2}$

Take : $sin3x = 0 \Rightarrow \sin 3x = \sin 0 \Rightarrow \sin\theta = \sin\alpha$

$\theta = 3x$, $\alpha = 0°$

$$\theta = n\pi + (-1)^n \alpha, n \epsilon Z$$

$$3x = n\pi + (-1)^n \times 0 \implies 3x = n\pi \implies x = \frac{n\pi}{3}$$

Take : $\cos 2x = \frac{1}{2}$

The principal value is in the *first* quadrant.

$$cos2x = cos60° \implies cos2x = cos\frac{\pi}{3}$$

$$\cos\theta = \cos\alpha$$

$$\theta = 2x, \ \alpha = \frac{\pi}{3}$$

$$\theta = 2n\pi \pm \alpha, n \epsilon Z$$

$$\therefore 2x = 2n\pi \pm \frac{\pi}{3} \implies x = n\pi \pm \frac{\pi}{6}$$

$\div 2$

The general solution $x = \frac{n\pi}{3}$ *and* $x = n\pi \pm \frac{\pi}{6}$

Example 3.47: Solve $\boldsymbol{\cos x + \sin x = \cos 2x + \sin 2x}$

$$\cos x - \cos 2x = \sin 2x - \sin x$$

$$2\sin\left(\frac{x+2x}{2}\right)\sin\left(\frac{2x-x}{2}\right) = 2\cos\left(\frac{2x+x}{2}\right)\sin\left(\frac{2x-x}{2}\right)$$

$$2\sin\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right) = 2\cos\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right)$$

$$2\sin\left(\frac{3x}{2}\right)\sin\left(\frac{3x}{2}\right) - 2\cos\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right) = 0$$

$$2\sin\left(\frac{x}{2}\right)\left[\sin\left(\frac{3x}{2}\right) - \cos\left(\frac{3x}{2}\right)\right] = 0 \implies \sin\left(\frac{x}{2}\right)\left[\sin\left(\frac{3x}{2}\right) - \cos\left(\frac{3x}{2}\right)\right] = 0$$

$$\sin\left(\frac{x}{2}\right) = 0, \ \sin\left(\frac{3x}{2}\right) - \cos\left(\frac{3x}{2}\right) = 0$$

Take: $\sin\left(\frac{x}{2}\right) = 0$,

$$\sin\theta = 0 \implies \theta = \frac{x}{2}$$

$$\theta = n\pi, n \in z$$

$$\frac{x}{2} = n\pi \implies x = 2n\pi$$

Take: $\sin\left(\frac{3x}{2}\right) - \cos\left(\frac{3x}{2}\right) = 0$

$$\sin\left(\frac{3x}{2}\right) = \cos\left(\frac{3x}{2}\right)$$

$$\frac{\sin\left(\frac{3x}{2}\right)}{\cos\left(\frac{3x}{2}\right)} = 1 \implies \tan\left(\frac{3x}{2}\right) = 1 \implies \tan\left(\frac{3x}{2}\right) = \tan\frac{\pi}{4}$$

$$\tan\theta = \tan\alpha$$

$$\theta = \frac{3x}{2}, \ \alpha = \frac{\pi}{4}$$

$$\theta = n\pi + \alpha \Longrightarrow \frac{3x}{2} = n\pi + {}^{\pi}/_{4} \Longrightarrow x = \frac{2}{3}\left(n\pi + \frac{\pi}{4}\right)$$

$$x = \frac{2n\pi}{3} + \frac{\cancel{2}\pi}{\cancel{12}_{\,6}} \Longrightarrow x = \frac{2n\pi}{3} + \frac{\pi}{6}, n \in z$$

Example 3.48: ***Solve the equation $\sin 9\theta = \sin\theta$***

$$\sin 9\theta - \sin\theta = 0$$

$$2\cos\left(\frac{9\theta+\theta}{2}\right)\sin\left(\frac{9\theta-\theta}{3}\right) = 0 \Longrightarrow 2\cos\left(\frac{10\,\theta}{2}\right)\sin\left(\frac{8\,\theta}{2}\right) = 0$$

$$2\cos(5\,\theta)\ \sin(4\,\theta) = 0 \Longrightarrow \cos 5\,\theta\ \sin 4\,\theta = 0$$

$$\cos 5\theta = 0,\ \sin 4\theta = 0$$

$$\cos\theta = 0,\ \sin\theta = 0$$

$$\boxed{\theta = 5\,\theta} \quad \boxed{\theta = 4\theta}$$

General Solution: $\boldsymbol{\theta = (2n+1)\frac{\pi}{2}, n \in z}$

$$Here\ \theta = 5\,\theta$$

$$5\,\theta = (2n+1)\frac{\pi}{2}, n \in z \Longrightarrow \theta = (2n+1)\frac{\pi}{10}, n \in z$$

General Solution: $\boldsymbol{\theta = n\pi}$

$$4\,\theta = n\pi \Longrightarrow \theta = \frac{n\pi}{4}$$

Example 3.50: ***Solve $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$***

$$\sin x - 3\sin 2x + \sin 3x = \cos 3x + \cos x - 3\cos 2x$$

$$2\sin\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) - 3\sin 2x = 2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) - 3\cos 2x$$

$$2\sin\left(\frac{4x}{2}\right)\cos\left(\frac{2x}{2}\right) - 3\sin 2x = 2\cos\left(\frac{4x}{2}\right)\cos\left(\frac{2x}{2}\right) - 3\cos 2x$$

$$2\sin 2x\ \cos x - 3\sin 2x = 2\cos 2x\ \cos x - 3\cos 2x$$

$$\sin 2x\,(2\cos x - 3) = \cos 2x(2\cos x - 3)$$

$$\sin 2x\,(2\cos x - 3) - \cos 2x\,(2\cos x - 3) = 0$$

$$(2\cos x - 3)(\sin 2x - \cos 2x) = 0$$

$$2\cos x - 3 = 0,\quad \sin 2x - \cos 2x = 0$$

$$2\cos x - 3 = 0,\quad \sin 2x = \cos 2x$$

$(not\ possible)$

$$\frac{\sin 2x}{\cos 2x} = 1 \Longrightarrow \tan 2x = 1$$

$$\tan 2x = \tan 45° \Longrightarrow \tan 2x = \tan {}^{\pi}/_{4}$$

$$\tan\theta = \tan\alpha \Longrightarrow \theta = 2x,\ \alpha = \frac{\pi}{4}$$

$$\theta = n\pi + \alpha$$

$$2x = n\pi + {}^{\pi}/_{4} \Longrightarrow x = \frac{n\pi}{2} + \frac{\pi}{8}$$

Example 3.51: ***Solve*** $\boldsymbol{\sin x + \cos x = 1 + \sin x \cos x}$

Let $\sin x + \cos x = t$

$t = \sin x + \cos x$

squaring on both side

$$t^2 = (\sin x + \cos x)^2 \Longrightarrow t^2 = \sin^2 x + \cos^2 x + 2\sin x \cos x$$

$$t^2 = 1 + 2\sin x \cos x \Longrightarrow t^2 - 1 = 2\sin x \cos x$$

$$\frac{t^2 - 1}{2} = \sin x \cos x$$

$$\sin x + \cos x = 1 + \sin x \cos x \Longrightarrow t = 1 + \frac{t^2 - 1}{2}$$

$$t = \frac{2 + t^2 - 1}{2} \Longrightarrow 2t = 2 + t^2 - 1$$

$$2t = t^2 + 1 \Longrightarrow t^2 - 2t + 1 = 0$$

$$(t-1),(t-1) = 0 \Longrightarrow t - 1 = 0$$

$$t = 1 \Longrightarrow \sin x + \cos x = 1$$

$$a = 1, \quad b = 1$$

$$\sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\sin x + \cos x = 1 \quad \div \sqrt{2} \Longrightarrow \frac{\sin x}{\sqrt{2}} + \frac{\cos x}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\sqrt{2}\left(\frac{\sin x}{\sqrt{2}} + \frac{\cos x}{\sqrt{2}}\right) = 1 \Longrightarrow \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) = 1$$

$$\sqrt{2}\left(\cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4}\right) = 1 \Longrightarrow \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) = 1$$

$$\cos\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \Longrightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$$

$$\cos\theta = \cos\alpha \Longrightarrow \theta = x - \frac{\pi}{4}, \ \alpha = {}^{\pi}/_{4}$$

General Solution: $\boldsymbol{\theta = (2n+1)\,\alpha, n \in z}$

$$x - \frac{\pi}{4} = 2n\pi \pm \alpha, \ n \in z \Longrightarrow x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$$

$$x - \frac{\pi}{4} = 2n\pi + \frac{\pi}{4}, \quad x - \cancel{\frac{\pi}{4}} = 2n\pi - \cancel{\frac{\pi}{4}}$$

$$x = 2n\pi + \frac{\pi}{4} + \frac{\pi}{4}, \ x = 2n\pi \Longrightarrow x = 2n\pi + \frac{2\pi}{\cancel{4}_{\,2}} \Longrightarrow x = 2n\pi + \frac{\pi}{2}$$

Example 3.52: ***Solve***: $\boldsymbol{2\sin^2 x + \sin^2 2x = 2}$

$$2\sin^2 x + (\sin 2x)^2 = 2 \Longrightarrow 2\sin^2 x + (2\sin x \cos x)^2 = 2$$

$$2\sin^2 x + 4\sin^2 x \cos^2 x = 2 \Longrightarrow 2(1 - \cos^2 x) + 4\sin^2 x \cos^2 x = 2$$

$$2 - 2\cos^2 x + 4\sin^2 x \cos^2 x = 2 \Longrightarrow 4\sin^2 x \cos^2 x - 2\cos^2 x = 2 - 2$$

$$2cos^2x(2\,sin^2x-1)=0 \implies cos^2x\,(2\,sin^2x-1)=0$$
$$cos^2x=0,\ 2\,sin^2x-1=0$$
$$cos\,x=0,\ 2\,sin^2x=1$$
$$[cos\,\theta=0]$$

General Solution: $\theta=(2n+1)\,\dfrac{\pi}{2}\,,n\in z \implies x=(2n+1)\dfrac{\pi}{2},n\in z$

$$sin^2 2x={}^1\!/_2 \implies sin\,x=\pm\frac{1}{\sqrt{2}} \implies sin\,x=\frac{1}{\sqrt{2}}\ or\ sin\,x=-\frac{1}{\sqrt{2}}$$
$$sin\,x=sin\frac{\pi}{4}\ or\ sin\,x=-\,sin\frac{\pi}{4}$$

$Take: sin\,x=sin\frac{\pi}{4}$ $sin\,\theta=sin\,\alpha$ $\theta=x\,,\alpha={}^{\pi}\!/_4$ $\theta=n\pi+(-1)^n\alpha$ $x=n\pi+(-1)^n\,{}^{\pi}\!/_4$	$Take: sin\,x=-\,sin\frac{\pi}{4}$ $\theta=x\,,\ \alpha=-{}^{\pi}\!/_4$ $\theta=n\pi+(-1)^n\,\alpha \implies \theta=n\pi+(-1)^n\left(-{}^{\pi}\!/_4\right)$ $\theta=n\pi+(-1)^{n+1}\,{}^{\pi}\!/_4 \implies \theta=n\pi\pm{}^{\pi}\!/_4$

Example* 3.54**: ***Solve $\sqrt{3}\ sin\,\theta-\ cos\,\theta\ =\sqrt{2}$

$$a=\sqrt{3},\ \ b=-1$$
$$r\ =\sqrt{a^2+b^2}\implies r\ =\sqrt{(\sqrt{3})^2+1^2}\implies r=\sqrt{3+1}$$
$$r=\sqrt{4}\ \implies \boxed{r=2}$$
$$\sqrt{3}\ sin\,\theta-cos\,\theta=\sqrt{2} \implies \frac{\sqrt{3}}{2}\ sin\,\theta-\frac{1}{2}cos\,\theta=\frac{\sqrt{2}}{2}$$
$$\div\,2$$
$$\frac{\sqrt{3}}{2}sin\,\theta-\frac{1}{2}cos\,\theta=\frac{\cancel{\sqrt{2}}}{\sqrt{2}\times\cancel{\sqrt{2}}} \implies \frac{\sqrt{3}}{2}sin\,\theta-\frac{1}{2}cos\,\theta=\frac{1}{\sqrt{2}}$$
$$sin\,\theta\,cos\,30^\circ-cos\,\theta\,sin\,30^\circ=\frac{1}{\sqrt{2}} \implies sin(\theta-30^\circ)=\frac{1}{\sqrt{2}}$$
$$sin\left(\theta-\frac{\pi}{6}\right)=sin45^\circ \implies sin\left(\theta-\frac{\pi}{6}\right)=sin\frac{\pi}{4}$$
$$sin\,\theta\ =sin\,\alpha \implies \theta\ =\ \theta-\frac{\pi}{6},\ \alpha=\frac{\pi}{6}$$

General solution : $\theta=\ n\pi\pm\ (-1)^n\alpha,n\in\mathbb{Z}$

$$\theta\ =\ n\pi\pm\ (-1)^n\frac{\pi}{4}\ \implies \theta\ =\ n\pi+\ (-1)^n\frac{\pi}{4}\ ,n\in\mathbb{Z}$$

Example* 3.55**: ***Solve $\boldsymbol{\sqrt{3}tan^2\theta+\left(\sqrt{3}-1\right)tan\,\theta-1=0}$

$$\sqrt{3}\,tan^2\,\theta+\sqrt{3}\ tan\,\theta-tan\,\theta-1=0$$
$$\sqrt{3}\,tan^2\,\theta\,(tan\,\theta\ -1)-(tan\,\theta\ +1)=0$$

$(tan\,\theta\ +1)(\sqrt{3}\,tan\,\theta\ -1) = 0 \implies tan\,\theta\ +1 = 0, (\sqrt{3}\,tan\,\theta\ -1) = 0$

$tan\,\theta\ = -1,\ \sqrt{3}\,tan\,\theta\ = 1$

$$tan\,\theta\ = \frac{1}{\sqrt{3}}$$

$Take : tan\,\theta = -1$

$tan\,\theta = tan(-45^\circ) \implies tan\,\theta = tan\left(-\frac{\pi}{4}\right),$

$tan\,\theta = tan\,\alpha \implies \theta = \theta,\ \alpha = -\frac{\pi}{4}$

$\boldsymbol{General\ solution}$: $\theta = n\pi + \alpha\ , n \in \mathbb{Z}$

$$\theta = n\pi - \frac{\pi}{4}$$

$Take : tan\,\theta = \frac{1}{\sqrt{3}}$

$tan\,\theta = tan\,30^\circ \implies tan\,\theta = tan\frac{\pi}{6}$

$tan\,\theta = tan\,\alpha \implies \theta = \theta, \alpha = \frac{\pi}{6}$

$$\theta = n\pi + \alpha \implies \theta = n\pi + \frac{\pi}{6}$$

Ex : 1 (i) Find the principal value and General solution : $\boldsymbol{sin\theta = -\frac{1}{\sqrt{2}}}$

$$sin\theta = -\frac{1}{\sqrt{2}}$$

$Since\ sinx\ is\ negative\ ,x\ lies\ in\ the\ third\ or\ \boxed{fourth}\ quadrant$

$Principal\ value\ of\ must\ be\ in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ i.e. $first\ or\ \boxed{fourth}\ quadrant$

The principal value is in the $fourth$ quadrant. $\boxed{sin(-\theta) = -sin\theta}$

$$sin\theta = -\frac{1}{\sqrt{2}} \implies sin\theta = sin(-45°)$$

$$sin\theta = sin\left(-\frac{\pi}{4}\right)$$

$\mathbf{sin\ \theta = sin\ \alpha} \implies \theta = \theta,\ \alpha = -\frac{\pi}{4}$

∴ The principal value of α $= -\boldsymbol{\frac{\pi}{4}}$

General solutions : sin θ = sin α is $\theta = n\pi + (-1)^n\alpha, n \epsilon Z$

$$\theta = n\pi + (-1)^n\left(-\frac{\pi}{4}\right), n\,\epsilon\, Z$$

$(\boldsymbol{ii})$ **Find the principal value and General solution :** $\boldsymbol{cot\theta = \sqrt{3}}$

$$cot\theta = \sqrt{3} \implies tan\theta = \frac{1}{\sqrt{3}}$$

$Since\ tan\theta\ is\ positive\ ,\theta\ lies\ in\ the\ \boxed{First}\ or\ third\ quadrant$

Principal value of must be in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ i.e. $\boxed{first}$ *or fourth quadrant*

The principal value is in the *first* quadrant.

$$tan\theta = \frac{1}{\sqrt{3}} \implies \theta = 30°$$

∴ The principal value of $\theta = \frac{\pi}{6}$

$$tan\theta = \frac{1}{\sqrt{3}} \implies tan\theta = tan30° \implies \begin{matrix} tan\theta = tan\frac{\pi}{6} \\ tan\,\theta = tan\,\alpha \end{matrix} \implies \theta = \theta, \alpha = \frac{\pi}{6}$$

$\boldsymbol{General\ solution: \theta = n\pi + \alpha}$

$$\theta = n\pi + \frac{\pi}{6}$$

$(\boldsymbol{iii})$ **Find the principal value and General solution:** $\boldsymbol{tan\theta = -\frac{1}{\sqrt{3}}}$

$$tan\theta = -\frac{1}{\sqrt{3}}$$

Since $tan\theta$ *is negative* , θ *lies in the Second or* $\boxed{fourth}$ *quadrant*

Principal value of must be in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ i.e. *first or* $\boxed{fourth}$ *quadrant*

The principal value is in the *fourth* quadrant.

$$tan\theta = -\frac{1}{\sqrt{3}} \implies tan\theta = tan(-30°) \implies tan\theta = tan\left(-\frac{\pi}{6}\right)$$

∴ **The principal value of** $\boldsymbol{\theta = -\frac{\pi}{6}}$

$$\begin{matrix} tan\theta = tan\left(-\frac{\pi}{6}\right) \\ tan\,\theta = tan\,\alpha \end{matrix} \implies \theta = \theta,\ \ \alpha = -\frac{\pi}{6}$$

$\boldsymbol{General\ solution: \theta = n\pi + \alpha}$

$$\theta = n\pi - \frac{\pi}{6}$$

$\boldsymbol{Ex}$: 2. $\boldsymbol{Solve\ the\ following\ equations\ for\ which\ solutions\ lies\ in\ the\ interval\ 0° \le \theta\ 360°}$

$(\boldsymbol{i})$ $\boldsymbol{sin^4x = sin^2x}$

$$sin^4x - sin^2x = 0 \implies sin^2x\,(sin^2x - 1) = 0$$

$$sin^2x = 0,\quad sin^2x - 1 = 0 \implies sin\,x = 0,\ sin^2x = 1$$

$$sin\,x = \sqrt{1} \implies sin\,x = \pm 1$$

$$sin\,x = 1,\ \ sinx = -1,$$

$\boldsymbol{\sin x = 0}$

$\theta = x$

General solution $: \sin\theta = 0$

$\theta = n\pi, n \in \mathbb{Z}$

$\boldsymbol{\sin x = 1}$

$$\sin x = \sin 90^\circ \Longrightarrow \sin x = \sin\frac{\pi}{2}$$
$$\sin\theta = \sin\alpha \quad \Longrightarrow \quad \theta = x,\ \alpha = \frac{\pi}{2}$$

General solution $: \boldsymbol{\sin\theta = \sin\alpha}$

$$\theta = n\pi + (-1)^n\alpha,\ n \in \mathbb{Z}$$

$$x = n\pi + (-1)^n\frac{\pi}{2},$$

$\sin x = -1$

$\sin x = \sin(-90^\circ)$

$$\sin x = \sin\left(-\frac{\pi}{2}\right) \Longrightarrow \theta = x,\ \alpha = \frac{\pi}{2}$$

General solution: $\boldsymbol{\sin x = \sin\alpha}$

$$\theta = n\pi + (-1)^n\alpha$$

$$x = n\pi + (-1)^n\left(-\frac{\pi}{2}\right),\ n \in \mathbb{Z}$$

(ii) $\boldsymbol{2\cos^2 x + 1 = -3\cos x}$

$2\cos^2 x + 3\cos x + 1 = 0$

Let $y = \cos x$

$$2y^2 + 3y + 1 = 0 \Longrightarrow 2y^2 + y + 2y + 1 = 0$$

$$y(2y+1) + 1(2y+1) = 0 \Longrightarrow (2y+1)(y+1) = 0$$

$$2y + 1 = 0,\ y + 1 = 0 \Longrightarrow 2y = 1,\ y = -1$$

$$y = -\frac{1}{2}$$

$$\boxed{120^\circ = \cancel{120^\circ}^{\,2} \times \frac{\pi}{\cancel{180^\circ}_{\,3}} = \frac{2\pi}{3}}$$

$$\cos x = -\frac{1}{2},\ \cos x = -1$$

Take $: \cos x = -\frac{1}{2},$

$$\cos x = \cos(180^\circ - 60^\circ) \Longrightarrow \cos x = \cos 120^\circ$$

$$\cos x = \cos\frac{2\pi}{3}$$
$$\cos\theta = \cos\alpha \quad \Longrightarrow \quad \theta = x,\ \alpha = \frac{2\pi}{2}$$

General solution: $\boldsymbol{\theta = n\pi \pm \alpha, n} \in \mathbb{Z} \Longrightarrow \theta = 2n\pi \pm \frac{2\pi}{3}$

$Take : \cos x = -1$

$$\cos x = \cos 180^{\circ} \implies \cos x = \cos \pi$$

$$\cos \theta = \cos \alpha$$

$\theta = x,\ \alpha = \pi$

$\boldsymbol{General\ solution}: \boldsymbol{\theta = 2n\pi \pm \alpha, n \in \mathbb{Z}}$

$x = 2n\pi \pm \pi,$

$(\boldsymbol{iii})\ \boldsymbol{2sin^2 x + 1 = 3\sin x}$

$-$ $\times$

3 2

-1 -2

$2sin^2 x - 3\sin x + 1 = 0$

$Let \quad y = \sin x$

$2y^2 - 3y + 1 = 0 \implies 2y^2 - y - 2y + 1 = 0$

$y\,(2y - 1) - 1\,(2y - 1) = 0 \implies (2y - 1)\ (y - 1) = 0$

$2y - 1 = 0, y - 1 = 0 \implies 2y = 1\,, y = 1$

$$y = \frac{1}{2}$$

$\therefore y = \dfrac{1}{2}\ or\ y = 1$

$sinx = \dfrac{1}{2}\ or\ sinx = 1$

$Take : sinx = \dfrac{1}{2}$

$\sin x = \sin 30^{\circ} \implies \sin x = \sin\dfrac{\pi}{6} \implies \theta = x,\ \alpha = \dfrac{\pi}{6}$

$\sin\theta = \sin\alpha$

General solutions : $\boldsymbol{\theta = n\pi + (-1)^n \alpha, n\ \epsilon\ Z}$

$x = n\pi + (-1)^n \dfrac{\pi}{6}$

Take : $\sin x = 1$

$\sin x = \sin\dfrac{\pi}{2} \implies \theta = x,\ \alpha = \dfrac{\pi}{2}$

$\sin\theta = \sin\alpha$

General solutions : $\boldsymbol{\theta = n\pi + (-1)^n \alpha}$

$x = n\pi + (-1)^n \dfrac{\pi}{2}$

$(\boldsymbol{iv})\ \cos \boldsymbol{2x = 1 - 3\sin x}$

$1 - 2sin^2 x + 3\sin x - 1 = 0 \implies -2sin^2 x + 3\sin x = 0$

$3\sin x - 2sin^2 x = 0 \implies \sin x\,(3 - 2\sin x) = 0$

$\sin x = 0\,, 3 - 2\sin x = 0$

$\sin\theta = 0\,, -2\sin x = -3$

$$2\sin x = 3 \Longrightarrow \sin x = \frac{3}{2}$$

$$\sin x = \frac{3}{2} \notin [-1,1]$$

$\boldsymbol{General\ solutions}$: $\mathbf{\sin\theta = 0}$

$\boldsymbol{\theta = n\pi,\ n \in Z}$

$x = n\pi$

3. $\boldsymbol{Solve\ the\ following\ equations.}$

$\boldsymbol{(i)}$ $\mathbf{\sin 5x - \sin x = \cos 3x}$

$$2\cos\left(\frac{5x+x}{2}\right)\sin\left(\frac{5x-x}{2}\right) = \cos 3x \Longrightarrow 2\cos\left(\frac{\overset{3}{\cancel{6}}x}{\cancel{2}}\right)\sin\left(\frac{\overset{2}{\cancel{4}}x}{\cancel{2}}\right) = \cos 3x$$

$$2\cos 3x \sin 2x = \cos 3x \Longrightarrow 2\cos 3x\sin 2x - \cos 3x = 0$$

$$\cos 3x\,(2\sin 2x - 1) = 0 \Longrightarrow \cos 3x = 0,\ 2\sin 2x - 1 = 0$$

$$\cos 3x = \cos 90°, \qquad 2\sin 2x = 1$$

$$\cos 3x = \cos\frac{\pi}{2}, \qquad \sin 2x = \frac{1}{2}$$

$$Take: \cos 3x = \cos\frac{\pi}{2} \Longrightarrow \theta = 3x,\ \alpha = \frac{\pi}{2}$$

$$\cos\theta = \cos\alpha$$

$\boldsymbol{General\ Solution}$: $\boldsymbol{\theta = 2n\pi \pm \alpha, n \in Z}$

$$3x = 2n\pi \pm \frac{\pi}{2} \Longrightarrow x = \frac{2n\pi}{3} + \frac{\pi}{6}$$

$$Take: \sin 2x = \frac{1}{2}$$

$$\sin 2x = \sin 30° \Longrightarrow \sin 2x = \sin\frac{\pi}{6}$$

$$\sin\theta = \sin\alpha \Longrightarrow \theta = 2x,\ \alpha = \frac{\pi}{6}$$

$\boldsymbol{General\ Solution}$: $\boldsymbol{\theta = n\pi + (-1)^n\alpha}$

$$2x = n\pi + (-1)^n\frac{\pi}{6} \Longrightarrow x = \frac{n\pi}{2} + (-1)^n\frac{\pi}{12}$$

$\boldsymbol{(ii)}$ $\mathbf{2\cos^2\theta + 3\sin\theta - 3 = 0}$

$$2(1-\sin^2\theta) + 3\sin\theta - 3 = 0 \Longrightarrow 2 - 2\sin^2\theta + 3\sin\theta - 3 = 0$$

$$-2\sin^2\theta + 3\sin\theta - 1 = 0 \Longrightarrow 2\sin^2\theta - 3\sin\theta + 1 = 0$$

$$Let\ x = \sin\theta$$

$$2x^2 - 3x + 1 = 0 \Longrightarrow 2x^2 - x - 2x + 1 = 0$$

$+$ $\times$
-3 2
-1 -2

$$x(2x-1) - 1(2x-1) = 0$$

$$(2x-1)(x-1) = 0$$

$2x-1=0, x-1=0$

$2x=1,\ x=1$

$x=\frac{1}{2}$

$\therefore x=\frac{1}{2}, x=1 \Rightarrow\ sin\theta=\frac{1}{2}, sinx=1$

$Take:\ sin\theta=\frac{1}{2}$

$sin\,\theta = sin\,30° \ \Rightarrow sin\,\theta = sin\frac{\pi}{6}$
$sin\,\theta = sin\,\alpha \ \Rightarrow\ \theta=\theta, \alpha=\frac{\pi}{6}$

$\boldsymbol{General\ Solution:\ \theta = n\pi + (-1)^n\alpha, n\,\epsilon\, Z}$

$$\theta = n\pi + (-1)^n\frac{\pi}{6}, n\,\epsilon\, Z$$

$Take:\ sin\theta = 1$

$sin\,\theta = sin 90° \Rightarrow sin\,\theta = \ sin\frac{\pi}{2}$
$sin\,\theta = \ sin\,\alpha \ \Rightarrow\ \theta=\theta, \alpha=\frac{\pi}{2}$

$\boldsymbol{General\ Solution: \theta = n\pi + (-1)^n\alpha, n\,\epsilon\, Z}$

$$\theta = n\pi + (-1)^n\frac{\pi}{2}, n\,\epsilon\, Z$$

$\boldsymbol{(iii)\ cos\,\theta + cos\,3\theta\ = 2\,cos\,2\theta}$

$\cos\theta + cos\,3\theta\ = 2\,cos\,2\theta$

$2\,cos\left(\frac{\theta+3\theta}{2}\right)cos\left(\frac{\theta-3\theta}{2}\right) = 2\cos 2\theta$

$2\ cos\left(\frac{4\theta}{2}\right)\ cos\left(\frac{-2\theta}{2}\right)\ = 2\cos 2\theta \Rightarrow 2\,cos2\theta\ \cos(-\theta)\ = 2\cos 2\theta$

$2\,cos2\theta\,cos\theta\ = 2\cos 2\theta \Rightarrow 2\cos2\theta\,cos\theta\ - 2\,cos2\theta = 0$

$2\,cos2\theta\ (cos\theta - 1) = 0 \Rightarrow 2\,cos2\theta\ = 0,\ cos\theta - 1 = 0$

$cos2\theta = 0,\ cos\theta = 1$

$Take: cos2\theta = 0,$

$\cos 2\theta = cos90° \Rightarrow\ cos2\theta = cos\frac{\pi}{2}$
$cos\theta = cos\alpha \Rightarrow \theta = 2\theta, \alpha = \frac{\pi}{2}$

$\boldsymbol{General\ Solution: \theta = 2n\pi \pm \alpha}$

$2\theta = 2n\pi \pm \frac{\pi}{2} \Rightarrow\ \theta = \frac{2n\pi}{2} \pm \frac{\pi}{4} \Rightarrow\ \theta = n\pi \pm \frac{\pi}{4}$
$\div 2$

$Take : cos\theta = 1$

$$\begin{matrix} cos\theta = cos0 \\ cos\theta = cos\alpha \end{matrix} \implies \theta = \theta, \alpha = 0$$

$\boldsymbol{General\ Solution: \theta = 2n\pi \pm \alpha}$

$$\theta = 2n\pi \pm 0 \implies \theta = 2n\pi$$

(iv) $\boldsymbol{\sin\theta + sin\,3\theta + sin5\theta = 0}$

$$sin\,5\theta + sin\,\theta + sin\,3\theta = 0$$

$$2\,sin\left(\frac{5\theta + \theta}{2}\right) cos\left(\frac{5\theta - \theta}{2}\right) + \sin 3\theta = 0$$

$$2\,sin\left(\frac{6\theta}{2}\right) cos\left(\frac{4\theta}{2}\right) + sin3\theta = 0$$

$$2\,sin3\theta\ \cos2\theta + sin3\theta = 0 \implies sin3\theta\,(2cos\theta + 1) = 0$$

$$sin3\theta = 0,\ 2\,cos2\theta + 1 = 0$$

$$,2\,cos2\theta = -1$$

$$,cos2\theta = -\frac{1}{2}$$

$Take : sin3\theta = 0$

$$\begin{matrix} sin3\theta = sin0 \\ sin\theta = sin\alpha \end{matrix} \implies \theta = 3\theta,\ \alpha = 0$$

$\boldsymbol{General\ Solution: \theta = n\pi + (-1)^n\alpha}$

$$3\theta = n\pi + (-1)^n(0) \implies 3\theta = n\pi \implies \theta = \frac{n\pi}{3}$$

$Take:\ cos2\theta = -\frac{1}{2}$

$$cos2\theta = cos(180° - 60°) \implies cos2\theta = \cos120°$$

$$\begin{matrix} cos2\theta = cos\frac{2\pi}{3} \\ cos\theta = cos\alpha \end{matrix} \implies \theta = 2\theta,\ \alpha = \frac{2\pi}{3}$$

$\boldsymbol{General\ Solution: \theta = 2n\pi \pm \alpha, n \in Z}$

$$2\theta = 2n\pi \pm \frac{2\pi}{3} \implies \theta = n\pi \pm \frac{\pi}{3}$$

(v) $\boldsymbol{sin\,2\theta - cos2\theta - sin\theta + cos\theta = 0}$

$$sin\,2\theta - sin\,\theta + cos\,\theta - cos2\theta = 0$$

$$2\,cos\left(\frac{2\theta + \theta}{2}\right) sin\left(\frac{2\theta - \theta}{2}\right) + 2\sin\left(\frac{\theta + 2\theta}{2}\right) + \sin\left(\frac{2\theta - \theta}{2}\right) = 0$$

$$2\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)+2\sin\frac{3\theta}{2}\ \sin\frac{\theta}{2}=0$$

$$2\sin\frac{\theta}{2}\left[\cos\left(\frac{3\theta}{2}\right)+\sin\left(\frac{3\theta}{2}\right)\right]=0 \implies 2\sin\frac{\theta}{2}=0,\ \cos\frac{3\theta}{2}+\sin\frac{3\theta}{2}=0$$

$$\sin\frac{\theta}{2}=0,\ \sin\frac{3\theta}{2}=-\cos\frac{3\theta}{2}$$

$$Take:\sin\frac{\theta}{2}=0$$

$$\sin\frac{\theta}{2}=\sin 0$$
$$\sin\theta=\sin\alpha, \implies \theta=\frac{\theta}{2},\ \alpha=0,$$

$$\textbf{General Solution}: \boldsymbol{\theta = n\pi + (-1)^n\alpha}$$

$$\frac{\theta}{2}=n\pi+(-1)^n(0) \implies \frac{\theta}{2}=n\pi \implies \theta=2n\pi$$

$$Take:\sin\frac{3\theta}{2}=-\cos\frac{3\theta}{2}$$

$$\frac{\sin\frac{3\theta}{2}}{\cos\frac{3\theta}{2}}=-1 \implies \tan\frac{3\theta}{2}=-1 \implies \tan\frac{3\theta}{2}=\tan(-45°)$$

$$\tan\frac{3\theta}{2}=\tan\left(-\frac{\pi}{4}\right)$$
$$\tan\theta=\tan\alpha \implies \theta=\frac{3\theta}{2},\ \alpha=-\frac{\pi}{4}$$

$$\textbf{General Solution}: \boldsymbol{\theta = n\pi + \alpha, n \in Z}$$

$$\frac{3\theta}{2}=n\pi-\frac{\pi}{4} \implies \theta=\frac{n\pi}{\frac{3}{2}}-\frac{\frac{\pi}{4}}{\frac{3}{2}} \implies \theta=\frac{2n\pi}{3}-\frac{\cancel{2}}{3}\times\frac{\pi}{\cancel{4}_{\,2}}$$
$$\div\frac{3}{2}$$

$$\theta=\frac{2n\pi}{3}-\frac{\pi}{6}$$

3. (vi) Solve : $\cos\theta + \sin\theta = \sqrt{2}$

$$\cos\theta + \sin\theta = \sqrt{2}$$
$$a = 1, b = 1$$

$$\sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{1+1} = \sqrt{2}$$

$$\boxed{\cos A \cos B + \sin A \sin B = \cos(A-B)}$$

$\cos\theta + \sin\theta = \sqrt{2}$
$\div \sqrt{2}$ $\Rightarrow \cos\theta \frac{1}{\sqrt{2}} + \sin\theta \frac{1}{\sqrt{2}} = 1$

$$\cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4} = 1 \Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = 1$$

$\cos\left(\theta - \frac{\pi}{4}\right) = \cos 0$
$\cos\theta = \cos\alpha$ $\Rightarrow \theta = \theta - \frac{\pi}{4}, \alpha = 0$

General solutions of : $\cos\theta = \cos\alpha$

$$\boldsymbol{\theta = 2n\pi \pm \alpha, n \in Z}$$

$$\theta - \frac{\pi}{4} = 2n\pi \pm 0 \Rightarrow \theta - \frac{\pi}{4} = 2n\pi \Rightarrow \theta = 2n\pi + \frac{\pi}{4}$$

3. (vii) Solve : $\sin x + \sqrt{3}\cos x = 1$

$$a = \sqrt{3}, b = 1$$
$$\sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$\sin x + \sqrt{3}\cos x = 1$
$\div 2$

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{1}{2}$$

$$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{1}{2} \Rightarrow \cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6} = \frac{1}{2}$$

$\cos\left(x - \frac{\pi}{6}\right) = \frac{1}{2} \Rightarrow \cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{3}$
$\cos\theta = \cos\alpha$ $\Rightarrow \theta = x - \frac{\pi}{6}, \alpha = \frac{\pi}{3}$

General solutions of $\cos\theta = \cos\alpha$

$$\boldsymbol{\theta = 2n\pi \pm \alpha, n \in Z}$$

$$x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3} \Rightarrow x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3} \Rightarrow x = 2n\pi \pm \frac{\pi}{3} - \frac{\pi}{6}$$

3. (viii) Solve: $\cot\theta + \mathrm{cosec}\,\theta = \sqrt{3}$

$$\cot\theta + \mathrm{cosec}\,\theta = \sqrt{3}$$

$$\frac{\cos\theta}{\sin\theta}+\frac{1}{\sin\theta}=\sqrt{3} \implies \frac{\cos\theta+1}{\sin\theta}=\sqrt{3}$$

$$\cos\theta+1=\sqrt{3}\sin\theta \implies 1=\sqrt{3}\sin\theta-\cos\theta$$

$$\sqrt{3}\sin\theta-\cos\theta=1$$

$\div 2$

$$a=\sqrt{3}, b=1$$

$$\sqrt{\left(\sqrt{3}\right)^2+1^2}=\sqrt{3+1}=\sqrt{4}=2$$

$$\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta=\frac{1}{2}$$

$$\sin\frac{\pi}{3}\sin\theta-\cos\frac{\pi}{3}\cos\theta=\frac{1}{2} \implies \cos\frac{\pi}{3}\cos\theta-\sin\frac{\pi}{3}\sin\theta=-\frac{1}{2}$$

$$\cos\left(\theta+\frac{\pi}{3}\right)=-\frac{1}{2} \implies \cos\left(\theta+\frac{\pi}{3}\right)=\cos(180^\circ-60^\circ)$$

$$\cos\left(\theta+\frac{\pi}{3}\right)=\cos 120^\circ \implies \cos\left(\theta+\frac{\pi}{3}\right)=\cos\frac{2\pi}{3}$$

$$\cos\theta=\cos\alpha \qquad \implies \theta=\theta+\frac{\pi}{3}, \alpha=\frac{2\pi}{3}$$

General solutions of $\cos\theta=\cos\alpha$

$$\boldsymbol{\theta=2n\pi\pm\alpha, n\in Z}$$

$$\theta+\frac{\pi}{3}=2n\pi\pm\frac{2\pi}{3} \implies \theta=2n\pi\pm\frac{2\pi}{3}-\frac{\pi}{3}$$

$$(ix)\ \boldsymbol{\tan\theta+\tan\left(\theta+\frac{\pi}{3}\right)+\tan\left(\theta+\frac{2\pi}{3}\right)=\sqrt{3}}$$

$$\tan\theta+\frac{\tan\theta+\tan\frac{\pi}{3}}{1-\tan\theta\tan\frac{\pi}{3}}+\frac{\tan\theta+\tan\frac{2\pi}{3}}{1-\tan\theta\tan\frac{2\pi}{3}}=\sqrt{3}$$

$$\tan 120^\circ=\tan(180^\circ-60^\circ)$$
$$\tan 120^\circ=-\tan 60^\circ$$
$$=\sqrt{3}$$

$$\tan\theta+\frac{\tan\theta+\sqrt{3}}{1-\tan\theta\tan\frac{\pi}{3}}+\frac{\tan\theta+\tan 120^\circ}{1-\tan\theta\tan 120^\circ}=\sqrt{3}$$

$$\tan\theta+\frac{\tan\theta+\sqrt{3}}{1-\sqrt{3}\tan\theta}+\frac{\tan\theta-\sqrt{3}}{1-\tan\theta\,(-\sqrt{3})}=\sqrt{3}$$

$$\tan\theta+\frac{\tan\theta+\sqrt{3}}{1-\sqrt{3}\tan\theta}+\frac{\tan\theta-\sqrt{3}}{1+(\sqrt{3})\tan\theta}=\sqrt{3}$$

$$\frac{\tan\theta+\sqrt{3}}{1-\sqrt{3}\tan\theta}+\frac{\tan\theta-\sqrt{3}}{1+(\sqrt{3})\tan\theta}=\sqrt{3}-\tan\theta$$

$$\frac{(tan\theta+\sqrt{3})\,(1+\sqrt{3}tan\theta)+\ (tan\theta-\sqrt{3})\,(1-\sqrt{3}tan\theta)}{(1-\sqrt{3}tan\theta)\,(1+\sqrt{3}tan\theta)}=\sqrt{3}-tan\theta$$

$$\frac{tan\theta+\cancel{\sqrt{3}tan^2\theta}+\cancel{\sqrt{3}}+3\ tan\theta+\ tan\theta-\cancel{\sqrt{3}tan^2\theta}\ -\cancel{\sqrt{3}}+3\ tan\theta}{1^2-(\sqrt{3}tan\theta)^2}$$

$$\frac{tan\theta+3tan\theta+\ tan\theta+3tan\theta}{1-3tan^2\theta}=\sqrt{3}-tan\theta$$

$$2tan\theta+\ 6\ tan\theta\ =\ (\sqrt{3}-tan\theta)\,(1-3tan^2\theta)$$

$$8\ tan\theta\ =\sqrt{3}-3\sqrt{3}tan^2\theta\ -\ tan\theta+3tan^2\theta$$

$$8\ tan\theta+tan\theta\ =3tan^3\theta\ -3\sqrt{3}tan^2\theta\ +\sqrt{3}$$

$$9\ tan\theta-3tan^3\theta=\sqrt{3}-3\sqrt{3}tan^2\theta$$

$$9\ tan\theta-3tan^3\theta=\ \sqrt{3}(1-3tan^2\theta)\implies\frac{3(3\ tan\theta-3tan^3\theta)}{1-3tan^2\theta}=\sqrt{3}$$

$$\cancel{\sqrt{3}}\times\sqrt{3}\ \ (tan3\theta)\ \ =\underset{1}{\cancel{\sqrt{3}}}\ \implies\ tan3\theta=\frac{1}{\sqrt{3}}$$

$$tan3\theta\ =\ tan\frac{\pi}{6}\implies\ \ \theta\ =3\theta,\ \alpha=\frac{\pi}{6}$$

$$\boldsymbol{General\ Solution:\ \theta=n\pi+\alpha,n\ \epsilon\ Z}$$

$$3\theta=n\pi+\frac{\pi}{6}\ \implies\ \ \theta=\frac{n\pi}{3}+\frac{\pi}{18}$$

$$(\boldsymbol{x})\ \boldsymbol{cos2\theta}=\frac{\boldsymbol{\sqrt{5}+1}}{\boldsymbol{4}}$$

$$cos2\theta\ =cos36°$$

$$\boxed{\overset{\overset{1}{\cancel{4}}}{\cancel{36}}°\times\frac{\pi}{\underset{\underset{5}{\cancel{20}}}{\cancel{180}}°}\ =\frac{\pi}{5}}$$

$$cos2\theta\ =cos\frac{\pi}{5}\ \implies\ \ \theta\ =2\theta,\ \ \alpha=\frac{\pi}{5}$$

$$cos\theta\ =cos\alpha$$

$$\boldsymbol{General\ Solution:\ \theta=2n\pi\pm\alpha,n\ \epsilon\ Z}$$

$$2\theta\ =2n\pi\pm\frac{\pi}{5}\ \ \implies\ \theta=\frac{2n\pi}{2}\pm\frac{\pi}{10}\implies\ \theta=\frac{n\pi}{2}+\frac{\pi}{10}$$

$$(\boldsymbol{xi})\ \boldsymbol{2cos^2\theta-7cosx+3=0}$$

$$Let\ y=cos\ x$$

+ 7 × 6
-1 -6

$$2y^2-7y+3=0$$

$$2y^2-1y-6y+3=0$$

$$y(2y-1)-3(2y-1)=0\ \implies(y-3)\,(2y-1)=0$$

$y-3=0,\ 2y-1=0$

$y=3,\ \ 2y=1$

$$y=\frac{1}{2}$$

$\therefore y=3, y=\frac{1}{2}$

$cos x = 3,$ $\qquad \cos x=\frac{1}{2}$

$(Is\ not\ possible)$

Take: $\cos x=\frac{1}{2}$

$\cos x=\cos\frac{\pi}{3} \Longrightarrow \theta = x,\ \alpha=\frac{\pi}{3}$

$cos\theta = cos\alpha$

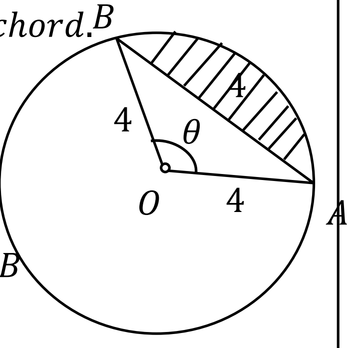
General solutions: $\boldsymbol{\theta = 2n\pi \pm \alpha, n\ \epsilon\ Z}$

$x=2n\pi \pm \frac{\pi}{3}, n\ \epsilon\ Z$

EXERCISE : 3.9

***Example 3.56:** The Government plans to have a circular zoological park of diameter 8km. A separate area in the form of a segment formed by a chord of length 4 km is to be allotted exclusively for a veterinary hospital in the park. Find the area of the segment to be allotted for the veterinary hospital.*

Let O be the centre of the circular park and AB be the chord.



$$\text{Let } \angle AOB = \theta$$

$$\text{Diameter} = 8km$$

$$\text{radius} = \frac{d}{2} = \frac{\cancel{8}}{\cancel{2}} = 4$$

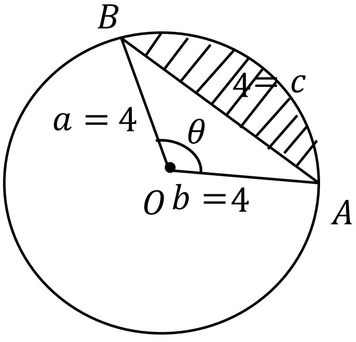
Area of the segment =

Area of the sector − Area of ΔOAB

$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta \quad = \frac{1}{2}r^2\,[\theta - \sin\theta]$$

$$= \frac{1}{2}4^2\,[\theta - \sin\theta] = \frac{1}{\cancel{2}}\overset{8}{\cancel{16}}\,[\theta - \sin\theta] = 8[\theta - \sin\theta] \quad \dots\dots\dots (1)$$

By cosine formula



$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\theta = \frac{4^2 + \cancel{4^2} - \cancel{4^2}}{2\times 4\times 4} = \frac{\cancel{4}\times\cancel{4}}{2\times\cancel{4}\times\cancel{4}}$$

$$\cos\theta = \frac{1}{2} \implies \theta = \frac{\pi}{3}$$

From (1)

$$\left.\begin{array}{r}\text{area of the segment to be allotted}\\ \text{for the veterinary hospital}\end{array}\right\} = 8[\theta - \sin\theta]$$

$$= 8\left[\frac{\pi}{3} - \sin\frac{\pi}{3}\right] = 8\left[\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right] = \overset{4}{\cancel{8}}\left[\frac{2\pi - 3\sqrt{3}}{3\times\cancel{2}}\right] = \frac{4}{3}\left[2\pi - 3\sqrt{3}\right]m^2$$

***Example 3.57:** In a ΔABC, prove that* $b^2\sin 2C + c^2\sin 2B = 2bc\sin A$.

By using sine formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\frac{a}{\sin A} = 2R \quad\Big|\quad \frac{b}{\sin B} = 2R \quad\Big|\quad \frac{c}{\sin C} = 2R$$

$$a = 2R\sin A \quad\Big|\quad b = 2R\sin B \quad\Big|\quad c = 2R\sin C$$

$$\text{L.H.S} = b^2\sin 2C + c^2\sin 2B$$

$$= (2R\sin B)^2\sin 2C + (2R\sin C)^2\sin 2B$$

$$= 4R^2\sin^2 B\,\sin 2C + 4R^2\sin^2 C\,\sin 2B$$

$$= 4R^2(\sin^2 B\sin 2C + \sin^2 C\sin 2B)$$

$$= 4R^2\ (\sin^2 B\ 2\sin C\cos C + \sin^2 C\ 2\sin B\cos B)$$
$$= 4R^2 \times 2\ \sin B\ \sin C\,(\sin B\cos C + \sin C\cos B)$$
$$= 8R^2 \sin B\sin C\ \sin(B+C)$$

$\boxed{\sin 2A = 2\sin A\cos A}$

$\boxed{\sin(A+B) = \sin A\cos B + \sin B\cos A}$

$\boxed{\begin{array}{l} A+B+C=\pi \\ B+C=\pi-A\end{array}}$

$$\sin(\theta - A) = \sin A \Rightarrow\ \ b = 2R\sin B \Longrightarrow \sin B = \frac{b}{2R}$$

$$c = 2R\sin C \Longrightarrow \sin C = \frac{c}{2R}\ \ = \overset{2}{\cancel{4}}\cancel{8}\overset{R}{\cancel{R^2}}\left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right)\sin(\pi - A) = 2bc\sin A = R.H.S$$

Example 3.58: ***In a*** $\boldsymbol{\Delta ABC}$***, prove that*** $\boldsymbol{\sin\left(\frac{B-C}{2}\right) = \frac{b-c}{a}\cos\frac{A}{2}}$.

By using sine formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\mathrm{R}$$

$$\frac{a}{\sin A} = 2\mathrm{R} \quad\Big|\quad \frac{b}{\sin B} = 2\mathrm{R} \quad\Big|\quad \frac{c}{\sin C} = 2\mathrm{R}$$

$$a = 2\mathrm{R}\sin A \quad\Big|\quad b = 2\mathrm{R}\sin B \quad\Big|\quad c = 2\mathrm{R}\sin C$$

$$\boxed{\begin{array}{l} A+B+C = 180^\circ \\ B+C = 180^\circ - A \\ \dfrac{B+C}{2} = \dfrac{180^\circ - A}{2} \\ \dfrac{B+C}{2} = \dfrac{180^\circ}{2} - \dfrac{A}{2} \\ \dfrac{B+C}{2} = 90^\circ - \dfrac{A}{2}\end{array}}$$

$$R.H.S = \frac{b-c}{a}\cos\frac{A}{2}$$

$$= \frac{2R\sin B - 2R\sin C}{2R\sin A}\ \cos\frac{A}{2} \quad = \frac{\cancel{2R}\,(\sin B - \sin C)}{\cancel{2R}\sin A}\ \cos\frac{A}{2}$$

$$= \frac{\cancel{2}\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{\cancel{2}\sin\frac{A}{2}\cancel{\cos\frac{A}{2}}}\ \cancel{\cos\frac{A}{2}} \quad = \frac{\sin\frac{B-C}{2}\cos\left(90^\circ - \frac{A}{2}\right)}{\sin\frac{A}{2}}$$

$\boxed{\cos(90^\circ - \theta) = \sin\theta}$

$$= \frac{\sin\frac{B-C}{2}\ \cancel{\sin\frac{A}{2}}}{\cancel{\sin\frac{A}{2}}} \ = \sin\frac{B-C}{2} = R.H.S$$

Example 3.59: ***If the three angles in a triangle are in the ratio*** $\boldsymbol{1:2:3}$***, then prove that the corresponding sides are in the ratio*** $\boldsymbol{1:\sqrt{3}:2}$.

Let the angles be $\theta, 2\theta, 3\theta$

$$\theta + 2\theta + 3\theta = 180^\circ$$

$$6\theta = 180^\circ$$

$$\theta = \frac{\cancel{180}^{\circ}\,30^\circ}{\cancel{6}} \implies \theta = 30^\circ$$

$By\ using\ sine\ formula$: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$A = \theta$	$B = 2\theta$	$C = 3\theta$
$A = 30^\circ$	$B = 2 \times 30^\circ$	$C = 3 \times 30^\circ$
	$B = 60^\circ$	$B = 90^\circ$

$$\frac{a}{\sin 30^\circ} = \frac{b}{\sin 60^\circ} = \frac{c}{\sin 90^\circ}$$

$$a : b : c = \sin 30^\circ : \sin 60^\circ : \sin 90^\circ = \frac{1}{2} : \frac{\sqrt{3}}{2} : 1 \quad \times 2$$

$$= \cancel{2} \times \frac{1}{\cancel{2}} : \cancel{2} \times \frac{\sqrt{3}}{\cancel{2}} : 2 \times 1 \quad = 1 : \sqrt{3} : 2$$

Example 3.60: In a ΔABC, prove that

$$(\boldsymbol{b} + \boldsymbol{c}) \cos \boldsymbol{A} + (\boldsymbol{c} + \boldsymbol{a}) \cos \boldsymbol{B} + (\boldsymbol{a} + \boldsymbol{b}) \cos \boldsymbol{C} = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$$

$$L.H.S = (b + c) \cos A + (c + a) \cos B + (a + b) \cos C$$

$$= b \cos A + c \cos A + c \cos B + a \cos B + a \cos C + b \cos C$$

$$= b \cos A + a \cos B + c \cos A + a \cos C + c \cos B + b \cos C$$

$$= c + b + a \qquad [By\ projection\ formula] \quad \boxed{a = b \cos C + c \cos B}$$

$$\implies a + b + c = R.H.S$$

Example 3.61: In a ΔABC, prove that $\dfrac{\boldsymbol{a}^2 + \boldsymbol{b}^2}{\boldsymbol{a}^2 + \boldsymbol{c}^2} = \dfrac{\boldsymbol{1} + \cos(\boldsymbol{A} - \boldsymbol{B}) \cos \boldsymbol{C}}{\boldsymbol{1} + \cos(\boldsymbol{A} - \boldsymbol{C}) \cos \boldsymbol{B}}$

$By\ using\ sine\ formula$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\mathrm{R}$$

$\frac{a}{\sin A} = 2\mathrm{R}$	$\frac{b}{\sin B} = 2\mathrm{R}$	$\frac{c}{\sin C} = 2\mathrm{R}$
$a = 2\mathrm{R} \sin A$	$b = 2\mathrm{R} \sin B$	$c = 2\mathrm{R} \sin C$

$$L.H.S = \frac{a^2 + b^2}{a^2 + c^2}$$

$$= \frac{(2R \sin A)^2 + (2R \sin B)^2}{(2R \sin A)^2 + (2R \sin C)^2} = \frac{4R^2 \sin^2 A + 4R^2 \sin^2 B}{4R^2 \sin^2 A + 4R^2 \sin^2 C}$$

$$= \frac{\cancel{4R^2}(\sin^2 A + \sin^2 B)}{\cancel{4R^2}(\sin^2 A + \sin^2 C)} = \frac{1 - \cos^2 A + \sin^2 B}{1 - \cos^2 A + \sin^2 C} = \frac{1 - (\cos^2 A - \sin^2 B)}{1 - (\cos^2 A - \sin^2 C)}$$

$$= \frac{1-\cos(A+B)\cos(A-B)}{1-\cos(A+C)\cos(A-C)} = \frac{1-\cos(\pi-C)\cos(A-B)}{1-\cos(\pi-B)\cos(A-C)}$$

$\boxed{\begin{array}{l}\sin^2 A+\cos^2 A=1\\ \sin^2 A = 1-\cos^2 A\end{array}}$ $\boxed{\cos(\pi-\theta) = -\cos\theta}$

$$= \frac{1-(-\cos C)\cos(A-B)}{1-(-\cos B)\cos(A-C)} = \frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B} = R.H.S$$

$\boxed{\begin{array}{l}A+B+C=\pi\\ A+B=\pi-C\end{array}}$

Example 3.62: Drive cosine formula using the law of sines in a ΔABC.

The law of sines :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\text{R}$$

$$\frac{a}{\sin A} = 2\text{R} \quad \Big| \quad \frac{b}{\sin B} = 2\text{R} \quad \Big| \quad \frac{c}{\sin C} = 2\text{R}$$

$$a = 2\text{R}\sin A \quad \Big| \quad b = 2\text{R}\sin B \quad \Big| \quad c = 2\text{R}\sin C$$

$$\textit{cosine formula: } \cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$R.H.S = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{(2R\sin B)^2+(2R\sin C)^2-(2R\sin A)^2}{2(2R\sin B)(2R\sin C)} = \frac{4R^2\sin^2 B+4R^2\sin^2 C-4R^2\sin^2 A}{8R^2\sin B\sin C}$$

$$= \frac{\cancel{4R^2}(\sin^2 B+\sin^2 C-\sin^2 A)}{\cancel{8R^2}_{2}\sin B\sin C} = \frac{\sin^2 B+\sin(C+A)\sin(C-A)}{2\sin B\sin C}$$

$$= \frac{\sin^2 B+\sin(\pi-B)\sin(C-A)}{2\sin B\sin C} = \frac{\sin^2 B+\sin B\sin(C-A)}{2\sin B\sin C}$$

$\boxed{\begin{array}{l}A+B+C=\pi\\ A+C=\pi-B\end{array}}$

$$= \frac{\cancel{\sin B}[\sin B+\sin(C-A)]}{2\cancel{\sin B}\sin C} = \frac{\sin(C+A)+\sin(C-A)}{2\sin C}$$

$\boxed{\sin(\pi-\theta)=\sin\theta}$

$\boxed{\sin(C+A) = \sin(\pi-B) = \sin B}$

$$= \frac{\cancel{2\sin C}\cos A}{\cancel{2\sin C}} = \cos A$$

$\boxed{\sin(A+B)+\sin(A-B) = 2\sin A\cos B}$

1. ***In a ΔABC, if*** $\dfrac{\sin \boldsymbol{A}}{\sin \boldsymbol{C}} = \dfrac{\sin(\boldsymbol{A}-\boldsymbol{B})}{\sin(\boldsymbol{B}-\boldsymbol{C})}$***, prove that*** $\boldsymbol{a^2, b^2, c^2}$ ***are in Arithmetic Progression.***

$\boxed{\begin{array}{l}A+B+C=\pi\\ A=\pi-(B+C)\end{array}}$ $\boxed{\begin{array}{l}A+B+C=\pi\\ C=\pi-(A+B)\end{array}}$ $\boxed{\sin(\pi-\theta)=\sin\theta}$

$$\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$$

$$\sin(B-C)\sin A = \sin(A-B)\sin C$$

$$\sin(B-C)\sin(\pi-(B+C)) = \sin(A-B)\sin(\pi-(A+B))$$

$$\sin(B-C)\sin(B+C) = \sin(A-B)\sin(A+B)$$

$$\sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B \;\ldots (1)$$

By using sine formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\frac{a}{\sin A} = 2R \qquad \frac{b}{\sin B} = 2R \qquad \frac{c}{\sin C} = 2R$$

$$\frac{a}{2R} = \sin A \qquad \frac{b}{2R} = \sin B \qquad \frac{c}{2R} = \sin C$$

subs $\sin A, \sin B$ *and* $\sin C$ *in* (1) $\Rightarrow \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$

$$\left(\frac{b}{2R}\right)^2 - \left(\frac{c}{2R}\right)^2 = \left(\frac{a}{2R}\right)^2 - \left(\frac{b}{2R}\right)^2 \Rightarrow \frac{b^2}{4R^2} - \frac{c^2}{4R^2} = \frac{a^2}{4R^2} - \frac{b^2}{4R^2}$$

$$\frac{b^2 - c^2}{\cancel{4R^2}} = \frac{a^2 - b^2}{\cancel{4R^2}} \Rightarrow b^2 - c^2 = a^2 - b^2 \Rightarrow b^2 + b^2 = a^2 + c^2$$

$2b^2 = a^2 + c^2 \Rightarrow$ *Thus,* a^2, b^2, c^2 *are in Arithmetic Progression.*

2. *The angles of a triangle ABC, are in Arithmetic Progression and if* $\boldsymbol{b : c = \sqrt{3} : \sqrt{2}}$, ***find*** $\boldsymbol{\angle A}$.

$$\boxed{\sin B = \frac{b}{2R}}$$

$\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}}$ *and* $\angle A, \angle B, \angle C$ *are* in A. P.

$$2\angle B = \angle A + \angle C \Rightarrow \frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}} \Rightarrow \frac{b}{\sqrt{3}} = \frac{c}{\sqrt{2}} \Rightarrow \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{\frac{\sqrt{2}}{2}} \Rightarrow \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{\frac{1}{\sqrt{2}}}$$

$$\sin B = \frac{\sqrt{3}}{2} \Rightarrow B = 60^\circ \Rightarrow \sin C = \frac{1}{\sqrt{2}} \Rightarrow C = 45^\circ$$

$$\boxed{\frac{\sqrt{2}}{2} = \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}} \times \sqrt{2}} = \frac{1}{\sqrt{2}}}$$

$$A + B + C = 180^\circ$$

$$A + 60^\circ + 45^\circ = 180^\circ \Rightarrow A + 105^\circ = 180^\circ$$

$$\boxed{\sin 60^\circ = \frac{\sqrt{3}}{2}}$$

$$A = 180^\circ - 105^\circ \Rightarrow A = 75^\circ$$

A, B, C are in Arithmetic Progression

$$\boxed{\sin 45^\circ = \frac{1}{\sqrt{2}}}$$

3. *In a* $\boldsymbol{\Delta ABC}$, ***if*** $\boldsymbol{\cos C = \frac{\sin A}{2 \sin B}}$, ***show that the triangle is isoceles.***

$$\cos C = \frac{\sin A}{2 \sin B}$$

[*By cosine and sine formula*]

$$\frac{a^2 + b^2 - c^2}{2ab} = \frac{\frac{a}{\cancel{2R}}}{2\left(\frac{b}{\cancel{2R}}\right)} \Rightarrow \frac{a^2 + b^2 - c^2}{\cancel{2}a\cancel{b}} = \frac{a}{\cancel{2}b} \Rightarrow \frac{a^2 + b^2 - c^2}{a} = a$$

$$a^2 + b^2 - c^2 = a^2 \Rightarrow \cancel{a^2} + b^2 - c^2 - \cancel{a^2} = 0$$

$$b^2 - c^2 = 0 \Rightarrow b^2 = c^2$$

$$b = c$$

4. In a ΔABC, prove that $\dfrac{\sin B}{\sin C} = \dfrac{c - a\cos B}{b - a\cos C}$.

$$R.H.S = \frac{c - a\cos B}{b - a\cos C} = \frac{c - \cancel{a}\left(\frac{c^2+a^2-b^2}{2c\cancel{a}}\right)}{b - \cancel{a}\left(\frac{a^2+b^2-c^2}{2\cancel{a}b}\right)} \quad [By\ \ cosine\ formula]$$

$$= \frac{\frac{2c^2-(c^2+a^2-b^2)}{2c}}{\frac{2b^2-(a^2+b^2-c^2)}{2b}} = \frac{\frac{2c^2-c^2-a^2+b^2}{2c}}{\frac{2b^2-a^2-b^2+c^2}{2b}} = \frac{\frac{c^2-a^2+b^2}{2c}}{\frac{b^2-a^2+c^2}{2b}}$$

$$= \frac{\cancel{c^2-a^2+b^2}}{\cancel{2}c} \times \frac{\cancel{2}b}{\cancel{b^2-a^2+c^2}} = \frac{b}{c} \Rightarrow L.H.S = \frac{\sin B}{\sin C} = \frac{\frac{b}{\cancel{2R}}}{\frac{c}{\cancel{2R}}} = \frac{b}{c}$$

$$L.H.S = R.H.S$$

5. In a ΔABC, prove that $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$.

$$L.H.S = a\cos A + b\cos B + c\cos C \qquad \boxed{[By\ \ sine\ formula]}$$
$$= 2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C \qquad \boxed{a = 2R\sin A}$$
$$= R\,[2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C] \qquad \boxed{2\sin A\cos A = \sin 2A}$$
$$= R\,[\underbrace{\sin 2A + \sin 2B} + 2\sin C\cos C] \qquad \boxed{\begin{array}{c}A+B+C=\pi\\ A+B=\pi-C\end{array}}$$
$$= R\,[2\sin(A+B)\cos(A-B) + 2\sin C\cos C] \qquad \boxed{\sin(\pi-\theta) = \sin\theta}$$
$$= R\,[2\sin(\pi-C)\cos(A-B) + 2\sin C\cos C] \qquad \boxed{\begin{array}{c}A+B+C=\pi\\ C=\pi-(A+B)\end{array}}$$
$$= R\,[2\sin C\cos(A-B) + 2\sin C\cos C] \qquad \boxed{\cos(\pi-\theta) = -\cos\theta}$$
$$= R\,2\sin C\,[\cos(A-B) + \cos C] \qquad \boxed{a = 2R\sin A}$$
$$= R2\sin C\,[\cos(A-B) + \cos(\pi-(A+B))]$$
$$= R2\sin C\,[\cos(A-B) - \cos(A+B)] = R2\sin C\,[2\sin A\sin B]$$
$$= 2\times 2R\sin A\sin B\sin C = 2\,a\sin B\sin C = R.H.S$$

6. In a ΔABC, $\angle A = 60^\circ$. $+c =$ prove that $b + c = 2a\cos\left(\dfrac{B-C}{2}\right)$.

$$A+B+C = 180^\circ \Rightarrow 60^\circ + B + C = 180^\circ \Rightarrow B + C = 180^\circ - 60^\circ$$

$$= B + C = 120^\circ \Rightarrow L.H.S = b + c \Rightarrow [By\ \ sine\ formula] = 2R\sin B + 2R\sin C$$

$$= 2R[\sin B + \sin C] = 2R\left[2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\right]$$

$$= 4R\left[\sin\left(\frac{\overset{60^\circ}{\cancel{120^\circ}}}{\cancel{2}}\right)\cos\left(\frac{B-C}{2}\right)\right]$$

$$= 4R\left[\sin 60^\circ\cos\left(\frac{B-C}{2}\right)\right] = \overset{2}{\cancel{4}}R\left[\frac{\sqrt{3}}{\cancel{2}}\cos\left(\frac{B-C}{2}\right)\right] = 2\sqrt{3}R\cos\left(\frac{B-C}{2}\right)$$

$$R.H.S = 2a\cos\left(\frac{B-C}{2}\right) = 2\times 2R\sin A\cos\left(\frac{B-C}{2}\right)\ [By\ sine\ formula]$$

$$= 4R\ \sin 60^\circ \cos\left(\frac{B-C}{2}\right) = \overset{2}{\cancel{4}}R\left[\frac{\sqrt{3}}{\cancel{2}}\cos\left(\frac{B-C}{2}\right)\right] = 2\sqrt{3}R\cos\left(\frac{B-C}{2}\right)$$

$$L.H.S = R.H.S$$

7. In a ΔABC, prove the following:

$(i)\ a\sin\left(\frac{A}{2}+B\right) = (b+c)\sin\frac{A}{2}$ $\quad(ii)\ a(\cos B+\cos C) = 2(b+c)\sin^2\frac{A}{2}$

$(iii)\ \frac{a^2-c^2}{b^2} = \frac{\sin(A-C)}{\sin(A+C)}$ $\quad(iv)\ \frac{a\sin(B-C)}{b^2-c^2} = \frac{b\sin(C-A)}{c^2-a^2} = \frac{c\sin(A-B)}{a^2-b^2}$

$(v)\ \frac{a+b}{a-b} = \tan\left(\frac{A+B}{2}\right).\cot\left(\frac{A-B}{2}\right)$

$(i)\ a\ sin\left(\frac{A}{2}+B\right) = (b+c)\sin\frac{A}{2}$

$[By\ sine\ formula]$

$$consider\ \ \frac{b+c}{a} = \frac{k\sin B + k\sin C}{k\sin A}$$

$$\frac{\sin B+\sin C}{\sin A} = \frac{2\sin\left(\frac{B+C}{2}\right)cos\left(\frac{B-C}{2}\right)}{2\ sin\frac{A}{2}cos\frac{A}{2}}$$

$$= \frac{sin\left(90-\frac{A}{2}\right)cos\left(\frac{B-C}{2}\right)}{sin\frac{A}{2}cos\frac{A}{2}} = \frac{cos\frac{A}{2}cos\left(\frac{B-C}{2}\right)}{sin\frac{A}{2}cos\frac{A}{2}} = \frac{cos\left(\frac{B-C}{2}\right)}{sin\frac{A}{2}}$$

$$\left[\because A+B+C = 180 \Rightarrow \frac{B+C}{2} = 90-\frac{A}{2}\right]$$

$$= \frac{cos\left(\frac{B-(180-A-B)}{2}\right)}{sin\frac{A}{2}} \quad [\because C = 180-A-B]$$

$$= \frac{cos\left(-90+\frac{A}{2}+B\right)}{sin\frac{A}{2}} = \frac{cos\left[-\left(90-\left(\frac{A}{2}+B\right)\right)\right]}{sin\frac{A}{2}} \quad \boxed{[\because \cos(-\theta) = cos\theta]}$$

$$= \frac{cos\left(90-\left(\frac{A}{2}+B\right)\right)}{sin\frac{A}{2}}$$

$$\frac{b+c}{a} = \frac{sin\left(\frac{A}{2}+B\right)}{sin\frac{A}{2}} \Longrightarrow (b+c)sin\frac{A}{2} = a\ sin\left(\frac{A}{2}+B\right)$$

$$(ii)\ a(\cos B + \cos C) = 2(b+c)\sin^2\frac{A}{2}$$

$$L.H.S = a(\cos B + \cos C)$$

$$= a\left(\frac{c^2+a^2-b^2}{2ac} + \frac{a^2+b^2-c^2}{2ab}\right) = \frac{1}{2}\left[\frac{c^2+a^2-b^2}{c} + \frac{a^2+b^2-c^2}{b}\right]$$

$$= \frac{1}{2bc}[bc^2 + a^2b - b^3 + a^2c + b^2c - c^3]$$

$$= \frac{1}{2bc}[-(b^3 + c^3) + (a^2b + a^2c) + bc^2 + b^2c]$$

$$= \frac{1}{2bc}[-(b + c) + (b^2 - bc + c^2) + a^2(b + c) + bc(b + c)]$$

$$= \frac{b + c}{2bc}[\,-b^2 + bc - c^2 + a^2 + bc] = \frac{b + c}{2bc}[a^2 - b^2 - c^2 - 2bc]$$

$$= \left(\frac{b + c}{2bc}\right)[(a + b - c) + (a - b + c)] \qquad [since\ 2s = a + b + c]$$

$$= \left(\frac{b + c}{2bc}\right)(a + b + c - 2c)\,(a + b + c - 2b)$$

$$= \left(\frac{b + c}{2bc}\right)[(2s - 2c)(2s - 2b)] = \left(\frac{b + c(4)}{2bc}\right)[(s - c)(s - b)]$$

$$= 2(b - c)\left[\frac{(s - b)(s - c)}{bc}\right] = 2(b + c)\,sin^2\frac{A}{2} = RHS$$

$(\boldsymbol{iii})$ $\dfrac{\boldsymbol{a^2 - c^2}}{\boldsymbol{b^2}} = \dfrac{\boldsymbol{sin(A - C)}}{\boldsymbol{sin(A + C)}}$

$$Let\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin c} = k \qquad [Using\ sine\ formula]$$

$$a = k\sin A,\ b = k\sin B\ and\ c = k\sin C \qquad \ldots(1)$$

$$LHS = \frac{a^2 - c^2}{b^2} = \frac{k^2 sin^2 A - k^2 sin^2 C}{k^2 sin^2 B}$$

$$= \frac{sin^2 A - sin^2 C}{sin^2 B} = \frac{\sin(A + C)\sin(A - C)}{sin^2(180 - (A + C)}$$

$$= \frac{\sin(A + C)\sin(A - C)}{sin^2(A + C)} = \frac{\sin(A - C)}{\sin(A + C)} = RHS$$

$(\boldsymbol{iv})$ $\dfrac{\boldsymbol{a\ sin(B - C)}}{\boldsymbol{b^2 - c^2}} = \dfrac{\boldsymbol{b\ sin(C - A)}}{\boldsymbol{c^2 - a^2}} = \dfrac{\boldsymbol{c\ sin(A - B)}}{\boldsymbol{a^2 - b^2}}$

$$Let\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin c} = k \qquad [Using\ sine\ formula]$$

$$a = k\sin A,\ b = k\sin B\ and\ c = k\sin C \qquad \ldots(1)$$

$$consider\ \frac{a\sin(B - C)}{b^2 - c^2} = \frac{k\sin A\sin(B - C)}{k^2 sin^2 B - k^2 sin^2 C}$$

$$= \frac{k\sin(B+C)\sin(B-C)}{k^2(sin^2B - sin^2C)} = \frac{k(sin^2B - sin^2C)}{k^2(sin^2B - sin^2C)} = \frac{1}{k} \quad \dots(2)$$

$$and \ \frac{b\sin(C-A)}{c^2-a^2} = \frac{k\sin B\sin(C-A)}{k^2sin^2C - k^2sin^2A}$$

$$= \frac{\sin(C+A)\sin(C-A)}{k\sin(C+A) - \sin(C-A)} = \frac{1}{k} \quad \dots(3)$$

$$Similarly, \ \frac{C\sin(A-B)}{a^2-b^2} = \frac{1}{k} \quad \dots(4)$$

$From\ (2),(3) and\ (4),$

$$\frac{a\sin(B-C)}{b^2-c^2} = \frac{b\sin(C-A)}{c^2-a^2} = \frac{c\sin(A-B)}{a^2-b^2}$$

(v) $\boldsymbol{\frac{a+b}{a-b} = tan\left(\frac{A+B}{2}\right).cot\left(\frac{A-B}{2}\right)}$

$$Let \ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin c} = k \qquad [Using\ sine\ formula]$$

$$a = k\sin A, \ b = k\sin B \ and \ c = k\sin C \quad \dots(1)$$

$$L.H.S = \frac{a+b}{a-b} = \frac{k\sin A + k\sin B}{k\sin A - k\sin B} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

$$= \frac{2sin\left(\frac{A+B}{2}\right).cos\left(\frac{A-B}{2}\right)}{2cos\left(\frac{A+B}{2}\right).sin\left(\frac{A-B}{2}\right)} = tan\left(\frac{A+B}{2}\right).cot\left(\frac{A-B}{2}\right)$$

$$= RHS$$

8. $\boldsymbol{In\ \Delta ABC, prove\ that\ (a^2 - b^2 + c^2)\tan B = (a^2 + b^2 - c^2)\tan C.}$

$$Let\ us\ prove\ that\ \frac{a^2-b^2+c^2}{a^2+b^2-c^2} = \frac{\tan C}{\tan B} = \tan C.\cot B$$

$$L.H.S = \frac{a^2-b^2+c^2}{a^2+b^2-c^2} = \frac{k^2sin^2A - k^2sin^2B + k^2sin^2C}{k^2sin^2A + k^2sin^2B - k^2sin^2C}$$

$$= \frac{sin^2A - sin^2B + sin^2C}{sin^2A - sin^2C + sin^2B}$$

$$\left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \Rightarrow a = k\sin A, \ b = k\sin B, c = k\sin C\right]$$

$$= \frac{\sin(A+B)\sin(A-B) + sin^2C}{\sin(A+C)\sin(A-C) + sin^2B}$$

$$= \frac{\sin C\sin(A-B) + sin^2C}{\sin B\sin(A-C) + sin^2B} \qquad [\because A+B = 180 - C \ \& A + C = 180 - B]$$

$$= \frac{\sin C\,[\sin(A-B)+\sin C]}{\sin B\,[\sin(A-C)+\sin B]} = \frac{\sin C\,[\sin(A-B)+\sin(A+B)]}{\sin B\,[\sin(A-C)+\sin(A+C)]}$$

$$= \frac{\sin C\,.\,2\sin A\cos B}{\sin B\,.\,2\sin A\cos C} = \frac{\sin C}{\cos c}.\frac{\cos B}{\sin B} = \tan C\cot B = RHS$$

$$\boxed{\because \sin(A-B)+\sin(A+B) = 2\,sinAcosB}$$

9. *An Engineer has to develop a triangular shaped park with perimeter 120 m in a village . The park to be developed must be of maximum area. Fimd out the dimensions of the park.*

Using example 3.62. the equilateral triangle has the maximum area for any fixed perimeter.

$\because$ *Let a be the side if the equilateral triangle.*

$$Given\ Perimeter = 120\ m$$

$$a+a+a = 120\ m$$

$$\cancel{3}a = \cancel{12}0$$

$$a = 40$$

Hence the dimensions of the park are 40 m, 40 m amd 40 m.

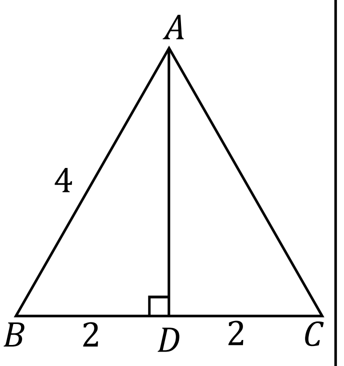
10. *A rope of length 12 m is given. Find the largest area of the triangle formed by this rope and find the dimensions of the triangler so formed.*

$$Length\ of\ the\ rope = 12\ m$$

$$\therefore Perimeter\ of\ the\ triangle = 12\ m$$

By example 3.62, the equilateral triangle hs the maximum area for any fixed perimeter

Let a be the side of the equilateral triangle.



$$a+a+a = 12$$

$$\cancel{3}a = \cancel{1}2$$

$$a = 4$$

Let AD be the height of the equilateral ΔABC.

$$\therefore AD = \sqrt{AB^2 - BD^2} = \sqrt{4^2 - 2^2} = \sqrt{16-4}$$

$$= \sqrt{12} = 2\sqrt{3}$$

$$\because Area\ of\ the\ largest\ triangle\ ABC = \frac{1}{2}\times BC\times AD$$

$$= \frac{1}{\sqrt{2}}\times 4\times 2\sqrt{3} = 4\sqrt{3}\ sq.m$$

11. *Derive projection formula form*
(*i*) *Law of sines*   (*ii*) *Law of cosines*

(*i*) *Law of sines*

$$Given\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\Longrightarrow a = k\sin A,\ b = k\sin B\ and\ c = k\sin C \quad \dots (1)$$

$To\ prove\ that$

$$a = b\cos C + c\cos B, b = c\cos A + a\cos C \quad and \quad c = a\cos B + b\cos A$$

$To\ prove\ that\ \ a = b\cos c + c\cos B$ $\qquad [Using(1)]$

$RHS = b\cos C + c\cos B$ $\qquad [\because A + B + C = 180]$

$= k\sin B\cos C + k\sin C\cos B = k[\sin B\cos C + \cos B\sin C]$

$= k\sin(B + C) = k\sin(180 - A) = k\sin A = a = LHS$

$Similarly\ b = c\cos A + a\cos C\ and\ c = a\cos B + b\cos A\ \ can\ be\ proved.$

$(ii)\ Law\ of\ cosines$

$RHS = b\cos C + c\cos B$

$$= b\left(\frac{a^2 + b^2 - c^2}{2ab}\right) + c\left(\frac{c^2 + a^2 - b^2}{2ca}\right) = \frac{a^2 + b^2 - c^2}{2a} + \frac{c^2 + a^2 - b^2}{2a}$$

$$= \frac{a^2 + \cancel{b^2} - \cancel{c^2} + \cancel{c^2} + a^2 - \cancel{b^2}}{2a} = \frac{\cancel{2}\cancel{a^2}}{\cancel{2}\cancel{a}} = a = LHS$$

EXERCISE 3.10

Eg 3.64: In a ΔABC, $a = 3, b = 5$ and $c = 7$. Find the value of $\cos A$, $\cos B$ and $\cos C$

Given $a = 3, b = 5, c = 7$

By Cosine formula, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$$= \frac{5^2 + 7^2 - 3^2}{2 \times 5 \times 7} = \frac{25 + 49 - 9}{2 \times 5 \times 7}$$

$$= \frac{74 - 9}{2 \times 5 \times 7} = \frac{\overset{13}{\cancel{65}}}{2 \times \cancel{5} \times 7} = \frac{13}{14}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{3^2 + 7^2 - 5^2}{2 \times 3 \times 7} = \frac{9 + 49 - 25}{2 \times 3 \times 7}$$

$$= \frac{58 - 25}{2 \times 3 \times 7} = \frac{\overset{11}{\cancel{33}}}{2 \times \cancel{3} \times 7} = \frac{11}{14}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{3^2 + 5^2 - 7^2}{2 \times 3 \times 5} = \frac{9 + 25 - 49}{2 \times 3 \times 5}$$

$$= \frac{34 - 49}{2 \times 3 \times 5} = \frac{-\overset{5}{\cancel{15}}}{2 \times 3 \times \cancel{5}} = -\frac{1}{2}$$

Example 3.65: In a ΔABC, $A = 30^\circ, B = 60^\circ$ and $c = 10$. Find a and b.

Given $A = 30^\circ, B = 60^\circ, c = 10$

$$\boxed{A + B + C = 180^\circ}$$

$$C = 180^\circ - (A + B)$$

$$C = 180^\circ - (30^\circ + 60^\circ)$$

$$C = 180^\circ - (90^\circ)$$

$$C = 90^\circ$$

By sine formula, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$$\frac{a}{\sin 30^\circ} = \frac{b}{\sin 60^\circ} = \frac{10}{\sin 90^\circ}$$

$$\frac{a}{\sin 30^\circ} = \frac{10}{\sin 90^\circ} \Rightarrow a = \frac{10 \sin 30^\circ}{\sin 90^\circ} \Rightarrow a = \frac{\overset{5}{\cancel{10}}\left(\frac{1}{\cancel{2}}\right)}{1}$$

$$\boxed{a = 5}$$

$$\frac{b}{\sin 60^\circ} = \frac{10}{\sin 90^\circ} \Rightarrow b = \frac{10\sin 60^\circ}{\sin 90^\circ}$$

$$b = \frac{\overset{5}{\cancel{10}}\left(\frac{\sqrt{3}}{\cancel{2}}\right)}{1} \Rightarrow \boxed{b = 5\sqrt{3}}$$

Example 3.66: In a ΔABC, if $a = 2\sqrt{2}, b = 2\sqrt{3}$ and $C = 75^\circ$, find the other side and the angles.

Given $a = 2\sqrt{2}, b = 2\sqrt{3}, C = 75^\circ$

By Cosine formula, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$$\cos 75^\circ = \frac{(2\sqrt{2})^2 + (2\sqrt{3})^2 - c^2}{2 \times (2\sqrt{2}) \times (2\sqrt{3})} = \frac{(4 \times 2) + (4 \times 3) - c^2}{8\sqrt{6}}$$

$$= \frac{8 + 12 - c^2}{8\sqrt{6}}$$

Eg. 3.67: Find the area of the triangle whose sides are 13cm, 14cm and 15cm.

Given $a = 13cm, b = 14cm, c = 15cm$

Using heron's formula,

area of $\Delta ABC, \Delta = \sqrt{s(s-a)(s-b)(s-c)}$ $\boxed{where\ s = \frac{a+b+c}{2}}$

$$\therefore s = \frac{13 + 14 + 15}{2} = \frac{\overset{21}{\cancel{42}}}{\cancel{2}} = 21$$

$$\therefore \Delta = \sqrt{21(21-13)(21-14)(21-15)}$$

$$= \sqrt{21 \times 8 \times 7 \times 6} = \sqrt{3 \times 7 \times 2 \times 2 \times 2 \times 7 \times 2 \times 3}$$

$$= 3 \times 7 \times 2 \times 2 = 84\ sq.cm$$

Example 3.68: In any ΔABC, prove that $a\cos A + b\cos B + c\cos C = \dfrac{8\Delta^2}{abc}$

$a\cos A + b\cos B + c\cos C = 2a\sin B \sin C$

$$= 2\cancel{a}\left(\frac{2\Delta}{\cancel{a}c}\right)\left(\frac{2\Delta}{ab}\right) = \frac{8\Delta^2}{abc}$$

Example 3.69: Suppose that there are two cell phone towers within range of a cell phone. The two towers are located at 6km apart along a straight highway, running east to west and the cell phone is north of the highway. The signal is 5km from the first tower and $\sqrt{31}$km from the second tower. Determine the position of the cell phone north and east of the first tower and how far it is from the highway.

Let θ be the position of the cell phone from north to east of the first tower.

Let $a = 5, b = 6$ *and* $c = \sqrt{31}$

Then, using the cosine formula,

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\left(\sqrt{31}\right)^2 = 5^2 + 6^2 - 2 \times 5 \times 6 \cos\theta$$

$$31 = 25 + 36 - 60\cos\theta$$

$$31 = 61 - 60\cos\theta$$

$$60\cos\theta = 61 - 31 = 30$$

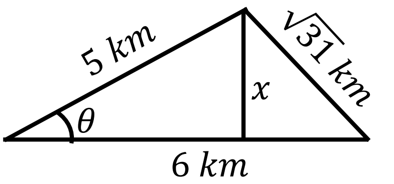
$$\cos\theta = \frac{\cancel{30}^{\,1}}{\cancel{60}_{\,2}} = \frac{1}{2} \Longrightarrow \cos\theta = \frac{1}{2} \Longrightarrow \theta = 60^\circ$$

Let x be the distance of the cell phone's position from the highway.

$$\sin\theta = \frac{x}{5} \Longrightarrow x = 5\sin\theta$$

$$x = 5\sin 60^\circ \qquad \boxed{\sin 60^\circ = \frac{\sqrt{3}}{2}}$$

$$x = \frac{5\sqrt{3}}{2}\,km$$

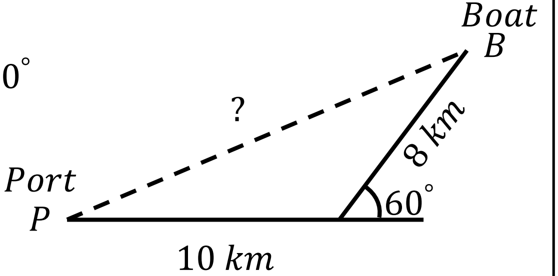


Example 3.70: *Suppose that a boat travels 10km from the port towards east and then turns 60° to its left. If the boat travels further 8km, how far from the port is the boat?*

Let BP be the required distance.

Let $a = 10, b = 8$ *and* $C = 180^\circ - 60^\circ = 120^\circ$

Then, using the cosine formula,



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$BP^2 = 10^2 + 8^2 - 2 \times 10 \times 8\cos 120^\circ$$

$$BP^2 = 100 + 64 - 160\cos(90^\circ + 30^\circ) \qquad \boxed{\cos(90^\circ + \theta) = -\sin\theta}$$

$$BP^2 = 164 - 160\,(-\sin 30^\circ)$$

$$BP^2 = 164 + \cancel{160}^{\,80}\left(\frac{1}{\cancel{2}}\right)$$

$$BP^2 = 244$$

$$\begin{array}{r|l} 2 & 244 \\ \hline 2 & 122 \\ \hline & 61 \end{array}$$

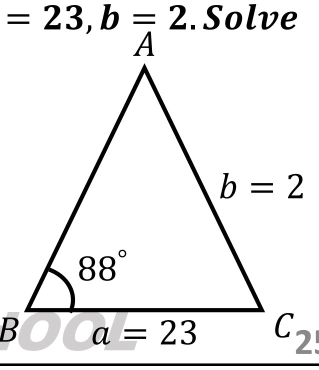
$$BP = \sqrt{244} = \sqrt{2 \times 2 \times 61} = 2\sqrt{61}\,km$$

1. *Determine whether the following measuremnetsproduce one triangle, two triangles or no triangle $\angle B = 88^\circ, a = 23, b = 2$. Solve if solution exists.*

Using sine formula, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

$$\frac{\sin A}{a} = \frac{\sin 88^\circ}{2}$$

$$\sin A = \frac{23(\sin 88^\circ)}{2}$$



$$\sin A = 23 \times 0.999$$

$$\sin A = 22.99 \text{ which is not possible.}$$

$$\therefore \text{Solution of the given triangle does not exist.}$$

2. If the sides of ΔABC are $a = 4, b = 6, c = 8$, then show that $4\cos B + 3\cos C = 2$.

$$LHS = 4\cos B + 3\cos C$$

$$= 4\left(\frac{a^2 + c^2 - b^2}{2ac}\right) + 3\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

$$= 4\left(\frac{16 + 64 - 36}{2(4)(8)}\right) + 3\left(\frac{16 + 36 - 64}{2(4)(6)}\right)$$

$$= 4\left(\frac{80 - 36}{64}\right) + 3\left(\frac{52 - 64}{48}\right)$$

$$= 4\left(\frac{44}{64}\right) + 3\left(\frac{-12}{48}\right)$$

$$= \frac{11}{4} - \frac{3}{4} = \frac{\cancel{8}}{\cancel{4}} = 2 = RHS \quad \text{Hence proved}$$

3. In ΔABC, if $a = \sqrt{3} - 1$, $b = \sqrt{3} + 1$ and $\angle C = 60^\circ$. Fimd the other side and the other two angles.

By Napier's formula, we have $\tan\left(\frac{A - B}{2}\right) = \frac{a - b}{a + b}\cot\frac{C}{2}$

$$= \frac{(\sqrt{3} + 1) - (\sqrt{3} - 1)}{(\sqrt{3} + 1) + (\sqrt{3} + 1)} \cot\frac{60^\circ}{2} = \frac{\cancel{2}}{\cancel{2}\sqrt{3}}\cot 30^\circ = \frac{1}{\cancel{\sqrt{3}}} \times \cancel{\sqrt{3}} = 1$$

$$\therefore \frac{A - B}{2} = 45^\circ \Longrightarrow A - B = 90^\circ \qquad \dots (1)$$

$$\text{Also } A + B = 180^\circ - C$$

$$= 180^\circ - 60^\circ = 120^\circ \qquad \dots (2)$$

$$\text{Adding } (1) \text{ and } (2),\ \cancel{2}A = \cancel{2}10^\circ \Longrightarrow \angle A = 105^\circ$$

Substituting $A = 105^\circ$ in (2) we get,

$$105^\circ + B = 120^\circ \Longrightarrow B = 120^\circ - 105^\circ = 15^\circ$$

Now, by sine formula, $\frac{c}{\sin C} = \frac{a}{\sin A} \Longrightarrow c = a\frac{\sin C}{\sin A}$

$$\Longrightarrow C = \frac{(\sqrt{3} + 1)\sin 60^\circ}{\sin 105^\circ} = \frac{(\sqrt{3} + 1)\frac{\sqrt{3}}{2}}{\frac{\sqrt{3} + 1}{2\sqrt{2}}}$$

$$= \frac{\sqrt{3}}{\cancel{2}} \times \cancel{2}\sqrt{2} = \sqrt{6}$$

$$\therefore \angle A = 105^\circ, \angle B = 15^\circ \text{ and } \angle C = \sqrt{6}$$

4. If any ΔABC, prove that the area $\Delta = \dfrac{b^2+c^2-a^2}{4\cot A}$

$$RHS = \frac{b^2+c^2-a^2}{4\cot A}$$

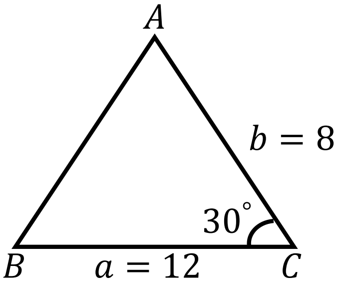
$$= \frac{b^2+c^2-a^2}{4\times\frac{\cos A}{\sin A}} = \frac{b^2+c^2-a^2}{4\cos A}\times\sin A$$

$$= \frac{\cancel{b^2+c^2-a^2}}{4\left(\frac{\cancel{b^2+c^2-a^2}}{2bc}\right)}\sin A = \frac{\cancel{2}bc}{{}_2\cancel{4}}\times\sin A$$

$$= \frac{1}{2}bc\sin A = area\ of\ \Delta ABC$$

$$\therefore \Delta = \frac{b^2+c^2-a^2}{4\cot A}$$

5. In a ΔABC, if $a = 12\ cm, b = 8\ cm$ and $\angle C = 30^\circ$, then show that its area is 24 sq. cm.



$$Area\ of\ \Delta ABC = \frac{1}{2}\ ab\sin C$$

$$= \frac{1}{\cancel{2}}\times \overset{6}{\cancel{12}}\times 8\sin 30^\circ$$

$$\therefore \Delta = 6\times 8\times\frac{1}{2} = 24\ sq.cm$$

Hence proved

6. In a ΔABC, if $a = 18\ cm, b = 24\ cm$ and $c = 30\ cm$, then show that its area is 216 sq. cm.

Given $a = 18, b = 24, c = 30$

Using heron's formula,

$$\boxed{where\ s = \frac{a+b+c}{2}}$$

$$area\ of\ \Delta ABC, \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore s = \frac{18+24+30}{2} = \frac{\cancel{72}}{\cancel{2}} = 36$$

$$\therefore \Delta = \sqrt{36(36-18)(36-24)(36-30)}$$

$$= \sqrt{36\times 18\times 12\times 6}$$

$$= \sqrt{36\times 9\times 2\times 4\times 3\times 3\times 2}$$

$$= \sqrt{36\times 9\times 4\times 4\times 9}$$

$$= 6\times 3\times 2\times 2\times 3$$

$$= 216\ sq.cm$$

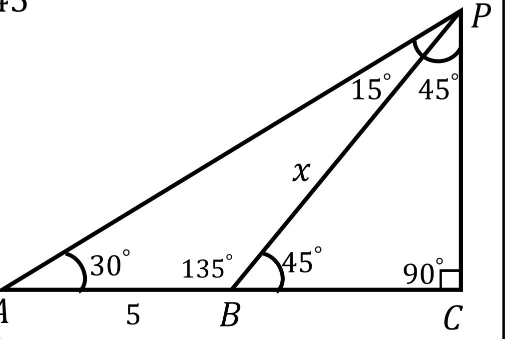
7. Two soldiers A and B in two different underground bunkers on a straght road, spot an intruder at the top of a hill. The angle of elevation of the intruder from A and B to the ground level in the eastern direction are 30° and 45° respectively. If A and B stand 5km apart, find the distance of the intruder from B.

Let P be the intryder, A and B are the soldiers,

Let x be the distance between the intruder and soldier B.

In ΔABP, Given $\angle PAB = 30^\circ$ and $\angle PBC = 45^\circ$

In ΔABP, $\angle APB = 15^\circ$



In ΔABP, using sine formula,

$$\frac{5}{\sin 15^\circ} = \frac{x}{\sin 30^\circ}$$

$$x = \frac{5}{\sin 15^\circ}\sin 30^\circ = 5 \times \frac{1}{2\sin 15^\circ}$$

$$Now, \sin 15^\circ = \sin(45^\circ - 30^\circ)$$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Substituting this value in (1) we get,

$$x = \frac{5}{\frac{\cancel{2}(\sqrt{3}-1)}{\cancel{2}\sqrt{2}}} = \frac{5\sqrt{2}}{\sqrt{3}-1}\ km$$

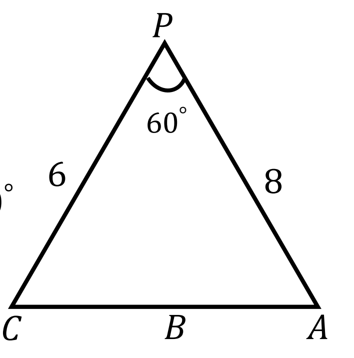
8. A researcher wants to determine the width of a pond from east to west, which cannot be done by actual measurement. From a point P, he finds the distance to the eastern − most point of the pond to be 8 km, while the distance of the western most point from P to be 6 km. If the angle between the two lines of sight is 60°. Find the width of the pond.

Let A be the point on the eastern side and B be the point on the western side.

Given $PA = 8, PB = 6$

Let $a = 6, b = 8$ and $\angle C = 60^\circ$

Using cosine formula,



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$= 36 + 64 + 2(6)(8)\cos 60^\circ$$

$$= 100 - \left(12 \times \cancel{8}^{4} \times \frac{1}{\cancel{2}}\right)$$

$$= 100 - 48 = 52$$

$$C = \sqrt{52} = \sqrt{4 \times 13}$$

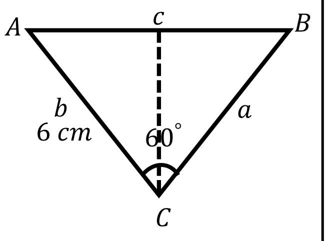
$$\therefore C = 2\sqrt{13}\ km$$

Hence the width of the river is $2\sqrt{13}$ km.

9. *Two navy helicopters A and B are flying over the bay of bengal at same altitude from the sea level to search a missing boat. Pilots of both the helicopters sight the boat at the same time while they are apart 10 km from each other. If the distance of the boat from A is 6 km and if the line segment AB subtends 60° at the boat, find the distance of the boat from B.*

Let c be the position of the boat and A and B are the positions of the pilot.

Using cosine formula, $c^2 = a^2 + b^2 - 2ab\cos C$



$$10^2 = a^2 + 6^2 - 2a(6)\cos 60^\circ$$

$$100 = a^2 + 36 - \cancel{12}\,a\left(\frac{1}{\cancel{2}}\right)$$

$$a^2 + 36 - 6a - 100 = 0$$

$$a^2 - 6a - 64 = 0$$

$$a^2 = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(-64)}}{2}$$

$$\left[\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$$

$$a = 1, b = -6, c = -64$$

$$a = \frac{6 \pm \sqrt{36 + 256}}{2}$$

$$\begin{array}{r|l} 2 & 292 \\ \hline 2 & 146 \\ \hline & 73 \end{array}$$

$$a = \frac{6 \pm \sqrt{292}}{2} \Rightarrow a = \frac{6 \pm \sqrt{2 \times 2 \times 73}}{2}$$

$$a = \frac{6 \pm 2\sqrt{73}}{2} \Rightarrow a = \frac{\cancel{2}(3 \pm \sqrt{73})}{\cancel{2}}$$

[*since* $3 - \sqrt{73}$ *is negative*]

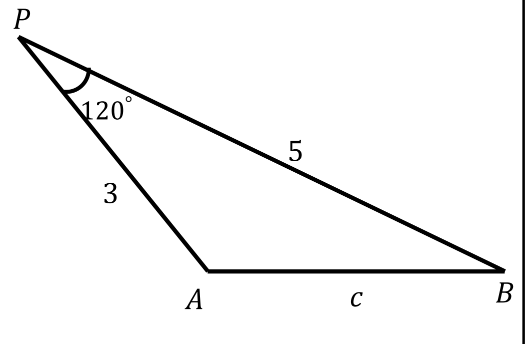
$$a = 3 \pm \sqrt{73} \Rightarrow a = \left(3 + \sqrt{73}\right) km$$

10. *A straight tunnel is to be made through a mountain. A surveyor observes the two extremities A and B of the tunnel to be built from a point P in front of the mountain. If AP = 3 km, BP = 5 km and $\angle APB = 120^\circ$, then find the length of the tunnel to be built.*

Let AB = C be the length of the tunnel.

Let $a = 5, b = 3$ *and* $\angle c = 120^\circ$

Using cosine formula,



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$= 25 + 9 - 2(5)(3)\cos 120^\circ$$

$$= 34 - 30\cos(180^\circ - 60^\circ)$$

$$= 64 - 30(-\cos 60^\circ)$$

$$= 34 - \overset{15}{\cancel{30}}\left(\frac{-1}{\cancel{2}}\right)$$

$$= 34 + 15 = 49$$

$$c^2 = 49$$

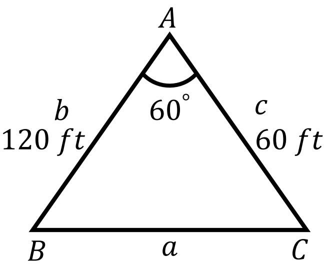
$$c = 7\ km$$

Hence, length of the tunnel = 7 km.

11. ***A farmer wants to purchase a triangular shaped land with sides 120 feet and 60 feet and the angle included between these two sides is 60°. If the land costs Rs. 500 per sq. ft, Find the amount he needed to purchase land. Also, find the perimeter of the land.***

$$\text{Given } b = 60, c = 120 \text{ and } \angle A = 60^\circ$$

Using cosine formula,



$$\begin{array}{r|l} 2 & 10800 \\ \hline 2 & 5400 \\ \hline 2 & 2700 \\ \hline 2 & 1350 \\ \hline 5 & 675 \\ \hline 5 & 135 \\ \hline 3 & 27 \\ \hline 3 & 9 \\ \hline & 3 \end{array}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$= 120^2 + 60^2 - 2(120)(60)\cos 60^\circ$$
$$= 14400 + 3600 - 14400\left(\frac{1}{2}\right)$$
$$= 18000 - 7200$$
$$a^2 = 10{,}800 \Longrightarrow a = \sqrt{10800}$$
$$a = \sqrt{2\times2\times2\times2\times5\times5\times3\times3\times3}$$
$$a = 2\times2\times5\times3\sqrt{3}$$
$$a = 60\sqrt{3}$$

$\therefore$ *Perimeter of the triangular land*

$$s = a + b + c$$
$$= 120 + 60 + 60\sqrt{3} = 180 + 60\sqrt{3}$$
$$= 60\left(3 + \sqrt{3}\right) \text{ feet}$$

Area of the triangular field ABC,

$$= \frac{1}{2}bc\sin A$$
$$= \frac{1}{2}\times 60\times 120\times \sin 60^\circ = 30\times 120\times \frac{\sqrt{3}}{2}$$
$$= 30\times 60\times\sqrt{3} = 1800\sqrt{3}$$
$$= 1800\times 1.732 = 3117.6$$

$$\text{Cost of land for } 1 \text{ sq.ft} = Rs. 500$$

$$\therefore \text{cost of } 3117.6 \text{ sq.ft land} = 500\times 3117.6 = Rs. 1{,}55{,}800$$

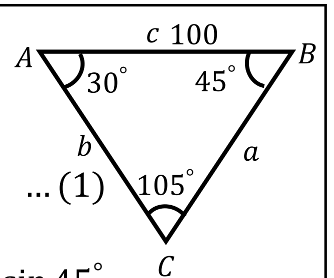
12. ***A fighter jet has a hit a small target by flying a horizontal distance. When the target is sighted the pilot measures the angle of depression to be 30°. If after 100 km, the target has an angle of depression of 45°, how far is the target from the fighter jet at the instant?***

Let C be the position of the target and A and B be the positions of the fighter jet.

$$\text{Given } \angle BAC = 30^\circ,\ \angle ABC = 45^\circ$$
$$\therefore \angle C = 180 - (30^\circ - 45^\circ) = 180 - 75^\circ = 105^\circ$$

$$\text{Given } AB = 100 \text{ km}$$

Using sine formula,



$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 30^\circ} = \frac{100}{\sin 105^\circ}$$

$$\Rightarrow \frac{a}{\frac{1}{2}} = \frac{100}{\sin 105^\circ} \Rightarrow 2a = \frac{100}{\sin 105^\circ} \Rightarrow a = \frac{50}{\sin 105^\circ} \quad \dots (1)$$

$$Now, \sin 105^\circ = \sin(60^\circ + 45^\circ) = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$$

$$= \frac{\sqrt{3}}{2} . \frac{1}{\sqrt{2}} + \frac{1}{2} . \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} \quad \dots (2)$$

Substituting (2) in (1) we get,

$$a = \frac{50}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{50(2\sqrt{2})}{\sqrt{3}+1} = \frac{100\sqrt{2}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

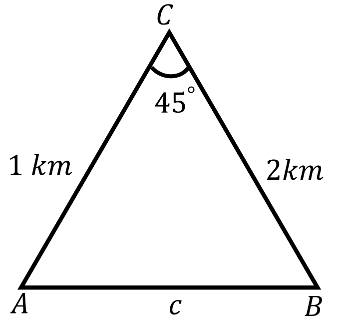
$$= \frac{100(\sqrt{6}-\sqrt{2})}{3-1} = \frac{100(\sqrt{6}-\sqrt{2})}{2} = 50(\sqrt{6}-\sqrt{2})km.$$

13. *A plane is 1 km from one landmark and 2 km from another. From the planes point of view the land between them subtends an angle of 45°. How for apart are the landmarks?*

Let A, B be the landmarks and C be the position of the plane.

Given $\angle ACB = 45^\circ$

Using cosine formula,



$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$c^2 = 2^2 + 1^2 - 2(2)(1) \cos 45^\circ$$

$$= 4 + 1 - 4\left(\frac{1}{\sqrt{2}}\right) = 5 - 2 \times \overset{\sqrt{2}}{\cancel{2}}\left(\frac{1}{\cancel{\sqrt{2}}}\right)$$

$$c^2 = 5 - 2\sqrt{2}$$

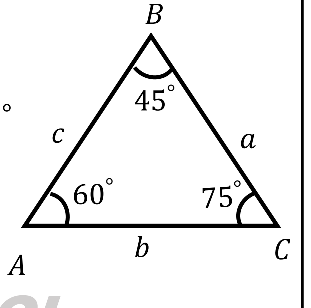
$$c^2 = \sqrt{5 - 2\sqrt{2}}\ km$$

14. *A man starts his morning walk at a point A reaches two points B and C and finally back to A such $\angle A = 60^\circ$ and $\angle B = 45^\circ$, $AC = 4km$ in ΔABC. Find the total distance he covered during his morning walk.*

Given $AC = 4\ km, \angle A = 60^\circ$ *and* $\angle B = 45^\circ$

$$\angle C = 180^\circ - (A + B)$$

$$= 180^\circ - (60^\circ + 45^\circ) = 180^\circ - 105^\circ = 75^\circ$$



Using sine formula,

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 60^\circ} = \frac{c}{\sin 45^\circ} \Longrightarrow \frac{a}{\frac{\sqrt{3}}{2}} = \frac{4}{\frac{1}{\sqrt{2}}}$$

$$\frac{2a}{\sqrt{3}} = 4\sqrt{2}$$

$$a = \frac{4\sqrt{6}}{2} = 2\sqrt{6}\ km.$$

$$Again\ using\ sine\ formula,\ \frac{c}{\sin c} = \frac{a}{\sin A}$$

$$\frac{c}{\sin 75^\circ} = \frac{2\sqrt{6}}{\sin 60^\circ} \Rightarrow \frac{c}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{2\sqrt{6}}{\frac{\sqrt{3}}{2}}$$

$$\frac{C.2\sqrt{2}}{\sqrt{30}+1} = \frac{4\sqrt{6}}{\sqrt{3}} \Rightarrow C = \frac{\overset{2}{\cancel{4\sqrt{6}}}(\sqrt{3}+1)}{\cancel{2\sqrt{6}}} = 2(\sqrt{3}+1)$$

$$[\therefore \sin 75^\circ = \sin(45^\circ + 30^\circ)$$

$$= sin45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$\boxed{= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}]}$$

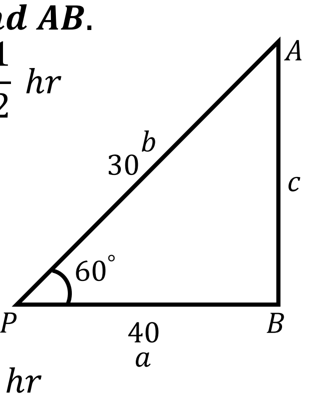
$$\therefore Total\ distance\ covered\ by\ the\ man = a + b + c$$

$$= 2\sqrt{6} + 4 + 2(\sqrt{3}+1)$$

$$= 2\sqrt{6} + 4 + 2\sqrt{3} + 2$$

$$= (6 + 2\sqrt{3} + 2\sqrt{6}\ )km.$$

15. *Two vehicles leave the same place P at the same time moving along two different roads. One vehicle moves at an average speed of 60km/hr and the other vehicle moves at an average speed of 80km/hrAfter half an hour the vehicle reach the destinationA and B. If AB subtends 60° at the initial point P, then find AB.*



$$Speed\ taken\ by\ the\ vehicle\ I = 60\ km/hr.\ Time = \frac{1}{2}\ hr$$

$$\therefore Distance = speed \times time$$

$$= \left(60 \times \frac{1}{2}\right) = 30\ km$$

$$PA = 30km$$

$$Speed\ taken\ by\ the\ vehicle\ II = 60\ km/hr.\ Time = \frac{1}{2}\ hr$$

$$\therefore Distance = speed \times time = \left(80 \times \frac{1}{2}\right) = 40\ km$$

$$PB = 40\ km$$

$$Given\ \angle APB = 60^{\circ}$$

$$Using\ cosine\ formula,\ c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 40^2 + 30^2 - 2(40)(30)\cos 60^{\circ}$$

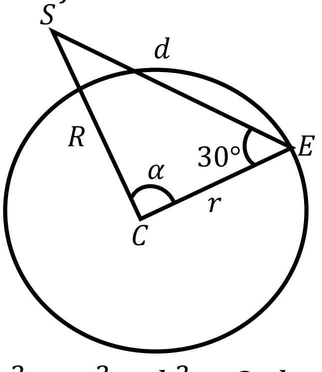
$$= 1600 + 900 - \cancel{2}(40)(60)\left(\frac{1}{\cancel{2}}\right)$$

$$= 2500 - 1200 = 1300$$

$$c = \sqrt{1300} = \sqrt{13 \times 100} = 10\sqrt{13}\ km$$

16. *Suppose that a satellite in sapce, an earth station and the center of earth all lie in the same plane. Let r be the radius of earth and R be the distance from the centre of earth to the satellite. Let d be the distancee from the earth station to the satellite. Let 30° be the angle of elevation from the earth station to the satellite. If the line segment connecting earth station and satellite subtends angle α at the centre of earth, then prove that* $\boldsymbol{d = \sqrt{1 + \left(\frac{r}{R}\right)^2 - 2\frac{r}{R}\cos\alpha}}$.

Let S be the position of the satellite, E be the position of the earth station and C be the centre of the earth



$$Given\ CE = r, CS = R\ and\ SE = d$$

$$Given\ \angle SCE = \alpha$$

In ΔSCE, applying cosine rule, we get.

$$d^2 = r^2 + R^2 - 2rR\cos\alpha$$

$$[\because\ c^2 = a^2 + b^2 - 2ab\cos C]$$

Dividing by R^2 *throughout we get,*

$$\frac{d^2}{R^2} = \frac{r^2}{R^2} + \frac{R^2}{R^2} - \frac{2r.R}{R^2}\cos\alpha$$

$$\frac{d^2}{R^2} = \frac{r^2}{R^2} + 1 - \frac{2r}{R}\cos\alpha$$

$$d^2 = R^2\left[\frac{r^2}{R^2} + 1 - \frac{2r}{R}\cos\alpha\right]$$

Taking positive square root both sides we get,

$$d = R\sqrt{\frac{r^2}{R^2} + 1 - \frac{2r}{R}\cos\alpha}$$

$$d = R\sqrt{1 + \left(\frac{r}{R}\right)^2 - 2\left(\frac{r}{R}\right)\cos\alpha}$$

Hence proved

EXERCISE : 5.1

Binomial Theorem:

A BINOMIAL is an algebraic expression of two terms which are connected by the operation + or −

For example : $x + 2y, x - y, x^3 + 4y, a + b$ *etc.. are binomials*

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

The expansion of binomials with higher powers such as $(x+a)^{10}, (x+a)^{17}$, etc becomes more difficult.

Therefore we look for a general formula which will help us in finding the expansion of binomials with higher powers.

$$(a+b)^n = nC_0a^nb^0 + nC_1a^{n-1}b^1 + nC_2a^{n-2}b^2 + \cdots + nC_na^0b^n$$

Particular term in the expansion: $T_{r+1} = nC_ra^{n-r}b^r$

Middle Term:

Case (i) : n is even

There is only one middle term and it is given by $\frac{n}{2} + 1$

Case (ii) : n is odd

There are two middle terms and they are given by $\left(\frac{n+1}{2}\right), \left(\frac{n+3}{2}\right)$

Example* 5.1**: ***Find the expansion of $\boldsymbol{(2x+3)^5}$

$$\boxed{(a+b)^n = a^n + nC_1a^{n-1}b^1 + nC_2a^{n-2}b^2 + \cdots + b^n}$$

$a = 2x,\ b = 3$ *and* $n = 5$

$$(2x+3)^5 = (2x)^5 + 5C_1(2x)^{5-1}(3)^1 + 5C_2(2x)^{5-2}(3)^2 + 5C_3(2x)^{5-3}(3)^3 + 5C_4(2x)^{5-4}(3)^4 + (3)^5$$

$$= (2x)^5 + 5(2x)^4(3) + 10(2x)^3(9) + 10(2x)^2(27) + 5(2x)(81) + 243$$

$$= 32x^5 + (15)16x^4 + 90(8x^3) + 270(4x^2) + 810x + 243$$

$$= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$$

$$5C_2 = \frac{5\times \cancel{4}^{\,2}}{\cancel{2}\times 1}$$

$$\because 5C_3 = 5C_2$$

$$\because 5C_4 = 5C_1$$

Example* 5.2**: ***Evaluate $\boldsymbol{98^4}$

$$98^4 = (100-2)^4$$

$$\boxed{(a-b)^n = a^n - nC_1a^{n-1}b^1 + nC_2a^{n-2}b^2 - \cdots + b^n}$$

$a = 100, b = 2$ *and* $n = 4$

$$(100-2)^4 = 100^4 - 4C_1100^{4-1}(2)^1 + 4C_2100^{4-2}(2)^2 - 4C_3100^{4-3}(2)^3 + (2)^4$$

$$= 100^4 - 4(100^3)(2) + 6(100^2)(4) - 4(100^1)(8) + 16$$

$= 100^4 - 8\,(1000000) + 24\,(10000) - 32\,(100) + 16$

$= 100000000 - 8000000 + 240000 - 3200 + 16$

$= 92236816$

Example 5.3: ***Find the middle term in the expansion of*** $(x+y)^6$

$a = x,\ b = y,\ n = 6$

if n is even there is only one middle term and it is given by $\frac{n}{2} + 1$

$middle\ term = \frac{6}{2} + 1 = 3 + 1 = 4$

$\boxed{6C_3 = \frac{6\times5\times4}{1\times2\times3} \quad 6C_3 = 20}$

∴4^{th} term is the middle term.

$T_{r+1} = nC_r a^{n-r} b^r \quad where\ r = 3$

$T_{3+1} = 6C_3\,(x)^{6-3}\,(y)^3$

$T_4 = 20\,x^3\,y^3$

4. *Find the middle terms in the expansion of* $(x+y)^7$

$a = x,\ b = y,\ n = 7$

if n is odd there are two middle terms and they are given by $\left(\frac{n+1}{2}\right), \left(\frac{n+3}{2}\right)$

$middle\ term = \frac{7+1}{2}, \frac{7+3}{2}$

$middle\ term = \frac{8}{2}, \frac{10}{2} = 4, 5$

$\boxed{7C_3 = \frac{7\times6\times5}{1\times2\times3}}$

4^{th} and 5^{th} term are the middle term.

To find 4^{th} ***term***

$T_{r+1} = nC_r a^{n-r} b^r \quad , where\ r = 3$

$T_{3+1} = 7C_3 \times x^{7-3} \times (y)^3$

$T_4 = 35\,x^4 y^3$

$To\ find\ 5^{th}\ term \implies T_{r+1} = nC_r a^{n-r} b^r \quad where\ r = 4$

$T_{4+1} = 7C_4 \times x^{7-4} \times (y)^4 \Rightarrow T_5 = 35\,x^3 y^4$

Example 5.5: ***Find the coefficient of*** x^6 ***in the expansion of*** $(3+2x)^{10}$

$a = 3,\ b = 2x,\ n = 10,\ r = 6$

$\boxed{T_{r+1} = nC_r a^{n-r} b^r}$

$\boxed{10C_4 = \frac{10\times\overset{3}{\cancel{9}}\times\overset{2}{\cancel{8}}\times7}{\cancel{4}\times\cancel{3}\times2\times1}}$

$T_{6+1} = 10C_6\,3^{10-6}\,(2x)^6$

$= 10C_{10-6}\,3^4\,(2x)^6 = 10C_4\,3^4\,(2x)^6$

$= 210 \times 3^4 \times 2^6 x^6$

$coefficient\ of\ x^6 = 210 \times 3^4 \times 2^6$

Example 5.6: Find the coefficient of x^3 in the expansion of $(2-3x)^7$

$$a = 2,\ b = -3x,\ n = 7,\ r = 3$$

$$\boxed{T_{r+1} = nC_r a^{n-r} b^r}$$

$$7C_3 = \frac{7 \times \cancel{6}^{3} \times 5}{3 \times \cancel{2} \times 1}$$

$$T_{3+1} = 7C_3\, 2^{7-3}(-3x)^3$$
$$= 35\, 2^4\,(-3x)^3 = 35 \times 2^4 \times -27\, x^3$$
$$= 35 \times 16 \times -27x^3 = -15120x^3$$

$Coefficient\ of\ x^3 = -15120$

Example 5.7: The 2nd, 3rd and 4th terms in the binomial expression of $(x+a)^n$ are 240, 720 and 1080 for a suitable of x. Find x, a and n.

$$(a+b)^n = a^n + nC_1 a^{n-1} b^1 + nC_2 a^{n-2} b^2 + \cdots + b^n$$

$$a = x,\quad b = a \quad and\ n = n$$

$$T_{r+1} = nC_r a^{n-r} b^r$$

$Given: T_2 = 240, T_3 = 720\ and\ T_4 = 1080$

$T_2 = 240$

$r = 1$

$$T_{1+1} = nC_1\, x^{n-1}\, a^1 \Longrightarrow T_2 = nC_1 x^{n-1} a$$

$$240 = nC_1 x^{n-1} a \quad \ldots(1)$$

$T_3 = 720$

$r = 2$

$$T_{2+1} = nC_2\, x^{n-2}\, a^2 \Longrightarrow T_3 = nC_2 x^{n-2} a^2$$

$$720 = nC_2 x^{n-2} a^2 \ldots(2)$$

$T_4 = 1080$

$r = 3$

$$T_{3+1} = nC_3\, x^{n-3}\, a^3 \Longrightarrow T_4 = nC_3 x^{n-3} a^3$$

$$1080 = nC_3 x^{n-3} a^3 \ldots(3)$$

$Dividing\ (2)\ by\ (1)$

$$\frac{nC_2 x^{n-2} a^2}{nC_1 x^{n-1} a} = \frac{\cancel{720}^{3}}{\cancel{240}} \Longrightarrow \frac{\frac{n(n-1)}{1 \times 2}\ x^{n-2} \cancel{a^2}^{a}}{\cancel{n}\ x^{n-1} a} = 3$$

$$\frac{n-1}{2} \times x^{n-2} \times x^{-n+1} \times a = 3 \Longrightarrow \frac{n-1}{2} \times x^{n-2-n+1} \times a = 3$$

$$\frac{n-1}{2} \times x^{-1} \times a = 3 \Longrightarrow \frac{n-1}{x} \times a = 6$$

$$\frac{a}{x} = \frac{6}{n-1} \quad \ldots(4)$$

Dividing (3) *by* (2)

$$\frac{nC_3 x^{n-3} a^3}{nC_2 x^{n-2} a^2} = \frac{1080}{720}$$

$$\frac{\frac{n\cancel{(n-1)}(n-2)}{1\times\cancel{2}\times 3}\times x^{n-3}\cancel{a^3}^{\,a}}{\frac{n\cancel{(n-1)}}{1\times\cancel{2}}\times x^{n-2}a^2} = \frac{\cancel{108}\ \cancel{12}\ 3}{\cancel{72}\ 8\ 2} \Rightarrow \frac{n-2}{3}\times x^{n-3}\times x^{-n+2}\times a = \frac{3}{2}$$

$$\frac{n-2}{3}\times x^{n-3-n+2}\times a = \frac{3}{2} \Rightarrow \frac{n-2}{3}\times x^{-1}\times a = \frac{3}{2}$$

$$\frac{n-2}{3}\times\frac{a}{x} = \frac{3}{2} \Rightarrow \frac{a}{x} = \frac{3}{2}\times\frac{3}{n-2}$$

$$\frac{a}{x} = \frac{9}{2(n-2)} \quad \dots(5)$$

From (4) *and* (5)

$$\frac{a}{x} = \frac{6}{n-1} \ \dots(4) \ \textit{and} \ \frac{a}{x} = \frac{9}{2(n-2)} \ \dots(5)$$

$$\frac{6}{n-1} = \frac{9}{2(n-2)}$$

$$6\times 2(n-2) = 9\,(n-1) \Rightarrow 12\,(n-2) = 9n-9$$

$$12n-24 = 9n-9 \Rightarrow 12n-24-9n+9 = 0$$

$$3n-15 = 0 \Rightarrow 3n = 15$$

$$n = \frac{15}{3} \Rightarrow n = 5$$

sub $n = 5$ *in* (1) $nC_1 x^{n-1} a = 240$

$$5C_1 x^{5-1} a = 240 \Rightarrow 5x^4 a = 240 \dots(6)$$

sub $n = 5$ *in* (4) $\frac{a}{x} = \frac{6}{n-1}$

$$\frac{a}{x} = \frac{6}{5-1} \Rightarrow \frac{a}{x} = \frac{6}{4} \Rightarrow \frac{a}{x} = \frac{3}{2} \ \dots(7)$$

Divide (6) *and* (7)

$$\frac{5x^4 a}{\frac{a}{x}} = \frac{240}{\frac{3}{2}} \Rightarrow 5x^4 a\times\frac{x}{\cancel{a}} = \overset{80}{\cancel{240}}\times\frac{2}{3} \Rightarrow 5x^5 = 160$$

$$x^5 = \frac{\overset{32}{\cancel{160}}}{\cancel{5}} \Rightarrow x^5 = 32 \Rightarrow x = \sqrt[5]{32}$$

$$x = \sqrt[5]{2\times 2\times 2\times 2\times 2} \implies x = 2$$

$$sub\ x = 2\ in\ (7) \implies \frac{a}{x} = \frac{3}{2}$$

$$\frac{a}{\cancel{2}} = \frac{3}{\cancel{2}} \implies \boxed{a = 3}$$

$$\frac{\overset{2}{\cancel{4}}\times 3}{\cancel{2}\times 1}$$

$$\because 4C_3 = 4C_1$$

Example 5.8: Expand $\left(\mathbf{2x - \frac{1}{2x}}\right)^{4}$

$$(a-b)^n = a^n - nC_1 a^{n-1}b^1 + nC_2 a^{n-2}b^2 - \cdots + b^n$$

$$a = 2x,\ b = \frac{1}{2x},\ n = 4$$

$$\left(2x - \frac{1}{2x}\right)^4 = (2x)^4 - 4C_1(2x)^{4-1}\left(\frac{1}{2x}\right)^1 + 4C_2(2x)^{4-2}\left(\frac{1}{2x}\right)^2$$

$$- 4C_3\ (2x)^{4-3}\left(\frac{1}{2x}\right)^3 + \left(\frac{1}{2x}\right)^4$$

$$= (2x)^4 - 4\,(2x)^3\left(\frac{1}{2x}\right) + 6\,(2x)^2\left(\frac{1}{4x^2}\right) - 4\,(2x)^1\left(\frac{1}{8x^3}\right) + \left(\frac{1}{16x^4}\right)$$

$$= 16x^4 - \overset{2}{4}\times 8\overset{x^2}{x^3}\left(\frac{1}{2x}\right) + 6\times 4x^2\left(\frac{1}{4x^2}\right) - 4\times\cancel{2x}\left(\frac{1}{\underset{2\ \ x^2}{\cancel{8x^3}}}\right) + \left(\frac{1}{16x^4}\right)$$

$$= 16x^4 - 16x^2 + 6 - \frac{1}{x^2} + \frac{1}{16x^4}$$

Example 5.9: Expand $\left(\mathbf{x^2 + \sqrt{1-x^2}}\right)^{5} + \left(\mathbf{x^2 - \sqrt{1-x^2}}\right)^{5}$

$$Let\ y = \sqrt{1-x^2} \implies (x^2+y)^5 + (x^2-y)^5$$

$$(a+b)^n = a^n + nC_1 a^{n-1}b^1 + nC_2 a^{n-2}b^2 + \cdots + b^n$$

$$a = x^2,\ b = y\ and\ n = 5$$

$$(x^2+y)^5 = (x^2)^5 + 5C_1(x^2)^{5-1}(y)^1 + 5C_2(x^2)^{5-2}(y)^2 + 5C_3(x^2)^{5-3}(y)^3 + 5C_4(x^2)^{5-4}(y)^4 + (y)^5$$

$$(x^2+y)^5 = (x^2)^5 + 5C_1(x^2)^4(y)^1 + 5C_2(x^2)^3(y)^2 + 5C_3(x^2)^2(y)^3 + 5C_4(x^2)^1(y)^4 + (y)^5$$

$$(x^2+y)^5 = x^{10} + 5x^8y + 10x^6y^2 + 10x^4y^3 + 5x^2y^4 + y^5$$

$$(x^2-y)^5 = x^{10} - 5x^8y + 10x^6y - 10x^4y^3 + 5x^2y^4 - y^5$$

$$(x^2+y)^5 + (x^2-y)^5 = x^{10} + 5x^8y + 10x^6y^2 + 10x^4y^3 + 5x^2y^4 + y^5 + x^{10} - 5x^8y + 10x^6y^2 - 10x^4y^3 + 5x^2y^4 - y^5$$

$$(x^2+y)^5 + (x^2-y)^5 = x^{10} + 10x^6y^2 + 5x^2y^4 + x^{10} + 10x^6y^2 + 5x^2y^4$$

$= 2x^{10} + 20x^6y^2 + 10x^2y^4$

$= 2x^{10} + 20x^6\left(\sqrt{1-x^2}\right)^2 + 10x^2\left(\sqrt{1-x^2}\right)^4$

$= 2x^{10} + 20x^6(1-x^2) + 10x^2\left(\sqrt{1-x^2}\right)^4$

$\left(\sqrt{1-x^2}\right)^4 = \left(\sqrt{1-x^2}\right)^2\left(\sqrt{1-x^2}\right)^2$

$\left(\sqrt{1-x^2}\right)^4 = (1-x^2)(1-x^2)$

$= (1-x^2)^2$

$= 2x^{10} + 20x^6 - 20x^8 + 10x^2(1-x^2)^2$

$= 2x^{10} + 20x^6 - 20x^8 + 10x^2(1^2 - 2\times 1\times x^2 + x^4)$

$= 2x^{10} + 20x^6 - 20x^8 + 10x^2(1 - 2x^2 + x^4)$

$= 2x^{10} + 20x^6 - 20x^8 + 10x^2 - 20x^4 + 10x^6$

$= 2x^{10} - 20x^8 + 30x^6 - 20x^4 + 10x^2 = 2[x^{10} - 10x^8 + 15x^6 - 10x^4 + 5x^2]$

***Example 5.10**: Using Binomial theorem, prove that $6^n - 5n$ always leaves remainder 1 when divided by 25 for all positive integer n.*

To prove this it is enough to prove, $6^n - 5n = 25k + 1$ for some integer k.

$(a+b)^n = a^n + nC_1a^{n-1}b^1 + nC_2a^{n-2}b^2 + \cdots + b^n$

$a = 1,\ b = 5,\ n = n,$

$(1+5)^n = (1)^n + nC_1\,(1)^{n-1}(5)^1 + nC_2\,(1)^{n-2}(5)^2 + \cdots + 5^n$

$(1+5)^n = 1 + nC_1 5 + nC_2 5^2 + nC_3 5^3 + \cdots + 5^n$

$6^n = 1 + 5n + 25\,nC_2 + 125\,nC_3 + \cdots + 5^n$

$6^n = 1 + 5n + 25(nC_2 + 5\,nC_3 + \cdots + 5^{n-2})$

$6^n - 5n = 1 + 25(nC_2 + 5nC_3 + \cdots + 5^{n-2})$

$= 1 + 25k,\ k \in \mathrm{N}$

Thus $6^n - 5n$ always leaves remainder 1 when divided by 25 for all positive integer n.

***Example 5.11**: Find the last two digits of the number 7^{400}*

$7^{400} = (7^2)^{200} = (49)^{200} = (50-1)^{200}$

$(a-b)^n = a^n - nC_1a^{n-1}b^1 + nC_2a^{n-2}b^2 - \cdots + b^n$

$a = 50,\ b = 1$ *and* $n = 200$

$(50-1)^{200} = 50^{200} - 200C_1\,50^{200-1}\,(1)^1 + \ldots\ldots + 200C_{198}50^{200-198}(1)^{198} - 200C_{199}50^{200-199}(1)^{199} + (1)^{200}$

$= 50^{200} - 200C_1\,50^{199} + \ldots\ldots + 200C_{198}50^2 - 200(50) + 1$

The last two digits: 0, 1.

1. *Expand* $(i)\ \left(2x^2 - \frac{3}{x}\right)^3\ (ii)\ \left(2x^2 - 3\sqrt{1-x^2}\right)^4 + \left(2x^2 + 3\sqrt{1-x^2}\right)^4$

$(i)\ \left(2x^2 - \frac{3}{x}\right)^3$

$$(a-b)^n = a^n - nC_1 a^{n-1} b^1 + nC_2 a^{n-2} b^2 - \cdots + b^n$$

$$\boxed{3C_2 = \frac{3\times 2}{1\times 2}}$$

$$a = 2x^2,\ b = \frac{3}{x}, n = 3$$

$$\left(2x^2 - \frac{3}{x}\right)^3 = (2x^2)^3 - 3C_1(2x^2)^{3-1}\left(\frac{3}{x}\right)^1 + 3C_2\,(2x^2)^{3-2}\left(\frac{3}{x}\right)^2 - \left(\frac{3}{x}\right)^3$$

$$= 8x^6 - 3\,(2x^2)^2\left(\frac{3}{x}\right) + 3\,(2x^2)^1\left(\frac{9}{x^2}\right) - \frac{27}{x^3}$$

$$= 8x^6 - 3\times 4\,x^4\left(\frac{3}{x}\right) + 3\times 2\,x^2\left(\frac{9}{x^2}\right) - \frac{27}{x^3}$$

$$= 8x^6 - 3\times 4\,\cancel{x^4}^{\,x^3}\left(\frac{3}{\cancel{x}}\right) + 3\times 2\,\cancel{x^2}\left(\frac{9}{\cancel{x^2}}\right) - \frac{27}{x^3}$$

$$= 8x^6 - 36\,x^3 + 54 - \frac{27}{x^3}$$

(ii) $\left(2x^2 - 3\sqrt{1-x^2}\right)^4 + \left(2x^2 + 3\sqrt{1-x^2}\right)^4$

$Put\ \sqrt{1-x^2} = a \quad \Longrightarrow \quad (2x^2-3a)^4 + (2x^2+3a)^4$

$$\boxed{(a-b)^n = a^n - nC_1 a^{n-1} b^1 + nC_2 a^{n-2} b^2 - \cdots + b^n}$$

$$a = 2x^2, b = 3a, n = 4$$

$$(2x^2-3a)^4 = (2x^2)^4 - 4C_1\,(2x^2)^{4-1}\,(3a)^1 + 4C_2\,(2x^2)^{4-2}\,(3a)^2 - 4C_3\,(2x^2)^{4-3}(3a)^3 + (3a)^4$$

$$\boxed{(a+b)^n = a^n + nC_1 a^{n-1} b^1 + nC_2 a^{n-2} b^2 + \cdots + b^n}$$

$$a = 2x^2, b = 3a, n = 4$$

$$(2x^2+3a)^4 = (2x^2)^4 + 4C_1\,(2x^2)^{4-1}\,(3a)^1 + 4C_2\,(2x^2)^{4-2}\,(3a)^2 + 4C_3\,(2x^2)^{4-1}(3a)^3 + (3a)^4$$

$$(2x^2-3a)^4 + (2x^2+3a)^4$$

$$= (2x^2)^4 - \cancel{4C_1\,(2x^2)^{4-1}\,(3a)^1} + 4C_2\,(2x^2)^{4-2}\,(3a)^2 - \cancel{4C_3\,(2x^2)^{4-1}(3a)^3} + (3a)^4$$
$$+ (2x^2)^4 + \cancel{4C_1\,(2x^2)^{4-1}\,(3a)^1} + 4C_2\,(2x^2)^{4-2}\,(3a)^2 + \cancel{4C_3\,(2x^2)^{4-1}(3a)^3} + (3a)^4$$

$$= 2(2x^2)^4 + 2\,[4C_2(2x^2)^2(3a)^2] + 2\,(3a)^4$$

$$= 2[16\,x^8 + 6\times 4\,x^4 \times 9\,a^2 + 81\,a^4]$$

$$= 2\left[16x^8 + 216\,x^4\left(\sqrt{1-x^2}\right)^2 + 81\left(\sqrt{1-x^2}\right)^4\right]$$

$$= 2\left[16x^8 + 216x^4\,(1-x^2) + 81\left(\left(\sqrt{1-x^2}\right)^2\right)^2\right]$$

$= 2\left[16x^8 + 216x^4 - 216x^6 + 81(1-x^2)^2\right]$

$= 2\left[16x^8 + 216x^4 - 216x^6 + 81(1-2x^2+x^4)\right]$

$= 2\left[16x^8 + 216x^4 - 216x^6 + 81 - 162x^2 + 81x^4\right]$

$= 2\left[16x^8 - 216x^6 + 297x^4 - 162x^2 + 81\right]$

$= 32x^8 - 432x^6 + 594x^4 - 324x^2 + 162$

2. Compute (i) 102^4 **(ii)** 99^4 **(iii)** 9^7

$(i)\ (102)^4 = (100+2)^4$

$\boxed{(a+b)^n = a^n + nC_1 a^{n-1}b^1 + nC_2 a^{n-2}b^2 + \cdots + b^n}$

$a = 100,\ b = 2 \ \ and\ n = 4$

$(100+2)^4 = 100^4 + 4C_1\,100^3\,(2)^1 + 4C_2\,100^2\,(2)^2 + 4C_3\,100^1\,(2)^3 + (2)^4$

$= 100^4 + 4\,(1000000)(2) + 6\,(10000)(4) + 4\,(100)\,(8) + 16$

$= 100000000 + 8000000 + 240000 + 3200 + 16$

$= 108243216$

$4C_2 = \frac{\cancel{4}^{2} \times 3}{1 \times 2}$

$\because 4C_3 = 4C_1$

$(ii)\ (99)^4 = (100-1)^4$

$\boxed{(a-b)^n = a^n - nC_1 a^{n-1}b^1 + nC_2 a^{n-2}b^2 - \cdots + b^n}$

$a = 100,\ b = 1 \ \ and\ n = 4$

$(100-1)^4 = 100^4 - 4C_1\,100^3\,(1)^1 + 4C_2\,100^2\,(1)^2 - 4C_3\,100^1\,(1)^3 + (1)^4$

$= 100^4 - 4\,(1000000)(1) + 6\,(10000)(1) - 4\,(100)\,(1) + 1$

$= 100000000 - 4000000 + 60000 - 400 + 1$

$= 96059601$

$4C_2 = \frac{\cancel{4}^{2} \times 3}{1 \times \cancel{2}}$

$\because 4C_3 = 4C_1$

$(iii)\ (9)^7 = (10-1)^7$

$(a-b)^n = a^n - nC_1 a^{n-1}b^1 + nC_2 a^{n-2}b^2 - \cdots + b^n$

$a = 10,\ \ b = 1 \ \ and\ n = 7$

$(10-1)^7 = 10^7 - 7C_1\,10^6\,(1)^1 + 7C_2\,10^5\,(1)^2 - 7C_3\,10^4\,(1)^3 + 7C_4\,10^3\,(1)^4$
$\quad - 7C_5\,10^2\,(1)^5 + 7C_6\,10^1\,(1)^7 - (1)^7$

$= 10000000 - 7\,(1000000)(1) + 21\,(100000)(1) - 35\,(10000)\,(1)$
$\quad + 35\,(1000)(1) - 21\,(100)(1) + 7\,(10)(1) - 1$

$= 10000000 - 7000000 + 2100000 - 350000 + 35000 - 2100 + 70 - 1$

$= 12135070 - 7352101 = 4782969$

3. *Using Binomial theorem, indicate which of the following two number is larger* : $(1.01)^{1000000}$, 10000.

$(1.01)^{1000000} = (1+0.01)^{1000000}$

$\boxed{(a+b)^n = a^n + nC_1 a^{n-1} b^1 + nC_2 a^{n-2} b^2 + \cdots + b^n}$

$a = 1$, $b = 0.01$, $n = 1000000$

$(1+0.01)^{1000000} = 1^{1000000} + 1000000C_1\,(1)^{1000000-1}(0.01)^1 + \ldots$

$= 1 + 1000000(1)^{999999}(0.01)^1 + \ldots$

$= 1 + 1000000 \times 1 \times 0.01 + \ldots$

$= 1 + 10000 + \cdots$ *which is greater than* 10,000

4. *Find the coefficient of* x^{15} *in* $\left(x^2 + \frac{1}{x^3}\right)^{10}$

$a = x^2,\ b = \frac{1}{x^3},\ \ n = 10,$

$\boxed{T_{r+1} = nC_r a^{n-r} b^r}$

$T_{r+1} = 10C_r\ (x^2)^{10-r}\left(\frac{1}{x^3}\right)^r$

$T_{r+1} = 10C_r\,(x^2)^{10-r}\,(x^{-3})^r$

$= 10C_r\,x^{20-2r}\,x^{-3r} = 10C_r\,x^{20-2r-3r}$

$T_{r+1} = 10C_r\ x^{20-5r}$

$20 - 5r = 15 \Longrightarrow -5r = 15 - 20$

$-5r = -5 \Longrightarrow \boxed{r = 1}$

Find the coefficient of x^{15}

sub $r = 1$ *in* $T_{r+1} = 10C_r\ x^{20-5r}$

$T_{1+1} = 10C_1 x^{20-5(1)}$

$T_2 = 10C_1 x^{20-5} = 10\,x^{15}$

coefficient of $x^{15} = 10$

5. *Find the coefficient of* x^6 *and the coefficient of* x^2 *in* $\left(x^2 - \frac{1}{x^3}\right)^6$

$T_{r+1} = nC_r a^{n-r} b^r$

$a = x^2,\ b = -\frac{1}{x^3},\ \ n = 6,$

$T_{r+1} = 6C_r (x^2)^{6-r}\left(-\frac{1}{x^3}\right)^r = 6C_r\ (x^2)^{6-r}\ \frac{(-1)^r}{x^{3r}}$

$$T_{r+1} = 6C_r x^{12-2r} \frac{(-1)^r}{x^{3r}} = 6C_r \ x^{12-2r} \ (-1)^r \ x^{-3r}$$

$$= 6C_r (-1)^r x^{12-2r-3r}$$

$$T_{r+1} = 6C_r (-1)^r x^{12-5r}$$

(i) ***To find the coefficient of*** $\boldsymbol{x^6}$

$$12 - 5r = 6 \implies -5r = 6 - 12$$

$$-5r = -6 \implies r = \frac{6}{5} \ (x^6 \ \textit{term is not possible})$$

(ii) ***To find the coefficient of*** $\boldsymbol{x^2}$

$$12 - 5r = 2 \implies -5r = 2 - 12$$

$$-5r = -10 \implies r = \frac{\cancel{-10}\ 2}{\cancel{-5}} \implies \boxed{r = 2}$$

$$T_{r+1} = 6C_r (-1)^r x^{12-5r}$$

$sub \ r = 2$

$$\boxed{6C_2 = \frac{\overset{3}{6 \times 5}}{1 \times \cancel{2}}}$$

$$T_{r+1} = 6C_2 \ (-1)^2 \ x^{12-5\times2} = 15 \times 1 \times x^{12-10}$$

$$= 15x^2$$

$Coefficient \ of \ x^2 = 15$

6. *Find the coefficient of* $\boldsymbol{x^4}$ ***in the expansion of*** $\boldsymbol{(1 + x^3)^{50} \left(x^2 + \frac{1}{x}\right)^5}$

$$\boxed{(a + b)^n = a^n + nC_1 a^{n-1} b^1 + nC_2 a^{n-2} b^2 + \cdots + b^n}$$

$$a = 1, \ b = x^3, \ n = 50$$

$$(1 + x^3)^{50} = 1^{50} + \ 50C_1 \, 1^{50-1} \, (x^3)^1 + \ 50C_2 \, 1^{50-2} \, (x^3)^2 + \ 50C_3 \, 1^{50-3} \, (x^3)^3 + \ \ldots + (x^3)^{50}$$

$$= 1 + 50C_1 \ (1) x^3 + \ 50C_2 (1) \, x^6 \ + 50C_3 (1) \, x^9 + \ \ldots + \ (x^3)^{50}$$

$$= 1 + \ 50C_1 x^3 + \ 50C_2 x^6 \ + 50C_3 x^9 \ + \ \ldots + \ (x^3)^{50}$$

$$a = x^2, \ b = \frac{1}{x}, \ n = 5$$

$$\boxed{(a + b)^n = a^n + nC_1 a^{n-1} b^1 + nC_2 a^{n-2} b^2 + \cdots + b^n}$$

$$\left(x^2 + \frac{1}{x}\right)^5 = (x^2)^5 + 5C_1 (x^2)^{5-1} \left(\frac{1}{x}\right)^1 + 5C_2 (x^2)^{5-2} \left(\frac{1}{x}\right)^2 + 5C_3 (x^2)^{5-3} \left(\frac{1}{x}\right)^3 + 5C_4 (x^2)^{5-4} \left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^5$$

$$= x^{10} + 5C_1 (x^2)^4 \frac{1}{x} + 5C_2 \, (x^2)^3 \ \frac{1}{x^2} + 5C_3 \, (x^2)^2 \ \frac{1}{x^3} + 5C_4 \, (x^2)^1 \ \frac{1}{x^4} + \ \frac{1}{x^5}$$

$$= x^{10} + 5C_1 \cancel{x^8}^{\,x^7} \frac{1}{\cancel{x}} + 5C_2 \cancel{x^6}^{\,x^4} \frac{1}{\cancel{x^2}} + 5C_3 \cancel{x^4}^{\,x} \frac{1}{\cancel{x^3}} + 5C_4 \cancel{x^2} \frac{1}{\cancel{x^4}_{\,x^2}} + \frac{1}{x^5}$$

$$= x^{10} + 5C_1 x^7 + 5C_2 x^4 + 5C_3 x + 5C_4 x^{-2} + x^{-5}$$

$$(1+x^3)^{50}\left(x^2+\frac{1}{x}\right)^5 = [1 + 50C_1 x^3 + 50C_2 x^6 + 50C_3 x^9 + \cdots + (x^3)^{50}]$$
$$[x^{10} + 5C_1 x^7 + 5C_2 x^4 + 5C_3 x^1 + 5C_4 (x^{-2}) + (x^{-5})]$$

$$Coefficient\ of\ x^4 = 1 \times 5C_2 + 50C_1 \times 5C_3 + 50C_2 \times 5C_4 + 50C_3 \times 1$$

$$= \frac{\overset{2}{5.4}}{1.2} + 50 \times \frac{\overset{2}{5.4}}{1.2} + \frac{\overset{25}{50.49}}{1.2} \times 5 + \frac{\overset{25\ \ 16}{50.49.48}}{1.2.3}$$

$$= 10 + 500 + 6125 + 19600$$

$$= 26235$$

7. Find the constant term of $\left(\mathbf{2x^3 - \frac{1}{3x^2}}\right)^5$

$$T_{r+1} = nC_r a^{n-r} b^r$$

$$a = 2x^3,\ b = -\frac{1}{3x^2},\ n = 5,$$

$$T_{r+1} = 5C_r (2x^3)^{5-r}\left(-\frac{1}{3x^2}\right)^r = 5C_r\, 2^{5-r}\, x^{15-3r} \frac{(-1)^r}{3^r x^{2r}}$$

$$= \frac{5C_r 2^{5-r}(-1)^r}{3^r} x^{15-3r} x^{-2r} = \frac{5C_r 2^{5-r}(-1)^r}{3^r} x^{15-3r-2r}$$

$$T_{r+1} = \frac{5C_r 2^{5-r}(-1)^r}{3^r} x^{15-5r}$$

$constant\ \ term\ independent\ of\ x\ is\ x^0$

$$15 - 5r = 0 \Longrightarrow -5r = \cancel{-15}^{\,3}$$

$$\boxed{r = 3}$$

$$\boxed{5C_3 = \frac{5 \times \overset{2}{\cancel{4}} \times 3}{1 \times \cancel{2} \times 3}}$$

Constant term

$$T_{3+1} = \frac{5C_3 2^{5-3}(-1)^3}{3^3} x^{15-5(3)} \Longrightarrow T_4 = \frac{5C_3 2^2 (-1)^3}{3^3} x^0$$

$$T_4 = \frac{10 \times 4 \times -1}{27} = \frac{-40}{27}$$

8. Find the last two digits of the number 3^{600}

$$3^{600} = (3^2)^{300} = 9^{300} = (10-1)^{300}$$

$$\boxed{(a-b)^n = a^n - nC_1 a^{n-1} b^1 + nC_2 a^{n-2} b^2 - \cdots + b^n}$$

$$a = 10,\ b = 1,\ n = 300$$

$$(10-1)^{300} = 10^{300} - 300C_1 10^{300-1}(1)^1 + \dots - 300C_{299}10^{300-299}(1)^{299} + (1)^{300}$$

$$= 10^{300} - 300C_1 10^{300-1}(1)^1 + \dots - 300C_1 10^1(1) + 1$$

$$= 10^{300} - 300C_1 10^{300-1}(1)^1 + \dots - 300(10)\,(1) + 1$$

$$= 10^{300} - 300C_1 10^{300-1}(1)^1 + \dots - 3000 + 1$$

Last two digits: 0, 1

9. *If n is positive integer, show that,* $9^{n+1} - 8n - 9$ *is always divisible by* 64.

$$9^{n+1} = (1+8)^{n+1}$$

$$\boxed{(a+b)^n = a^n + nC_1 a^{n-1}b^1 + nC_2 a^{n-2}b^2 + \dots + b^n}$$

$$a = 1,\quad b = 8,\quad n = n+1$$

$$(1+8)^{n+1} = (1)^{n+1} + (n+1)C_1\,(1)^{n+1-1}(8)^1 + (n+1)C_2\,(1)^{n+1-2}(8)^2 + \dots + 8^{n+1}$$

$$= 1 + (n+1)(1)8 + (n+1)C_2(1)(8)^2 + \dots + 8^{n+1}$$

$$= 1 + 8n + 8 + (n+1)C_2 8^2 + \dots + 8^{n+1}$$

$$9^{n+1} = 8n + 9 + (n+1)C_2 8^2 + (n+1)C_3 8^3 + \dots + 8^{n+1}$$

$$9^{n+1} - 8n - 9 = (n+1)C_2 8^2 + (n+1)C_3 8^3 + \dots + 8^{n+1}$$

$$= 8^2[(n+1)C_2 + (n+1)C_3 8 + \dots + 8^{n+1-2}]$$

$$= 8^2\,[(n+1)C_2 + (n+1)C_3 8 + \dots + 8^{n-1}]$$

$$9^{n+1} - 8n - 9 \text{ is divisible by } 64$$

10. *If n is an odd positive integer, prove that the coefficients of the middle terms in the epansion of* $(x+y)^n$ *are equal.*

$$(x+y)^n$$

$$a = x,\quad b = y,\quad n = n$$

$$\text{If } n \text{ is odd the middle terms are } \frac{n+1}{2} \text{ and } \frac{n+3}{2}$$

$$\text{To find } T_{\frac{n+1}{2}} \Rightarrow r = \frac{n+1}{2} - 1 \Rightarrow r = \frac{n+1-2}{2} \Rightarrow r = \frac{n-1}{2}$$

$$\boxed{T_{r+1} = nC_r a^{n-r} b^r}$$

$$T_{\frac{n-1}{2}+1} = nC_{\frac{n-1}{2}}(x)^{n-\frac{n-1}{2}}\,(y)^{\frac{n-1}{2}}$$

$$\text{coefficients of } T_{\frac{n+1}{2}} = nC_{\frac{n-1}{2}}$$

To find $T_{\frac{n+3}{2}}$

$$r = \frac{n+3}{2} - 1 \Rightarrow r = \frac{n+3-2}{2} \Rightarrow r = \frac{n+1}{2}$$

$$T_{\frac{n+1}{2}+1} = nC_{\frac{n+1}{2}}(x)^{n-\frac{n+1}{2}}\,(y)^{\frac{n+1}{2}} \implies \text{coefficients of } T_{\frac{n+3}{2}} = nC_{\frac{n+1}{2}}$$

To prove coefficients of $T_{\frac{n+1}{2}} =$ *coefficients of* $T_{\frac{n+3}{2}}$

$$nC_{\frac{n-1}{2}} = nC_{\frac{n+1}{2}}$$

To prove they are equal

$R.H.S = nC_{\frac{n+1}{2}}$ $\boxed{\because nC_r = nC_{n-r}}$

$$= nC_{n-\left(\frac{n+1}{2}\right)} = nC_{\frac{2n-n-1}{2}}$$

$$= nC_{\frac{n-1}{2}} = L.H.S$$

11. *If n is a positive integer and r is a non negative, prove that the coefficients of x^r and x^{n-r} in the expansion of $(1+x)^n$ are equal.*

$$(1+x)^n$$

$a = 1,\ \ b = x,\ \ n = n$

$\boxed{T_{r+1} = nC_r a^{n-r} b^r}$ $\boxed{nC_r = nC_{n-r}}$

$$T_{r+1} = nC_r\ (1)^{n-r}\ x^r \implies T_{r+1} = nC_r\ x^r$$

Coefficient of $x^r = nC_r$

$T_{r+1} = nC_r\ x^r$, *when* $r = n-r$

$T_{n-r+1} = nC_{n-r}\, x^{n-r} \implies$ *Coefficient of* $x^{n-r} = nC_{n-r}$

To prove: coefficients of $x^r =$ *coefficients of* x^{n-r}

12. *If a and b are distinct integers, prove that a − b is a factor of $a^n - b^n$, whenever n is a positive integer.*
[*Hint: write $a^n = (a-b+b)^n$ and expand*]

$$a^n - b^n = (\underbrace{a-b}_{a} + b)^n - b^n$$

$$\boxed{(a+b)^n = a^n + nC_1 a^{n-1} b^1 + nC_2 a^{n-2} b^2 + \cdots + b^n}$$

$a = a-b,\ b = b,\ n = n$

$$(a-b+b)^n - b^n = (a-b)^n + nC_1(a-b)^{n-1}.b + nC_2(a-b)^{n-2}.b^2 + \ \ldots + nC_{n-1}(a-b).b^{n-1} + \cancel{b^n} - \cancel{b^n}$$

$$= (a-b)^n + nC_1(a-b)^{n-1}.b + nC_2(a-b)^{n-2}.b^2 + \ \ldots + nC_{n-1}(a-b).b^{n-1}$$

$$= (a-b)\{(a-b)^{n-1} + nC_1(a-b)^{n-2}.b + \cdots + nC_{n-1}b^{n-1}\}$$

$a^n - b^n$ *is divisible by* $(a-b)$

13. *In the binomial expansion of $(a+b)^n$, the coefficient of the 4th and 13th terms are equal to each other, find n.*

$\boxed{T_{r+1} = nC_r a^{n-r} b^r}$

$\boldsymbol{To\ find}$ **4th term** $where\ r = 3$

$T_{3+1} = nC_3\ a^{n-3}\ b^3 \implies T_4 = nC_3\ a^{n-3}\ b^3$

$Coefficient\ of T_4 = nC_3$

$\boldsymbol{To\ find}$ **13th term** $where\ r = 12$

$T_{12+1} = nC_{12}\ a^{n-12}\ b^{12} \implies T_{13} = nC_{12}\ a^{n-12}\ b^{12}$

$Coefficient\ of T_{13} = nC_{12}$

$The\ coefficient\ are\ given\ to\ be\ equal$

$nC_3 = nC_{12} \implies nC_3 = nC_{n-12}$

$n - 12 = 3 \implies n = 3 + 12$

$\boxed{n = 15}$

14. $\boldsymbol{In\ the\ binomial\ coefficients\ of three\ consecutive\ terms\ in\ the\ expansion\ of\ (a+x)^n\ are\ in\ the\ ratio\ 1:7:42, then\ find\ n.}$

$Let\ T_{r+1}, T_{r+2}, T_{r+3}\ be\ the\ terms\ whose\ coefficients\ are\ in\ the\ ratio\ 1:7:42$

$(a+x)^n$

$a = a, b = x, n = n$

$\boxed{T_{r+1} = nC_r a^{n-r} b^r}$

$T_{r+1} = nC_r\ (a)^{n-r}\ x^r$

$Coefficient\ of\ T_{r+1} = nC_r$

$similiarly$

$Coefficient\ of\ T_{r+2} = nC_{r+1}$

$Coefficient\ of\ T_{r+3} = nC_{r+2}$

$nC_r : nC_{r+1} : nC_{r+2} = 1:7:42$

$nC_r : nC_{r+1} = 1:7 \quad \& \quad nC_{r+1} : nC_{r+2} = 7:42$

$$\frac{nC_r}{nC_{r+1}} = \frac{1}{7} \quad \& \quad \frac{nC_{r+1}}{nC_{r+2}} = \frac{7}{42} \implies \frac{nC_{r+1}}{nC_{r+2}} = \frac{1}{6}$$

6

$$\frac{nC_r}{nC_{r+1}} = \frac{1}{7} \implies \frac{\dfrac{n!}{r!\,(n-r)!}}{\dfrac{n!}{(r+1)!\ [n-(r+1)]!}} = \frac{1}{7} \implies \frac{\dfrac{n!}{r!\,(n-r)!}}{\dfrac{n!}{(r+1)!\ [n-r-1]!}} = \frac{1}{7}$$

$\boxed{n! = n(n-1)!}$

$\boxed{(n+1)! = (n+1)n!}$

$$\frac{n!}{r!\,(n-r)!} \times \frac{(r+1)!\ (n-r-1)!}{n!} = \frac{1}{7}$$

$$\frac{1}{r!\,(n-r)!} \times (r+1)!\,(n-r-1)! = \frac{1}{7} \qquad \boxed{(n-r)! = (n-r)(n-r-1)!}$$

$$\frac{1}{\cancel{r!}(n-r)\cancel{(n-r-1)!}} \times (r+1)\cancel{r!}\,\cancel{(n-r-1)!} = \frac{1}{7} \implies \frac{r+1}{n-r} = \frac{1}{7}$$

$$7r+7 = n-r \implies 7 = n-r-7r \qquad \boxed{(r+2)! = (r+2)(r+1)!}$$

$$7 = n-8r \qquad \boxed{(n-r-1)! = (n-r-1)(n-r-2)!}$$

$$n-8r = 7 \ldots (1)$$

$$\frac{nC_{r+1}}{nC_{r+2}} = \frac{1}{6}$$

$$\frac{\frac{n!}{(r+1)!\,[n-(r+1)]!}}{\frac{n!}{(r+2)!\;[n-(r+2)]!}} = \frac{1}{6} \implies \frac{\frac{n!}{(r+1)!\;(n-r-1)!}}{\frac{n!}{(r+2)!\;(n-r-2)!}} = \frac{1}{6}$$

$$\frac{n!}{(r+1)!\;(n-r-1)!} \times \frac{(r+2)!\,(n-r-2)!}{n!} = \frac{1}{6}$$

$$\frac{\cancel{n!}}{(r+1)!\,(n-r-1)(n-r-2)!} \times \frac{(r+2)(r+1)!\,(n-r-2)!}{\cancel{n!}} = \frac{1}{6}$$

$$\frac{r+2}{n-r-1} = \frac{1}{6} \implies 6r+12 = n-r-1$$

$$12+1 = n-r-6r$$

$$13 = n-7r \implies n-7r = 13 \ldots (2)$$

Solving (1) *and* (2)

$$\begin{array}{r} \cancel{n} - 8r = 7 \\ (-)\ \ (+)\ \ \ (-) \\ \cancel{n} - 7r = 13 \\ \hline \cancel{-}r = \cancel{-}6 \end{array} \implies \boxed{r=6}$$

Subs $r=6$ *in eqn* (1) $n-8r=7$

$$n - 8(6) = 7 \implies n-48 = 7$$

$$n = 7+48 \implies \boxed{n=55}$$

15. *In the binomial coefficients of* $(1+x)^n$, *the coefficient of the 5th, 6th and 7th terms are in A.P. Find all values of* n.

$$(1+x)^n$$

$$a=1,\ b=x,\ n=n$$

$$T_{r+1} = nC_r a^{n-r} b^r$$

To find* 5^{th} *term *where* $r=4$

$$T_{4+1} = nC_4\ (1)^{n-4}\ x^4 \implies T_5 = nC_4\ x^4$$

Coefficient of $T_5 = nC_4$

similiarly

$Coefficient\ of\ T_6 = nC_5$ $and\ Coefficient\ of\ T_7 = nC_6$

To find n.

$\underset{t_1}{nC_4}, \underset{t_2}{nC_5}, \underset{t_3}{nC_6}, are\ in\ AP$

$$t_2 - t_1 = t_3 - t_2 \implies nC_5 - nC_4 = nC_6 - nC_5$$

$$nC_5 + nC_5 = nC_6 + nC_4 \implies 2(nC_5) = nC_4 + nC_6$$

$$2 \times \frac{n!}{5!\,(n-5)!} = \frac{n!}{4!\,(n-4)!} + \frac{n!}{6!\,(n-6)!}$$

$$2 \times \frac{n!}{5!\,(n-5)!} = \frac{n!}{4!\,(n-4)!} + \frac{n!}{6!\,(n-6)!}$$

$$\frac{2}{5!\,(n-5)!} = \frac{1}{4!\,(n-4)!} + \frac{1}{6!\,(n-6)!}$$

$$\frac{2}{5 \times \cancel{4!}\,(n-5)\cancel{(n-6)!}} = \frac{1}{\cancel{4!}\,(n-4)(n-5)\cancel{(n-6)!}} + \frac{1}{6 \times 5 \times \cancel{4!}\,\cancel{(n-6)!}}$$

$$\frac{2}{5(n-5)} = \frac{1}{(n-4)(n-5)} + \frac{1}{30} \implies \frac{2}{5(n-5)} - \frac{1}{(n-4)(n-5)} = \frac{1}{30}$$

$$\frac{2(n-4)-5}{5(n-4)(n-5)} = \frac{1}{30} \implies \frac{2n-8-5}{5(n^2-5n-4n+20)} = \frac{1}{30}$$

$$\frac{2n-13}{\cancel{5}(n^2-9n+20)} = \frac{1}{\cancel{30}\ 6} \implies \frac{2n-13}{n^2-9n+20} = \frac{1}{6}$$

$$6(2n-13) = n^2 - 9n + 20 \implies 12n - 78 = n^2 - 9n + 20$$

$$n^2 - 9n + 20 - 12n + 78 = 0$$

$$n^2 - 21n + 98 = 0 \implies (n-14)(n-7) = 0$$

+ ×
-21 98
-14 -7

$$n - 14 = 0, n - 7 = 0 \implies n = 14, n = 7$$

16. $\boldsymbol{Prove\ that C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = \frac{2n!}{(n!)^2}}$

$$\boxed{(a+b)^n = nC_0 a^{n-0} b^0 + nC_1 a^{n-1} b^1 + \ \ldots + nC_n a^{n-n} b^n}$$

$$(1+x)^n = nC_0 (1)^{n-0} x^0 + nC_1 (1)^{n-1} x^1 + \ \ldots + nC_n (1)^{n-n} x^n$$

$$a = 1, b = x$$

$$(1+x)^n = nC_0 + nC_1 x + nC_2 x^2 + \cdots + nC_n x^n$$

$$(x+1)^n = nC_0 x^{n-0} 1^0 + nC_1 x^{n-1} (1)^1 + nC_2 x^{n-2} (1)^2 + \cdots + nC_n (x)^{n-n} 1^n$$

$$a = x, b = 1$$

$$(x+1)^n = nC_0 x^n + nC_1 x^{n-1} + nC_2 x^{n-2} + \cdots + nC_n$$

$$(1+x)^n (x+1)^n = nC_0^2 x^n + nC_1^2 x^n + \ \ldots + nC_n^2 x^n$$

$Coefficient\ of\ x^n\ of (1+x)^n(x+1)^n = nC_0^2 + nC_1^2 + nC_2^2 + \cdots + nC_n^2$

$(1+x)^{2n} = C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 \ \ldots (1)$

$To\ find\ General\ of\ (1+x)^{2n}$

$r = n, a = 1, b = x, n = 2n$

$\boxed{T_{r+1} = nC_r a^{n-r} b^r}$

$T_{n+1} = 2nC_n (1)^{n-n} x^n \implies T_{n+1} = 2nC_n x^n$

$Coefficient\ of\ x^n\ of\ T_{n+1} = 2nC_n \ldots (2)$

$(1)\ and\ (2)\ are\ equal$

$$C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = 2nC_n = \frac{2n!}{n!\,(2n-n)!} = \frac{2n!}{n!\,n!} = \frac{2n!}{(n!)^2}$$

Arithmetic and Geometric Progressions

$(\boldsymbol{i})$ ***Arithmetic Progression***

The general form of an A.P.

$a, a + d, a + 2d, a + 3d, \ldots\ldots\ldots, a + (n-1)d$

The general term of an Arithmetic sequence is $\boldsymbol{T_n = a + (n-1)d}$

$a = t_1 = First\ term$

Last term = $t_n = l$

Common difference : d = $t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \cdots$

Geometric Sequence or Geometric Progression (G.P.)

The general form of an G.P: $a, ar, ar^2, ar^3, \ldots\ldots$ $\boldsymbol{ar^{n-1}}$

Common ratio : $r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \cdots$

The general term of an Geometric sequence is $\boldsymbol{T_n = ar^{n-1}}$

Arithmetic − Geometric Progression

A sequence of the form

$a, (a+d)r, (a+2d)\,r^2, (a+3d)\,r^3, \ldots\ldots\ a + (n-1)d\ r^{n-1}$

is called an arithmetic geometric progression.

a is initial term, d is called common difference, r is called common ratio of AGP.

The $\boldsymbol{n^{th}}$ ***term of an Arithmetic sequence is*** $\boldsymbol{T_n = [a + (n-1)d]\ r^{n-1}}$

Harmonic Progression

A sequence $h_1, h_2, h_3, \ldots.$ *is said to be harmonic sequence or a harmonic progression*

$if \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \frac{1}{h_4}, \dots \dots is\ an\ arithmetic\ sequence$

$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d} \dots \quad are\ in\ H.P$

$a, a + d, a + 2d, a + 3d, \dots are\ in\ A.P$

Three quantities $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are in

$\mathbf{a, b, c\ are\ in\ A.P}$

$(i)\ A.P\ if\ b = \frac{a+c}{2} \Rightarrow b\ is\ called\ arithmetic\ mean\ of\ a\ and\ c.$

$\boldsymbol{a, b, c\ are\ in\ G.P}$

$(ii)\ G.P\ if\ \ \mathrm{b} = \sqrt{aC} \Rightarrow b\ is\ called\ geometric\ mean\ of\ a\ and\ c.$

$\boldsymbol{a, b, c\ are\ in\ H.P\ then\ \frac{1}{a}, \frac{1}{b}, \frac{1}{c} are\ in\ A.P}$

$(iii)\ H.P\ if\ b = \frac{2ac}{a+c} \Rightarrow b\ is\ called\ harmonic\ mean\ of\ a\ and\ c.$

$\boldsymbol{In\ general},$

(i) If $a_1, a_2, \dots a_n are\ in\ A.P, then\ the\ number \frac{a_1 + a_2 + \cdots + a_n}{n}\ is\ called$ $arithetic\ mean\ of\ a_1, a_2, \dots a_n$

(ii) If $a_1, a_2, \dots a_n are\ in\ G.P, then\ the \sqrt[n]{a_1, a_2, \dots a_n}\ is\ called\ geometric\ mean$ $of\ a_1, a_2, \dots a_n$

(iii) If $h_1, h_2, \dots h_n are\ in\ H.P, then\ the\ then\ their\ reciprocals\ are$ $\frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \dots, \frac{1}{h_n}\ is\ an\ arithmeticis\ sequence$

$$Arithmetic\ mean = \frac{\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} + \cdots + \frac{1}{h_n}}{n}$$

$$Harmonic\ mean = \frac{n}{h_1 + h_2 + \cdots + h_n}$$

$EXERCISE : 5.2$

Example 5.12: Prove that if a, b, c are in HP, if and only if $\frac{a}{c} = \frac{a-b}{b-c}$

$If\ a, b, c\ are\ in\ HP, then\ \frac{1}{a}, \frac{1}{b}, \frac{1}{c} are\ in\ AP$

$$then \quad b = \frac{2ac}{a+c}$$

$$b(a+c) = 2ac \Longrightarrow ab + bc = ac + ac$$

$$ab - ac = ac - bc \Longrightarrow a(b-c) = c(a-b)$$

$$\frac{a}{c} = \frac{a-b}{b-c}$$

$If\ \frac{a}{c} = \frac{a-b}{b-c}\ then\ prove\ a, b, c\ are\ in\ H.P$

$$\frac{a}{c} = \frac{a-b}{b-c} \Longrightarrow a(b-c) = c(a-b)$$

$$ab - ac = ca - bc \Longrightarrow ab + bc = ac + ac$$

$$b(a+c) = 2ac \Longrightarrow b = \frac{2ac}{a+c} \quad is\ a\ harmonic\ mean$$

$$Hence\ a, b, c\ is\ a\ H.P$$

Example 5.13 If the 5th and 9th terms of a HP are $\frac{1}{19}$ and $\frac{1}{35}$, find the 12th term of the sequence.

$Given: T_5 = \frac{1}{19}, T_9 = \frac{1}{35}\ are\ H.P$

$$\boxed{T_n = a + (n-1)d}$$

$T_5 = 19,\ T_9 = 35\ are\ in\ \mathrm{A.P}$

$$T_5 = 19 \Longrightarrow a + (5-1)d = 19$$

$$a + 4d = 19 \ldots (1)$$

$$T_9 = 35 \Longrightarrow a + (9-1)d = 35$$

$$a + 8d = 35 \ldots (2)$$

$solve\ (1)\ \&\ (2)$

$$\begin{array}{r} \cancel{a} + 4d = 19 \\ \underset{(-)}{\cancel{a}} \underset{(-)}{+} 8d = \underset{(-)}{35} \\ \hline -4d = -16 \end{array} \Longrightarrow d = \frac{\overset{4}{\cancel{-16}}}{\cancel{-4}} \Longrightarrow \boxed{d = 4}$$

$$Subs\ d = 4\ in\ eqn\ (1) \Longrightarrow a + 4d = 19$$

$$a + 4(4) = 19 \Longrightarrow a + 16 = 19$$

$$a = 19 - 16 \Longrightarrow a = 3$$

To find 12th term of sequence

$T_n = a + (n-1)d$

$T_{12} = 3 + (12-1)\,4 = 3 + (11)\,4$

$T_{12} = 3 + 44$

$T_{12} = 47 \implies 12th\ term\ of\ HP\ is\ \frac{1}{47}$

Example 5.14: ***Find seven numbers $A_1, A_2 \ldots A_7$ so that the sequence $4, A_1, A_2, \ldots A_7, 7$ is in AP and also 4 numbers G_1, G_2, G_3, G_4 so that the sequence $12, G_1, G_2, G_3, G_4, \frac{3}{8}$ is in GP.***

Given: $4, A_1, A_2, \ldots A_7, 7$ *is an AP*

$a = 4,\ T_9 = 7$

$a + 8d = 7 \implies 4 + 8d = 7 \implies 8d = 7 - 4$

$8d = 3 \implies d = \frac{3}{8}$

$A_1 = T_2 = a + d = 4 + \frac{3}{8} = 4\frac{3}{8}$

$A_2 = T_3 = a + 2d = 4 + 2 \times \frac{3}{8} = 4 + \frac{6}{8} = 4\frac{6}{8}$

$A_3 = T_4 = a + 3d = 4 + 3 \times \frac{3}{8} = 4 + \frac{9}{8} = 4 + 1 + \frac{1}{8} = 5 + \frac{1}{8} = 5\frac{1}{8}$

$A_4 = T_5 = a + 4d = 4 + 4 \times \frac{3}{8} = 4 + \frac{12}{8} = 4 + 1 + \frac{4}{8} = 5 + \frac{4}{8} = 5\frac{4}{8}$

$A_5 = T_6 = a + 5d = 4 + 5 \times \frac{3}{8} = 4 + \frac{15}{8} = 4 + 1 + \frac{7}{8} = 5 + \frac{7}{8} = 5\frac{7}{8}$

$A_6 = T_7 = a + 6d = 4 + 6 \times \frac{3}{8} = 4 + \frac{18}{8} = 4 + 2 + \frac{2}{8} = 6 + \frac{2}{8} = 6\frac{2}{8}$

$A_7 = T_8 = a + 7d = 4 + 7 \times \frac{3}{8} = 4 + \frac{21}{8} = 4 + 2 + \frac{5}{8} = 6 + \frac{5}{8} = 6\frac{5}{8}$

$\therefore$ *So the required 7 numbers are* $4\frac{3}{8}, 4\frac{6}{8}, 5\frac{1}{8}, 5\frac{4}{8}, 5\frac{7}{8}, 6\frac{2}{8}, 6\frac{5}{8}$

Given: $12, G_1, G_2, G_3, G_4, \frac{3}{8}$ *are an G.P*

$a = 12$

$T_6 = \frac{3}{8} \implies ar^{6-1} = \frac{3}{8} \implies ar^5 = \frac{3}{8}$

$12r^5 = \frac{3}{8} \implies r^5 = \frac{\cancel{3}}{8} \times \frac{1}{\cancel{12}_4} = \frac{1}{32}$

$r = \sqrt[5]{\frac{1}{32}} = \sqrt[5]{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}} \implies \boxed{r = \frac{1}{2}}$

$$G_1 = T_2 = ar = \overset{6}{\cancel{12}} \times \frac{1}{\cancel{2}} = 6$$

$$G_2 = T_3 = ar^2 = 12 \times \left(\frac{1}{2}\right)^2 = \overset{3}{\cancel{12}} \times \frac{1}{\cancel{4}} = 3$$

$$G_3 = T_4 = ar^3 = 12 \times \left(\frac{1}{2}\right)^3 = \overset{3}{\cancel{12}} \times \frac{1}{\underset{2}{\cancel{8}}} = \frac{3}{2} = 1\frac{1}{2}$$

$$G_4 = T_5 = ar^4 = 12 \times \left(\frac{1}{2}\right)^4 = \overset{3}{\cancel{12}} \times \frac{1}{\underset{4}{\cancel{16}}} = \frac{3}{4}$$

$\therefore$ *The required 4 numbers are* $6, 3, 1\frac{1}{2}, \frac{3}{4}$

***Example* 5.13**: **If the product of the 4th, 5th and 6th terms of a GP is 4096 and if the product of the 5th, 6th and 7th terms of it is 32768, find the sum of first 8 terms of the GP.**

Let $a, ar, a\,r^2, a\,r^3$ …… be the geometric sequence

$\boxed{T_n = ar^{n-1}}$

Product of the 4th, 5th and 6th terms is 4096

$$ar^{4-1} \times ar^{5-1} \times ar^{6-1} = 4096$$

$$ar^3 \times ar^4 \times ar^5 = 4096$$

$$a^3\, r^{3+4+5} = 4096$$

$$a^3\, r^{12} = 4096 \ldots (1)$$

Product of the 5th, 6th and 7th terms is 32768

$$ar^{5-1} \times ar^{6-1} \times ar^{7-1} = 32768$$

$$ar^4 \times ar^5 \times ar^6 = 32768$$

$$a^3\, r^{4+5+6} = 32768$$

$$a^3\, r^{15} = 32768 \ldots (2)$$

solve $(2) \div (1)$

$$\frac{\cancel{a^3}\cancel{r^{15}}\, r^3}{\cancel{a^3}\cancel{r^{12}}} = \frac{\overset{8}{\cancel{32768}}}{\cancel{4096}} \Longrightarrow r^3 = 8$$

$$r = \sqrt[3]{8} = \sqrt[3]{2 \times 2 \times 2} \Longrightarrow \boxed{r = 2}$$

Subs $r = 2$ *in eqn* (1) $a^3 r^{12} = 4096$

$$a^3\,(2)^{12} = 4096$$

$$a^3 \times 4096 = 4096 \Longrightarrow a^3 = \frac{\cancel{4096}\ 1}{\cancel{4096}}$$

$$a^3 = 1 \Longrightarrow \boxed{a = 1}$$

The sum of first 8 terms $, r = 2 > 1,$

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$

$where\ n = 8, a = 1$

$$S_8 = 1\left(\frac{2^8-1}{2-1}\right) = 2^8 - 1 = 256 - 1 = 255$$

1.Write the first 6 terms of the sequences whose nth terms are given below and classify them as AP, GP, arithmetico - geometric progression, HP and none of them.

$(i)\ \frac{1}{2^{n+1}}$, $(ii)\ \frac{(n+1)(n+2)}{(n+3)(n+4)}$, (iii) $4\left(\frac{1}{2}\right)^n$ (iv) $\frac{(-1)^n}{n}$ (v) $\frac{2n+3}{3n+4}$ (vi) 2018 (vii) $\frac{3n-2}{3^{n-1}}$.

$(i)\ n^{th}\ term\ is\ \frac{1}{2^{n+1}} \Rightarrow T_n = \frac{1}{2^{n+1}}$

$$T_1 = \frac{1}{2^{1+1}} = \frac{1}{2^2} = \frac{1}{4} \qquad T_2 = \frac{1}{2^{2+1}} = \frac{1}{2^3} = \frac{1}{8}$$

$$T_3 = \frac{1}{2^{3+1}} = \frac{1}{2^4} = \frac{1}{16} \qquad T_4 = \frac{1}{2^{4+1}} = \frac{1}{2^5} = \frac{1}{32}$$

$$T_5 = \frac{1}{2^{5+1}} = \frac{1}{2^6} = \frac{1}{64} \qquad T_6 = \frac{1}{2^{6+1}} = \frac{1}{2^7} = \frac{1}{128}$$

$It\ is\ a\ GP\ whose\ 1^{st} term\ is\ \frac{1}{4}\ \ and\ common\ ratio\ is\ \frac{1}{2}$

$\therefore The\ first\ 6\ terms\ are\ \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$

$(ii)\ n^{th}\ term\ is\ \frac{(n+1)(n+2)}{(n+3)(n+4)}$

$$T_n = \frac{(n+1)(n+2)}{(n+3)(n+4)}$$

$$T_1 = \frac{(1+1)(1+2)}{(1+3)(1+4)} = \frac{2\times 3}{4\times 5} = \frac{6}{20} \qquad T_2 = \frac{(2+1)(2+2)}{(2+3)(2+4)} = \frac{3\times 4}{5\times 6} = \frac{12}{30}$$

$$T_3 = \frac{(3+1)(3+2)}{(3+3)(3+4)} = \frac{4\times 5}{6\times 7} = \frac{20}{42} \qquad T_4 = \frac{(4+1)(4+2)}{(4+3)(4+4)} = \frac{5\times 6}{7\times 8} = \frac{30}{56}$$

$$T_5 = \frac{(5+1)(5+2)}{(5+3)(5+4)} = \frac{6\times 7}{8\times 9} = \frac{42}{72} \qquad T_6 = \frac{(6+1)(6+2)}{(6+3)(6+4)} = \frac{7\times 8}{9\times 10} = \frac{56}{90}$$

$\therefore The\ first\ 6\ terms\ are\ \frac{6}{20}, \frac{12}{30}, \frac{20}{42}, \frac{30}{56}, \frac{42}{72}, \frac{56}{90}$

$Not\ an\ AP/\ GP\ /\ HP$

(iii) $\boldsymbol{n^{th} term\ is\ 4\left(\frac{1}{2}\right)^n}$

$$T_n = 4\left(\frac{1}{2}\right)^n$$

$$T_1 = 4\left(\frac{1}{2}\right)^1, T_2 = 4\left(\frac{1}{2}\right)^2, T_3 = 4\left(\frac{1}{2}\right)^3, T_4 = 4\left(\frac{1}{2}\right)^4, T_5 = 4\left(\frac{1}{2}\right)^5, T_6 = 4\left(\frac{1}{2}\right)^6$$

$\therefore$ *The first 6 terms are* $4\left(\frac{1}{2}\right), 4\left(\frac{1}{2}\right)^2, 4\left(\frac{1}{2}\right)^3, 4\left(\frac{1}{2}\right)^4, 4\left(\frac{1}{2}\right)^5, 4\left(\frac{1}{2}\right)^6$

It is a GP with first term is $4\frac{1}{2}$ *and common ratio* $\frac{1}{2}$

(iv) $\boldsymbol{n^{th} term\ is\ \frac{(-1)^n}{n}}$

$$T_n = \frac{(-1)^n}{n}$$

$$T_1 = \frac{(-1)^1}{1} = \frac{-1}{1}, T_2 = \frac{(-1)^2}{2} = \frac{1}{2}, T_3 = \frac{(-1)^3}{3} = \frac{-1}{3}$$

$$T_4 = \frac{(-1)^4}{4} = \frac{1}{4}, T_5 = \frac{(-1)^5}{5} = \frac{-1}{5}, T_6 = \frac{(-1)^6}{6} = \frac{1}{6}$$

$\therefore$ *The first 6 terms are* $\frac{-1}{1}, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \frac{1}{6}$

It is not an AP / GP / HP

(v) $\boldsymbol{n^{th} term\ is\ \frac{2n+3}{3n+4}}$

$$T_n = \frac{2n+3}{3n+4}$$

$$T_1 = \frac{2(1)+3}{3(1)+4} = \frac{5}{7}, T_2 = \frac{2(2)+3}{3(2)+4} = \frac{7}{10}, T_3 = \frac{2(3)+3}{3(3)+4} = \frac{9}{13}$$

$$T_4 = \frac{2(4)+3}{3(4)+4} = \frac{11}{16}, T_5 = \frac{2(5)+3}{3(5)+4} = \frac{13}{19}, T_6 = \frac{2(6)+3}{3(6)+4} = \frac{15}{22}$$

$\therefore$ *The first 6 terms are* $\frac{5}{7}, \frac{7}{10}, \frac{9}{13}, \frac{11}{16}, \frac{13}{19}, \frac{15}{22}$

It is not an AP / GP / HP

(vi) $\boldsymbol{n^{th} term\ is\ 2018}$

$\therefore$ *The first 6 terms are* 2018, 2018, 2018, 2018, 2018, 2018.

It is not an AP / GP / HP

(vii) n^{th} term is $\frac{3n-2}{3^{n-1}}$

$$T_n = \frac{3n-2}{3^{n-1}}$$

$$T_1 = \frac{3(1)-2}{3^{1-1}} = \frac{3-2}{3^0} = \frac{1}{1} \qquad T_2 = \frac{3(2)-2}{3^{2-1}} = \frac{6-2}{3^1} = \frac{4}{3}$$

$$T_3 = \frac{3(3)-2}{3^{3-1}} = \frac{9-2}{3^2} = \frac{7}{3^2} \qquad T_4 = \frac{3(4)-2}{3^{4-1}} = \frac{12-2}{3^3} = \frac{10}{3^3}$$

$$T_5 = \frac{3(5)-2}{3^{5-1}} = \frac{15-2}{3^4} = \frac{13}{3^4} \qquad T_6 = \frac{3(6)-2}{3^{6-1}} = \frac{18-2}{3^5} = \frac{16}{3^5}$$

$\therefore$ *The first 6 terms are* $\frac{1}{1}, \frac{4}{3}, \frac{7}{3^2}, \frac{10}{3^3}, \frac{13}{3^4}, \frac{16}{3^5}$

It is Arithmetico Geometric Progression.

2. Write the first 6 terms of the sequences whose nth term a_n is given below.

(i) $a_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$

$a_n = (n+1) \Longrightarrow$ *if n is odd* $\Longrightarrow$ $a_n = (n) \Longrightarrow$ *if n is even*

$a_1 = n+1 = 1+1 = 2 \ (odd) \Longrightarrow a_2 = n = 2 \ (even)$

$a_3 = n+1 = 3+1 = 4 \ (odd) \Longrightarrow a_4 = n = 4 \ (even)$

$a_5 = n+1 = 5+1 = 6 \ (odd) \Longrightarrow a_6 = n = 6 \ (even)$

The first six terms are $2, 2, 4, 4, 6, 6$

(ii) $a_n = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ a_{n-1} + a_{n-2} & \text{if } n > 2 \end{cases}$

$a_1 = 1 \ (n=1) \Longrightarrow a_2 = 2 \ (n=2)$

$a_3 = a_{n-1} + a_{n-2} \ (n > 2) = a_{3-1} + a_{3-2} = a_2 + a_1 = 2 + 1 = 3$

$a_4 = a_{n-1} + a_{n-2} \ (n > 2) = a_{4-1} + a_{4-2} = a_3 + a_2 = 3 + 2 = 5$

$a_5 = a_{n-1} + a_{n-2} \ (n > 2) = a_{5-1} + a_{5-2} = a_4 + a_3 = 5 + 3 = 8$

$a_6 = a_{n-1} + a_{n-2} \ (n > 2) = a_{6-1} + a_{6-2} = a_5 + a_4 = 8 + 5 = 13$

The first six terms are $1, 2, 3, 5, 8, 13$

(iii) $a_n = \begin{cases} n & \text{if } n \text{ is } 1, 2, \text{or } 3 \\ a_{n-1} + a_{n-2} + a_{n-3} & \text{if } n > 2 \end{cases}$

$a_1 = n = 1 \ (n=1) \Longrightarrow a_2 = n = 2 \ (n=2) \Longrightarrow a_3 = n = 3 \ (n=3)$

$a_4 = a_{n-1} + a_{n-2} + a_{n-3}\ (n > 2) = a_{4-1} + a_{4-2} + a_{4-3}$

$= a_3 + a_2 + a_1 = 3 + 2 + 1 = 6$

$a_5 = a_{n-1} + a_{n-2} + a_{n-3}\ (n > 2) = a_{5-1} + a_{5-2} + a_{5-3} = a_4 + a_3 + a_2$

$= 6 + 3 + 2 = 11$

$a_6 = a_{n-1} + a_{n-2} + a_{n-3}\ (n > 2) = a_{6-1} + a_{6-2} + a_{6-3} = a_5 + a_4 + a_3$

$= 11 + 6 + 3 = 20$

The first six terms are $1, 2, 3, 6, 11, 20$

3. *Write the nth term of the following sequences.*

(i) 2, 2, 4, 4, 6,6…., (ii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$, **(iii)** $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}, \dots,$

(iv) 6, 10,4,12,2,14,0,16,−2,..

(*i*) *The given sequence is* 2, 2, 4, 4, 6, 6 …. ,

$a_n = (n+1)\ if\ n\ is\ odd$

$a_n = n\ if\ n\ is\ even$

(*ii*) *The given sequence is* $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$

$$a_n = \frac{n}{n+1}$$

(*iii*) *The given sequence is* $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}, \dots$

$$a_n = \frac{2n-1}{2n}$$

(*iv*) *The given sequence is* 6, 10, 4, 12, 2, 14, 0, 16, −2, …

$$a_n = \begin{cases} 7 - n & if\ n\ is\ odd \\ 8 + n & if\ n\ is\ even \end{cases}$$

4. The product of three increasing numbers in GP is 5832. If we add 6 to the second number and 9 to the third number, then resulting numbers form an A.P. Find the numbers in GP.

Let the three numbers in GP be $\frac{a}{r}, a, ar$

Given: *Product of these 3 numbers* $= 5832$

$$\left(\frac{a}{r}\right)(a)\,(ar) = 5832 \Rightarrow a^3 = 5832$$

$$a = \sqrt[3]{5832} = \sqrt[3]{2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3}$$

$$= 2 \times 3 \times 3$$

$$\boxed{a = 18}$$

2	5832
2	2916
2	1458
3	729
3	243
3	81
3	27
3	9
	3

$\frac{18}{r}, 18+6,\ 18r+9$ *are in A.P* $\Longrightarrow$ $\frac{18}{r}, 24, 18r+9$ *are in A.P*

$$t_2 - t_1 = t_3 - t_2$$

$$24 - \frac{18}{r} = 18r + 9 - 24 \Longrightarrow 24 + 24 = 18r + 9 + \frac{18}{r}$$

$$48 = \frac{18r^2 + 9r + 18}{r} \Longrightarrow 48r = 18r^2 + 9r + 18$$

$$18r^2 + 9r + 18 - 48r = 0$$

$$18r^2 - 39r + 18 = 0 \Longrightarrow 6r^2 - 13r + 6 = 0$$
$$\div 3$$

$$(3r-2)(2r-3) = 0 \Longrightarrow 3r - 2 = 0,\ 2r - 3 = 0$$

$$3r = 2,\ 2r = 3 \Longrightarrow r = \frac{2}{3},\ r = \frac{3}{2}$$

$\therefore$ *The GP are* $\frac{a}{r}, a, ar.$

$$a = 18,\ r = \frac{2}{3}$$

$$\frac{a}{r} = \frac{18}{\frac{2}{3}} = \overset{9}{\cancel{18}} \times \frac{3}{\cancel{2}} = 27$$

$$ar = \overset{6}{\cancel{18}} \times \frac{2}{\cancel{3}} = 12$$

$\therefore$ *The numbers are* $27, 18, 12.$

5. *Write the nth term of the sequence* $\frac{3}{1^2 2^2}, \frac{5}{2^2 3^2}, \frac{7}{3^2 4^2} ..$ *as a difference of 2 terms.*

$$\frac{3}{1^2.2^2}, \frac{5}{2^2.3^2}, \frac{7}{3^2.4^2}, \ldots$$

numerator $= 3, 5, 7, \ldots$ *are in A.P* $\Longrightarrow a = 3,\ d = 5 - 3 = 2$

$$T_n = a + (n-1)d$$
$$= 3 + (n-1)2 = 3 + 2n - 2$$

$$T_n = 2n + 1\ \ (numerator)$$

Denominator $= 1^2.2^2, 2^2.3^2, 3^2.4^2 \ldots \Longrightarrow T_n = n^2.(n+1)^2$

$$n^{th}\ term = \frac{2n+1}{n^2.(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

6. *If* t_k *is the kth term of a GP, then show that* t_{n-k}, t_n, t_{n+k} *also form a GP for anypositive integer k.*

$$k^{th}\ term\ in\ GP = t_k = ar^{k-1}$$

$$t_{n-k} = ar^{n-k-1} \Longrightarrow t_n = ar^{n-1} \Longrightarrow t_{n+k} = ar^{n+k-1}$$

To prove that t_{n-k}, t_n, t_{n+k} are in GP.

$$\frac{t_2}{t_1} = \frac{t_3}{t_2} \implies \frac{t_n}{t_{n-k}} = \frac{t_{n+k}}{t_n} \implies (t_n)^2 = (t_{n-k})(t_{n+k})$$

$$(ar^{n-1})^2 = (ar^{n-k-1})(ar^{n+k-1})$$

$$a^2 r^{2n-2} = a^2\, r^{n\cancel{-k}-1+n\cancel{+k}-1} \implies a^2 r^{2n-2} = a^2 r^{2n-2}$$

$\therefore\ t_{n-k}, t_n, t_{n+k}$ are in GP.

7. If a, b, c are in GP, and if $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$, then prove that x, y, z are in AP.

a, b, c are in GP

$$\frac{b}{a} = \frac{c}{b} \implies b^2 = ac$$

Given: $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$

$$a^{\frac{1}{x}} = b^{\frac{1}{y}} \implies a = \left(b^{\frac{1}{y}}\right)^x$$

$$\boxed{a = b^{\frac{x}{y}}}$$

$$c^{\frac{1}{z}} = b^{\frac{1}{y}} \implies c = \left(b^{\frac{1}{y}}\right)^z \implies \boxed{c = b^{\frac{z}{y}}}$$

$$b^2 = ac \implies b^2 = b^{\frac{x}{y}}.b^{\frac{z}{y}}$$

$$b^2 = b^{\left(\frac{x}{y}+\frac{z}{y}\right)} \implies 2 = \frac{x}{y} + \frac{z}{y} \implies 2 = \frac{x+z}{y} \implies 2y = x + z$$

$$y + y = x + z \implies y - x = z - y$$

$$t_2 - t_1 = t_3 - t_2 \implies \therefore x, y, z \text{ are in AP}$$

8. The AM of two numbers exceeds their GM by 10 and HM by 16. Find the numbers.

$$AM - GM = 10 \text{ and } AM - HM = 16$$

$$\text{Let } AM = \frac{a+b}{2} = x \quad \text{and } GM = \sqrt{ab} = y$$

$$\text{Then } HM = \frac{2ab}{a+b} = \frac{\left(\sqrt{ab}\right)^2}{\frac{a+b}{2}} = \frac{y^2}{x}$$

$$AM - GM = 10$$

$$x - y = 10 \implies x = 10 + y \quad \dots(1)$$

$$AM - HM = 16$$

$$x - \frac{y^2}{x} = 16 \implies \frac{x^2 - y^2}{x} = 16 \implies x^2 - y^2 = 16x$$

$(y+10)^2 - y^2 = 16(y+10) \Longrightarrow y^2 + 20y + 10^2 - y^2 = 16y + 160$

$20y + 100 - 16y - 160 = 0 \Longrightarrow 4y - 60 = 0$

$4y = 60 \Longrightarrow y = \frac{60}{4} \Longrightarrow y = 15$

$Subs\ y = 15\ in\ (1)\ \ x = y + 10$

$x = 15 + 10 \Longrightarrow x = 25$

$\frac{a+b}{2} = x \Longrightarrow \frac{a+b}{2} = 25 \Longrightarrow a + b = 50 \Longrightarrow a = 50 - b$

$\sqrt{ab} = y \Longrightarrow \sqrt{ab} = 15 \Longrightarrow ab = 225$

$Sub\ a = 50 - b$

$(50-b)b = 225 \Longrightarrow 50b - b^2 = 225 \Longrightarrow -b^2 + 50b - 225 = 0$

$b^2 - 50b + 225 = 0 \Longrightarrow (b-5)(b-45) = 0$

$b - 45 = 0,\ b - 5 = 0$

$b = 45,\ b = 5$

$If\ b = 5\ then\ a = 50 - b \Longrightarrow a = 50 - 5$

$\boxed{a = 45}$

$If\ b = 45\ then\ a = 50 - b$

$a = 50 - 45 \Longrightarrow \boxed{a = 5}$

$Hence\ the\ number\ are\ 45\ and\ 5$

$AM = \frac{a+b}{2} = \frac{45+5}{2} = \frac{50}{2} = 25$

$GM = \sqrt{ab} = \sqrt{5 \times 45} = \sqrt{5 \times 5 \times 9} = 5 \times 3 = 15$

$HM = \frac{2ab}{a+b} = \frac{\cancel{2} \times \overset{9}{\cancel{45}} \times \cancel{5}}{\underset{\underset{5}{\cancel{25}}}{\cancel{50}}} \Longrightarrow HM = 9$

$Verification:\quad GM - AM = 10$
$\qquad AM - HM = 16$

9. *If the root of the equation* $(q-r)x^2 + (r-p)x + p - q = 0$ *are equal, then Show that* p, q *and* r *are in AP.*

$If\ the\ root\ of\ the\ equation\ (q-r)x^2 + (r-p)x + p - q = 0\ are\ equal\ then$

$b^2 - 4ac = 0 \Longrightarrow b^2 = 4ac$
$a = q - r, b = r - p,\ c = p - q$

$(r-p)^2 = 4(q-r)(p-q)$

$r^2 + p^2 - 2rp = 4(pq - q^2 - pr + qr) = 4pq - 4q^2 - 4pr + 4qr$

$$r^2 + p^2 - 2rp + 4pr = 4pq - 4q^2 + 4qr$$
$$p^2 + r^2 + 2pr = 4pq + 4qr - 4q^2$$
$$p^2 + r^2 + 2rp = 4q(p + r) - 4q^2$$
$$(p + r)^2 - 4q(p + r) + 4q^2 = 0$$
$$(p + r)^2 - 4q(p + r) + 4q^2 = 0$$
$$a = 1, \quad b = -4q, c = 4q^2$$
$$p + r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$p + r = \frac{4q \pm \sqrt{(-4q)^2 - 4(1)(4q^2)}}{2(1)} = \frac{4q \pm \sqrt{16q^2 - 16q^2}}{2} = \frac{\cancel{4}^{2}q}{\cancel{2}}$$
$$p + r = 2q \implies p + r = q + q$$
$$q + q = p + r \implies q - p = r - q$$
$$\therefore p, q, r \text{ are in } AP$$

10. If a, b, c are respectively the p^{th}, q^{th} and r^{th} term of a GP Show that $(q - r)\log a + (r - p)\log b + (p - q)\log c = 0$

Given: $T_p = a$

$$AR^{p-1} = a \implies \log AR^{p-1} = \log a$$
$$T_q = b$$
$$AR^{q-1} = b \implies \log AR^{q-1} = \log b$$
$$T_r = c$$
$$AR^{r-1} = c \implies \log AR^{r-1} = \log c$$

Let A be the first term. R be the common ratio of GP.

$$L.H.S = (q - r)\log a + (r - p)\log b + (p - q)\log c$$
$$= (q - r)\log AR^{p-1} + (r - p)\log AR^{q-1} + (p - q)\log AR^{r-1}$$
$$= (q - r)\left[\log A + \log R^{p-1}\right] + (r - p)\left[\log A + \log R^{q-1}\right] + (p - q)\left[\log A + \log R^{r-1}\right]$$
$$= (q - r)\log A + (q - r)(p - 1)\log R + (r - p)\log A + (r - p)(q - 1)\log R + (p - q)\log A + (p - q)(r - 1)\log R$$
$$= \log A\left[q - r + r - p + p - q\right] + \log R\left[(q - r)(p - 1) + (r - p)(q - 1) + (p - q)(r - 1)\right]$$

$$= \log A\,[\cancel{q} - \cancel{r} + \cancel{r} - \cancel{p} + \cancel{p} - \cancel{q}]$$

$$+ \log R\,[q\cancel{p} - \cancel{q} - r\cancel{p} + \cancel{r} + r\cancel{q} - \cancel{r} - p\cancel{q} + \cancel{p} + p\cancel{r} - \cancel{p} - q\cancel{r} + \cancel{q}]$$

$$= \log A\ (0) + \log R\ (0) = 0$$

Finite Series

$If\ a_n\ is\ a\ sequence\ of\ numbers\ then\ a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k\ is\ called\ finite\ series.$

Arithmetic series a+ (a + d) +(a + 2d) + (a + 3d), ...

Sum of an A.P

Sum of an A.P upto n terms : $S_n = \frac{n}{2}[2a + (n-1)d]$

Sum of of an A.P upto last terms : $S_n = \frac{n}{2}(a + l)$

The general form of an Geometric series:
$a + ar + ar^2 + ar^3 + \cdots$

Sum to n terms of an G.P

$S_n = \frac{a(r^n - 1)}{r - 1}$	$if\ r > 1$
$S_n = \frac{a(1 - r^n)}{1 - r}$	$if\ r < 1$
$S_n = na$	$if\ r = 1$

Sum of Arithmetico-Geometric Progressions

(i) $A\ series\ is\ said\ to\ be\ an\ arithmetico - geometric\ series\ if\ the\ terms\ of\ the\ series\ form\ an\ arithmetico\ geometric\ sequence.$

(ii) $The\ sum\ S_n\ of\ the\ first\ n\ terms\ of\ arithmetico - geometric\ sequence\ is\ given\ by$

$$S_n = \frac{a - [a + (n-1)d]r^n}{1 - r} + dr\left[\frac{1 - r^{n-1}}{(1-r)^2}\right] \quad for\ r \neq 1$$

$Exercise\ -\ 5.3$

Example 5.16: Find the sum up to n terms of the series :

$$\mathbf{1+\frac{6}{7}+\frac{11}{49}+\frac{16}{343}+\cdots}$$

$Given\ that: 1+\frac{6}{7}+\frac{11}{49}+\frac{16}{343}+\cdots$

$Numerator: 1, 6, 11, 16 \ldots.$

$a=1, d=6-1=5$

$Denominator: 1, \frac{1}{7}, \frac{1}{49}, \frac{1}{343} \ldots\ldots$

$a=1,\ \ r=\frac{1}{7}$

$$S_n=\frac{a-[a+(n-1)d]r^n}{1-r}+dr\left[\frac{1-r^{n-1}}{(1-r)^2}\right]$$

$$S_n=\frac{1-(1+(n-1)5)\left(\frac{1}{7}\right)^n}{1-\frac{1}{7}}+5\times\frac{1}{7}\ \frac{1-\left(\frac{1}{7}\right)^{n-1}}{\left(1-\frac{1}{7}\right)^2}$$

$$S_n=\frac{1-\left[\frac{1+5n-5}{7^n}\right]}{\frac{7-1}{7}}+\frac{5}{7}\left(\frac{1-\left(\frac{1}{7^{n-1}}\right)}{\left(\frac{7-1}{7}\right)^2}\right)$$

$$S_n=\frac{1-\left[\frac{5n-4}{7^n}\right]}{\frac{6}{7}}+\frac{5}{7}\left(\frac{\frac{7^{n-1}-1}{7^{n-1}}}{\left(\frac{6}{7}\right)^2}\right)$$

$$S_n=\frac{\frac{7^n-5n-4}{7^n}}{\frac{6}{7}}+\frac{5}{7}\left(\frac{\frac{7^{n-1}-1}{7^{n-1}}}{\left(\frac{6}{7}\right)^2}\right)$$

$$=\frac{7^n-5n+4}{\cancel{7^n}}\times\frac{\cancel{7}}{6}+\frac{5}{7}\times\frac{(7^{n-1}-1)}{7^{n-1}}\times\frac{7^2}{36}$$

$$= \frac{7^n - 5n + 4}{7^{n-1} \times 6} + \frac{5(7^{n-1} - 1)}{\cancel{7^n}\ 7^{n-2}} \times \frac{\cancel{7^2}}{36}$$

$$= \frac{7^n - 5n + 4}{7^{n-1} \times 6} + \frac{5(7^{n-1} - 1)}{7^{n-2} \times 36}$$

Example* 5.17: *Find the sum of the first n terms of the series

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \dots\dots\dots$$

Let t_k denote the k^{th} term of the given series

$$t_k = \frac{1}{\sqrt{k}+\sqrt{k+1}}$$

$$= \frac{1}{\sqrt{k}+\sqrt{k+1}} \times \frac{\sqrt{k}-\sqrt{k+1}}{\sqrt{k}-\sqrt{k+1}} = \frac{\sqrt{k}-\sqrt{k+1}}{\left(\sqrt{k}\right)^2 - \left(\sqrt{k+1}\right)^2}$$

$$= \frac{\sqrt{k}-\sqrt{k+1}}{k-(k+1)} = \frac{\sqrt{k}-\sqrt{k+1}}{k-k-1} = \frac{\sqrt{k}-\sqrt{k+1}}{-1} = -\sqrt{k}+\sqrt{k+1}$$

$$= \sqrt{k+1} - \sqrt{k}$$

$$t_1 + t_2 + t_3 + \cdots t_n = \sqrt{2} - \sqrt{1} + \sqrt{3} - \sqrt{2} + \sqrt{4} - \sqrt{3} + \cdots + \left[\sqrt{n+1} - \sqrt{n}\right]$$

$$= \sqrt{n+1} - \sqrt{1} = \sqrt{n+1} - 1$$

Example*: 5.18: *Find $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$

Given that: $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} \Rightarrow \frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$

$$\frac{1}{\cancel{k(k+1)}} = \frac{A(k+1)+B(k)}{\cancel{k(k+1)}} \Rightarrow$$

put $k = -1$ *in* $1 = A(k+1) + B(k)$

$1 = A(-1+1) + B(-1) \Rightarrow 1 = A(0) + B(-1)$

$1 = -B \Rightarrow \boxed{B = -1}$

put $k = 0$

$1 = A(0+1) + B(0) \Rightarrow A(1) = 1 \Rightarrow \boxed{A = 1}$

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\sum \frac{1}{k(k+1)} = \sum \left(\frac{1}{k} - \frac{1}{k+1}\right) \Rightarrow t_1 + t_2 + t_3 + \cdots t_n$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \quad = 1 - \frac{1}{n+1}$$

1. Find the sum of the first 20-terms of the AP having the sum of first 10 terms as 52 and the sum of the first 15 terms as 77.

$Given\ that\ \ S_{10} = 52\ and\ \ S_{15} = 77$

$S_{10} = 52$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$n = 10$

$$S_{10} = \frac{10}{2}[2a + (10-1)d]$$

$$52 = 5[2a + 9d] \Rightarrow 5(2a + 9d) = 52$$

$$2a + 9d = \frac{52}{5} \ \dots (1)$$

$S_{15} = 77$

$n = 15$

$$S_{15} = \frac{15}{2}[2a + (15-1)d] \Rightarrow 77 = \frac{15}{2}[2a + 14d]$$

$$77 \times \frac{2}{15} = 2a + 14d \Rightarrow 2a + 14d = \frac{154}{15} \ \dots (2)$$

$Solve\ (1) and\ (2)$

$$\begin{array}{rcl} \cancel{2a} + 14d & = & \dfrac{154}{15} \\ (-)\quad (-) & & (-) \\ \cancel{2a} + 9d & = & \dfrac{52}{5} \\ \hline 5d & = & \dfrac{154}{15} - \dfrac{52}{5} \end{array} \Rightarrow 5d = \frac{154 - (52 \times 3)}{15}$$

$$5d = \frac{154 - 156}{15} \Rightarrow 5d = \frac{-2}{15} \Rightarrow d = \frac{-2}{15 \times 5} \Rightarrow d = \frac{-2}{75}$$

$$sub\ d = \frac{-2}{75}\ in\ (1)\ \ 2a + 9d = \frac{52}{5}$$

$$2a + 9\left(\frac{-2}{75}\right) = \frac{52}{5} \Rightarrow 2a - \frac{18}{75} = \frac{52}{5} \Rightarrow 2a = \frac{52}{5} + \frac{18}{75}$$

$$2a = \frac{(52 \times 15) + 18}{75} \Rightarrow 2a = \frac{780 + 18}{75}$$

$$2a = \frac{\cancel{798}^{266}}{\cancel{75}_{25}} \Rightarrow 2a = \frac{266}{25} \Rightarrow a = \frac{\cancel{266}^{133}}{25 \times \cancel{2}_{1}} \Rightarrow a = \frac{133}{25}$$

To find $\boldsymbol{S_{20}}$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\text{where}, n = 20, a = \frac{133}{25} \text{ and } d = \frac{-2}{75}$$

$$\boldsymbol{S_{20}} = \frac{20}{2}\left[2\left(\frac{133}{25}\right) + (20-1)\left(\frac{-2}{75}\right)\right] = \frac{\cancel{20}^{10}}{\cancel{2}}\left[2\left(\frac{133}{25}\right) + (19)\left(\frac{-2}{75}\right)\right]$$

$$= 10\left[\frac{266}{25} - \frac{38}{75}\right] = 10\left[\frac{3 \times 266 - 38}{75}\right] = 10\left[\frac{798 - 38}{75}\right]$$

$$\boldsymbol{S_{20}} = \cancel{10}^{2}\left[\frac{\cancel{760}^{152}}{\cancel{75}_{\cancel{15}\ 3}}\right] \Rightarrow S_{20} = \frac{304}{3}$$

2. *Find the sum up to the 17th terms of the series*

$$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \cdots$$

$$T_n = \frac{1^3 + 2^3 + 3^3 \ldots n^3}{1 + 3 + 5 + \cdots (2n-1)}$$

$$= \frac{\left[\frac{n(n+1)}{2}\right]^2}{n^2} = \frac{\frac{n^2(n+1)}{4}}{n^2} = \frac{n^2(n+1)^2}{4n^2} = \frac{1}{4}(n+1)^2 = \frac{1}{4}(n^2 + 2n + 1)$$

$$S_n = \frac{1}{4}\left[\sum n^2 + 2\sum n + \sum 1\right]$$

$\boxed{\sum 1 \text{ means adding } 1 \text{ for } n \text{ times}}$

$$= \frac{1}{4} \times \frac{n(n+1)(2n+1)}{6} + 2\,\frac{n(n+1)}{2} + n$$

$$S_{17} = \frac{1}{4} \times \frac{17(17+1)(2 \times 17 + 1)}{6} + 17(17+1) + 17$$

$$= \frac{1}{4}\left(\frac{17(18)(34+1)}{6} + 17(18) + 17\right) = \frac{1}{4}\left(\frac{17(\cancel{18}^{3})(35)}{\cancel{6}} + 17(18) + 17\right)$$

$$= \frac{1}{4}\left(1785 + 306 + 17\right) = \frac{1}{\cancel{4}}\left[\cancel{2108}^{527}\right] = 527$$

3. ***Compute the sum of first n terms of the following series:***
$(i) 8+88+888+8888+\cdots, \quad (ii) 6+66+666+6666+\ldots$

$(i)\ 8+88+888+8888+\cdots,$

$$\boxed{S_n = \frac{a(r^n-1)}{r-1}}$$

$$S_n = 8[1+11+111+1111+\cdots+n\ term]$$

$$= \frac{8}{9}[9+99+999+\cdots+n\ term]$$

$$= \frac{8}{9}[(10-1)+(100-1)+(1000-1)+\cdots+n\ terms]$$

$$= \frac{8}{9}[10+100+1000+\cdots nterms + (-1-1-1\ldots to\ n\ terms)]$$

Which are in G.P $a=10,\ r=\frac{100}{10}=10>1$

$$= \frac{8}{9}\times\frac{10(10^n-1)}{9} - n = \frac{8}{9}\left[\frac{10(10^n-1)-9n}{9}\right] = \frac{8}{81}[10(10^n-1)-9n]$$

(ii) ***Find the sum to n terms of the series*** $6+66+666+\cdots$

$$\boxed{S_n = \frac{a(r^n-1)}{r-1}}$$

$$\mathrm{S_n} = 6+66+666+\cdots \text{to n terms}$$

$$= 6\,(1+11+111+\cdots \text{to n terms})$$

$$= \frac{6}{9}(9+99+999+\cdots to\ n\ terms)$$

$$= \frac{2}{3}[(10-1)+(100-1)+(1000-1)+\cdots to\ n\ terms]$$

$$= \frac{2}{3}[10+100+1000+\cdots nterms + (-1-1-1\ldots to\ n\ terms)]$$

Which are in G.P $a=10,\ r=\frac{100}{10}=10>1$

$$= \frac{2}{3}\left[\frac{a(r^n-1)}{r-1}+(-n)\right] = \frac{2}{3}\left[\frac{10(10^n-1)}{10-1}-n\right]$$

$$\mathrm{S_n} = \frac{2}{3}\left[\frac{10(10^n-1)}{9}-n\right]$$

4. ***Compute the sum of first n terms of***
$\mathbf{1+(1+4)+\left(1+4+4^2\right)+\left(1+4+4^2+4^3\right)+\ldots\ldots\ldots}$

$$\boxed{S_n = \frac{a(r^n-1)}{r-1}}$$

Let T_n *be the* n^{th} *term of the given series*

$1+4+4^2+4^3+\cdots$

$$T_n = 1\left[\frac{4^k-1}{4-1}\right] \Longrightarrow T_n = \frac{4^k-1}{3} \Longrightarrow S_n = \sum_{k=1}^{n}\frac{4^k-1}{3}$$

$$S_n = \frac{1}{3}\left[\sum_{k=1}^{n} 4^k - \sum_{k=1}^{n} 1\right]$$

$$S_n = \frac{1}{3}\left[(4^1 + 4^2 + 4^3 + \cdots + 4^n) - n\right]$$

$$S_n = \frac{1}{3}\left[\frac{4(4^n - 1)}{3} - n\right] \Longrightarrow S_n = \frac{4}{9}(4^n - 1) - \frac{n}{3}$$

5. *Find the general term and sum to n terms of the sequence* $\mathbf{1, \frac{4}{3}, \frac{7}{9}, \frac{10}{27}, \ldots}$

Numerator: $1, 4, 7, 10 \ldots. \Longrightarrow a = 1, d = 4 - 1 = 3$

Denominator: $\frac{1}{1}, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3} \ldots \Longrightarrow a = 1, r = \frac{\frac{1}{3}}{1} = \frac{1}{3}$

The given sequence are: $a, (a+d)r, (a+2d)r^2, \ldots$

$\therefore$ *This is an A.G.P*

$$T_n = [a + (n-1)d]r^{n-1}$$
$$= [1 + (n-1)3]\left(\frac{1}{3}\right)^{n-1} = [1 + 3n - 3]\left(\frac{1}{3}\right)^{n-1}$$

$$T_n = 3n - 2\left(\frac{1}{3^{n-1}}\right) \Longrightarrow T_n = \frac{3n-2}{3^{n-1}}$$

S_n *be the sum of n term of given sequences*

$$S_n = \sum_{k=1}^{n} \frac{3k-2}{3^{k-1}} = \sum_{k=1}^{n}(3k-2)\sum \frac{1}{3^{k-1}} = \left[\sum_{k=1}^{n} 3k - \sum_{k=1}^{n} 2\right]\sum \frac{1}{3^{k-1}}$$

$$= \left[3\sum_{k=1}^{n} k - \sum_{k=1}^{n} 2\right]\sum \frac{1}{3^{k-1}}$$

$$\boxed{S_n = \frac{a(r^n - 1)}{r - 1}}$$

$a = 1, r = 3$

$$= [3(1 + 2 + 3 + \cdots n) - 2n]\left(\frac{1}{3^0 + 3^1 + 3^2 + \cdots 3^{n-1}}\right)$$

$$= \left[\frac{3n(n+1)}{2} - 2n\right]\left(\frac{1}{1\left(\frac{3^n - 1}{3 - 1}\right)}\right) = \left[\frac{3n^2 + 3n}{2} - 2n\right]\left(\frac{1}{\left(\frac{3^n - 1}{2}\right)}\right)$$

$$= \left[\frac{3n^2 + 3n - 4n}{\cancel{2}}\right]\left(\frac{\cancel{2}}{3^n - 1}\right) = \frac{3n^2 - n}{3^n - 1} = \frac{n(3n-1)}{3^n - 1}$$

6. *Find the value of n if the sum to n terms of the series* $\sqrt{3}+\sqrt{75}+\sqrt{243}+..$ ***is*** $435\sqrt{3}$

$Given: \sqrt{3}+\sqrt{75}+\sqrt{243}+\cdots is\ an\ A.P$

$\sqrt{3}+\sqrt{25\times 3}+\sqrt{81\times 3}+\ldots \Longrightarrow \sqrt{3}+5\sqrt{3}+9\sqrt{3}+\ldots$

$a=\sqrt{3},\quad d=t_2-t_1$

$=5\sqrt{3}-\sqrt{3}$

$d=4\sqrt{3}$

$S_n=435\sqrt{3}$

$S_n=\frac{n}{2}[2a+(n-1)d] \Longrightarrow 435\sqrt{3}=\frac{n}{2}[2\sqrt{3}+(n-1)4\sqrt{3}]$

$\frac{n}{2}\times 2[\sqrt{3}+(n-1)2\sqrt{3}]=435\sqrt{3} \Longrightarrow n[\sqrt{3}+(n-1)2\sqrt{3}]=435\sqrt{3}$

$\div\sqrt{3}$

$\sqrt{3}n+2\sqrt{3}(n-1)n=435\sqrt{3} \Longrightarrow n+2n(n-1)=435$

$n+2n^2-2n=435 \Longrightarrow 2n^2-n-435=0$

$2n^2-30n+29n-435=0 \Longrightarrow 2n(n-15)+29(n-15)=0$

$(2n+29)(n-15)=0 \Longrightarrow 2n+29=0\ , n-15=0$

$2n=-29\quad , n=15$

$n=-\frac{29}{2}$

7. *Show that the sum of* $(m+n)^{th}$ ***and*** $(m-n)^{th}$ ***term of AP is equal to twice the*** m^{th} ***term***

$\boxed{t_n=a+(n-1)d}$

$Given: \quad t_{m+n}+t_{m-n}=2t_m$

$L.H.S=t_{m+n}+t_{m-n}$

$=a+(m+n-1)d+a+(m-n-1)d$

$=a+md+\cancel{n}d-d+a+md-\cancel{n}d-d$

$=2a+2md-2d=2a+2(m-1)d$

$=2(a+(m-1)d)=2t_m=R.H.S$

8. A man repays an amount of Rs.3250 by paying Rs.20 in the first month and then increases the payment by Rs.15 per month. How long will it take him to clear the amount?

$Given: a=20,\ d=15,\ S_n=3250,\ n=?$

$S_n=\frac{n}{2}[2a+(n-1)d]$

$$3250 = \frac{n}{2}[2(20) + (n-1)15] \Longrightarrow \frac{n}{2}[40 + (n-1)15] = 3250$$

$$\frac{n}{2}[40 + 15n - 15] = 3250 \Longrightarrow \frac{n}{2}[15n + 25] = 3250$$

$$n[15n + 25] = 3250 \times 2 \Longrightarrow 15n^2 + 25n = 6500$$

$$\div 5$$

$$3n^2 + 5n = 1300 \Longrightarrow 3n^2 + 5n - 1300 = 0$$

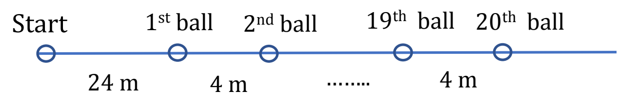
$$3n^2 - 60n + 65n - 1300 = 0 \Longrightarrow 3n(n-20) + 65(n-20) = 0$$

$$(3n + 65)(n - 20) = 0 \Longrightarrow 3n + 65 = 0, n - 20 = 0$$

$$3n = -65, n = 20$$

$$n = -\frac{65}{3}$$

9. In a race, 20 balls are placed in a line at intervals of 4 meters with the first ball, 24 meters away from the starting point. A contestant is required to bring the balls back to the starting place one at a time. How far would the contestant run to bring back all balls?



$$Distance\ travelled\ to\ bring\ 1st\ ball = 2(24) = 48m$$

$$Distance\ travelled\ to\ bring\ 2^{nd}\ ball = 2(24 + 4) = 2(28) = 56m$$

$$Distance\ travelled\ to\ bring\ 3^{rd}\ ball = 2(24 + 4 + 4) = 2(32) = 64m$$

$$The\ series\ 48, 56, 64, \ldots in\ an\ A.P$$

$$a = 48, \quad d = t_2 - t_1 = 56 - 48 = 8$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{20} = \frac{20}{2}[2(48) + (20-1)8]$$

$$= 10[96 + 19 \times 8] = 10[96 + 152]$$

$$= 10[248]$$

$$S_{20} = 2480m$$

10. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and nth hour?

$$The\ number\ of\ bacteria\ at\ the\ end\ of\ different\ hours\ from\ an\ G.P$$

$$a = 30, r = 2$$

$$Number\ of\ 1^{st}\ hour, T_2 = (2)30 = 60$$

$$Number\ of\ 2^{nd}\ hour, T_3 = ar^2 = 30 \times (2)^2 = 120$$

$Number\ of\ 4^{th}\ hour,\ T_5 = ar^4 = 30 \times (2)^4 = 30 \times 16 = 480$

$Number\ of\ n^{th}\ hour,\ T_{n+1} = ar^n = 30(2)^n = 30(2)^n$

11. What will Rs.500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

$Given\ that\ n = 10\ ,\ p = 500\ ,\ i = \frac{10}{100}$

$\text{Amount} = P(1+i)^n$

$$A = 500\left(1 + \frac{10}{100}\right)^{10} = 500\left(\frac{100+10}{100}\right)^{10}$$

$$\text{A}= 500\left(\frac{110}{100}\right)^{10} \Rightarrow A = 500\left(\frac{11}{10}\right)^{10} \Rightarrow A = 500\ (1.1)^{10}$$

A =1296.87

12. In a certain town, a viral disease caused severe health hazards upon its people disturbing their normal life. It was found that on each day, the virus which caused the disease spread in GP. The amount of infectious virus particle gets doubled each day, being 5 particles on the first day. Find the day when the infectious virus particles just grow over 150000 times?

$Given\ that$

$a = 5,\ r = 2,\ T_n = 150000, n = ?$

$5, 10, 20, 40 \ldots 1{,}50{,}000$

$t_n = ar^{n-1}$

$$1{,}50{,}000 = 5(2)^{n-1} \Longrightarrow \frac{1{,}50{,}000}{5} = 2^{n-1}$$

$2^{n-1} = 30{,}000$

2	30,000
2	15,000
2	7500
2	3750
	1875

12. In a certain town, a viral disease caused severe health hazards upon its people disturbing their normal life. It was found that on each day, the virus which caused the disease spread in GP. The amount of infectious virus particle gets doubled each day, being 5 particles on the first day. Find the day when the infectious virus particles just grow over 150000 times?

$Given\ that$

$a = 5,\ r = 2,\ T_n = 150000, n = ? \Rightarrow 5, 10, 20, 40 \ldots 1{,}50{,}000$

$$t_n = ar^{n-1} = 5(2)^{n-1} > 1{,}50{,}000 \Rightarrow 2^{n-1} > \frac{1{,}50{,}000}{5}$$

$$= 30{,}000$$

$EXERCISE : 5.4$

Some special Finite Series

$$(i) \sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$(ii) \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(iii) \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

5.6 Infinite Sequence and Series

If (d_n) is a sequence and a is a number so that for any given small positive number, there is a stage after which the distance between a_n and a is smaller than that positive number, then we may say that a_n goes to a as n goes to infinity.

In technical terms that a_n tends to a as n tends to infinity. In other words , in the limiting case a_n becomes a or the limit of a_n is a as n tends to ∞ . We also say that the sequence (a_n)converges to a. If (a_n) converges to a, then we write $\lim_{n \to \infty} a_n = a$.

5.6.1 Fibonacci Sequence

The Fibonacci sequence is a sequence of numbers where a number other than first two terms, is found be adding up the two numbers before it.

Starting with 0 and 1, the sequence goes 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, and so forth. Written as a rule, the expression is $x_n = x_{n-1} + x_{n-2}, n \geq 3$ $with\ x_0 = 1, x_1 = 1.$

Definition

$Let \sum_{n=1}^{\infty} a_n$ *be a series of real numbers and let*

$$S_n = a_1 + a_2 + a_3 + \cdots + a_{n,}\ n \in N$$

The sequence (S_n)*is called the partical sum sequence of* $\sum_{n=1}^{\infty} a_n$

If (S_n) *converges and if* $\lim_{n \to \infty} S_n = S,$
then the series is said to be a convergent series and S *is called the sum of the series.*

5.6.2 Infinite Geometric Series

The series Σx^n *is called a geometric series or geometric progession.*

Let us start with the series: $\sum_{n=0}^{\infty} x^n, x \neq 1.$

$$If\ S_n = x_0 + x_1 + x_2 + \cdots + x_{n,}\ then\ S_n = \frac{1-x^n}{1-x}$$

$As x_n$ *tends to* 0 *if* $|x| < 1,$ *we say that* s_n *tends to* $\frac{1}{1-x}$ *if* $|x| < 1.$

- $\sum_{n=0}^{\infty} x^n$ *converges for all x with* $|x|$*with* < 1 *and the sum is* $\frac{1}{1-x}$*. That is, for all real numbers x satisfying* $|x| < 1.$

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots}$$

- $\sum_{n=0}^{\infty} (-1)^n x^n$ *converges for all x with* $|x| < 1$ *and the sum is* $\frac{1}{1+x}$*. That is for all real numbers x satisfying* $|x| < 1.$

$$\boxed{\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots}$$

- $\sum_{n=0}^{\infty} (2x)^n$ *converges for all x with* $|x| < \frac{1}{2}$ *and the sum is* $\frac{1}{1-2x}$*. That is for all real numbers x satisfying* $|x| < \frac{1}{2}.$

$$\boxed{\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots}$$

- $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ *converges for all x and the sum is* e^x*. That is for all real numbers x.*

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots$$

- $\sum_{n=0}^{\infty} (-1)^n x$ *converges only for* $x = 0.$

Let us discuss some special series.
By assuming the convergence of those series let us soleve some problems.

Infinite Arithmetico-Geometric Series

The sum of arithmetico − geometric series $\Sigma((a + (n-1)d)r^{n-1}$ *is given by*

$$S = \lim_{n\to\infty} S_n = \frac{a}{1-r} + \frac{dr}{(1-r)^2} for -1 < r < 1.$$

$\boldsymbol{Example}$: **5.19.** $\boldsymbol{Find\ the\ sum}$: $\mathbf{1+\frac{4}{5}+\frac{7}{25}+\frac{10}{125}+\cdots}$

$$Here\ a=1, d=3\ and\ r=\frac{1}{5}$$

$$S_n=\frac{a}{1-r}+\frac{dr}{(1-r)^2}$$

$$=\frac{1}{1-\frac{1}{5}}+\frac{3\times\frac{1}{5}}{\left(1-\frac{1}{5}\right)^2}\qquad=\frac{1}{\frac{4}{5}}+\frac{\frac{3}{5}}{\left(\frac{5-1}{5}\right)^2}$$

$$=\frac{5}{4}+\frac{\frac{3}{5}}{\left(\frac{4}{5}\right)^2}\quad=\frac{5}{4}+\frac{\frac{3}{5}}{\frac{16}{25}}\quad=\frac{5}{4}+\frac{3}{\cancel{5}}\times\frac{\overset{5}{\cancel{25}}}{16}=\frac{5}{4}+\frac{15}{16}$$

$$=\frac{20+15}{16}\ =\frac{35}{16}$$

Telescopic Summation for Infinite Series

$\boldsymbol{Example}$: **5.20.** Find $\sum_{n=1}^{\infty}\frac{1}{n^2+5n+6}$

$Let\ a_n\ denote\ the\ n^{th}\ term\ of\ the\ given\ series.$

$$a_n=\frac{1}{n^2+5n+6}\ =\frac{1}{(n+3)(n+2)}\quad (By\ using\ partial\ \ fraction)$$

$$\frac{1}{(n+3)(n+2)}=\frac{A}{(n+3)}+\frac{B}{(n+2)}$$

$$\frac{1}{\cancel{(n+3)(n+2)}}=\frac{A(n+2)+B(n+3)}{\cancel{(n+3)(n+2)}}$$

$$1=A(n+2)+B(n+3)$$

$$Put\ n=-2:\ 1=A(-2+2)+B(-2+3)$$

$$1=A(0)+B(1)\ \Longrightarrow\ B=1$$

$$Put\ n=-3: 1=A(-3+2)+B(-3+3)$$

$$-A=1\Longrightarrow A=-1$$

$$\frac{1}{(n+3)(n+2)}=\frac{-1}{n+3}+\frac{1}{n+2}$$

$$a_n = \frac{1}{n+2} - \frac{1}{n+3}$$

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$S_n = \left(\frac{1}{3} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$S_n = \frac{1}{3} - \frac{1}{n+3} \quad If n \to \infty \; then \; \frac{1}{n+3} \to 0$$

$$\boxed{\frac{1}{n+3} = \frac{1}{\infty + 3} = 0}$$

$$S_n = \frac{1}{3} - 0 \implies S_n = \frac{1}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 5n + 6} = \frac{1}{3}$$

5.6.5 Binomial Series

$For \; any \; rational \; number \; n,$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$for \; all \; real \; number \; x \; satisfying |x| < 1.$

As the proof involves higher mathematical concepts, let us assume the theorem without proof and see some particular cases and solve some problems. In the theorem.

1. $By \; taking \; -x \; in \; the \; place \; of \; x, we \; get$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \quad (|x| < 1)$$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \; |x| < 1$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \quad |x| < 1$$

$\boldsymbol{Example \; 5.21. Expand \; (1+x)^{\frac{2}{3}} \; upto \; four \; terms \; for \; |x| < 1}$

$$Here \; n = \frac{2}{3}$$

$$(1+x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$\frac{n(n-1)}{2!} = \frac{\frac{2}{3}\left(\frac{2}{3} - 1\right)}{1 \times 2} = \frac{\frac{2}{3} \times \frac{-1}{3}}{2} = \frac{\frac{-2}{9}}{2} = \frac{-1}{\frac{\cancel{-2}}{9}} \times \frac{1}{\cancel{2}} = \frac{-1}{9}$$

$$\frac{n(n-1)(n-2)}{3!}=\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{1\times2\times3}=\frac{\frac{2}{3}\left(\frac{2-3}{3}\right)\left(\frac{2-6}{3}\right)}{6}=\frac{\frac{2}{3}\times\frac{-1}{3}\times\frac{-4}{3}}{6}$$

$$=\frac{\frac{8}{27}}{6}=\frac{\cancel{8}^{4}}{27}\times\frac{1}{\cancel{6}_{3}}=\frac{4}{81}$$

$$(1+x)^{\frac{2}{3}}=1-\frac{2}{3}x+\left(\frac{-1}{9}\right)x^2+\frac{4}{81}x^3+\cdots$$

$$=1-\frac{2}{3}x-\frac{1}{9}x^2+\frac{4}{81}x^3+\cdots$$

Example 5.22 Expand $\frac{1}{(1+3x)^2}$ in powers of x. Find a condition on x for which the expansion is valid.

$$\frac{1}{(1+3x)^2}=(1+3x)^{-2}$$

$$(1+x)^{-n}=1-nx+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3+\cdots$$

$$x=3x,\ n=2$$

$$(1+3x)^{-2}=1-2(3x)+\frac{2(2+1)}{1\times2}(3x)^2-\frac{2(2+1)(2+2)}{1\times2\times3}(3x)^3$$

$$+\frac{2(2+1)(2+2)(2+3)}{1\times2\times3\times4}(3x)^4+\cdots$$

$$=1-6x+\frac{\cancel{2}\times3}{1\times\cancel{2}}9x^2-\frac{\cancel{2}\times\cancel{3}\times4}{1\times\cancel{2}\times\cancel{3}}27x^3+\frac{\cancel{2}\times\cancel{3}\times\cancel{4}\times5}{1\times\cancel{2}\times\cancel{3}\times\cancel{4}}81x^4+\cdots$$

$$=1-6x+27x^2-108x^3+405x^4+\cdots$$

Example 5.23. Expand $\frac{1}{(3+2x)^2}$ in powers of x. Find a condition on x for which expansion is valid

$$\frac{1}{(3+2x)^2}=\frac{1}{3^2\left(1+\frac{2}{3}x\right)^2}=\frac{1}{9\left(1+\frac{2x}{3}\right)^2}$$

$$\frac{1}{(3+2x)^2}=\frac{1}{9}\left(1+\frac{2}{3}x\right)^{-2}$$

$$Let\ y=\frac{2}{3}x$$

$$\frac{1}{(3+2x)^2} = \frac{1}{9}(1+y)^{-2} \quad |y| < 1$$

$$\boxed{(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots} \quad x = y, \; n = 2$$

$$= \frac{1}{9}\left[1 - 2y + \frac{2(2+1)}{1\times 2}y^2 - \frac{2(2+1)(2+2)}{1\times 2\times 3}y^3 + \frac{2(2+1)(2+2)(2+3)}{1\times 2\times 3\times 4}y^4 + \cdots\right]$$

$$= \frac{1}{9}\left[1 - 2y + 3y^2 - \frac{\cancel{2}\times\cancel{3}\times 4}{1\times\cancel{2}\times\cancel{3}}y^3 + \frac{\cancel{2}\times\cancel{3}\times\cancel{4}\times 5}{1\times\cancel{2}\times\cancel{3}\times\cancel{4}}y^4 + \cdots\right]$$

$$= \frac{1}{9}\left[1 - 2y + 3y^2 - 4y^3 + 5y^4 + \cdots\right]$$

$$Sub\ y = \frac{2}{3}x$$

$$= \frac{1}{9}\left[1 - 2\left(\frac{2}{3}x\right) + 3\left(\frac{2}{3}x\right)^2 - 4\left(\frac{2}{3}x\right)^3 + 5\left(\frac{2}{3}x\right)^4 + \cdots\right]$$

$$= \frac{1}{9}\left[1 - \frac{4}{3}x + 3\left(\frac{4}{9}x^2\right) - 4\left(\frac{8}{27}x^3\right) + 5\left(\frac{16}{81}x^4\right) + \cdots\right]$$

$$= \frac{1}{9}\left[1 - \frac{4}{3}x + \frac{4}{3}x^2 - \frac{32}{27}x^3 + \frac{80}{81}x^4 + \cdots\right]$$

$$= \frac{1}{9} - \frac{4}{27}x + \frac{4}{27}x^2 - \frac{32}{243}x^3 + \frac{80}{729}x^4 + \cdots$$

Example5.24. Find $\sqrt[3]{65}$

$$\sqrt[3]{65} = 65^{\frac{1}{3}}$$

$$= (64+1)^{\frac{1}{3}} = 64^{\frac{1}{3}}\left(1 + \frac{1}{64}\right)^{\frac{1}{3}}$$

$$= \sqrt[3]{64}\left(1 + \frac{1}{64}\right)^{\frac{1}{3}} = 4\left(1 + \frac{1}{64}\right)^{\frac{1}{3}}$$

$$\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots}$$

$$x=\frac{1}{64}, n=\frac{1}{64}$$

$$=4\left(1+\frac{1}{3}\left(\frac{1}{64}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{64}\right)^2+\cdots\right)$$

$$=4\left[1+\frac{1}{3}\left(\frac{1}{64}\right)+\frac{\frac{1}{3}\left(\frac{1-3}{3}\right)}{1\times 2}\left(\frac{1}{64}\right)\left(\frac{1}{64}\right)+\cdots\right]$$

$$=4\left[1+\frac{1}{3}\left(\frac{1}{64}\right)+\frac{\frac{1}{3}\left(\frac{\cancel{-2}}{3}\right)}{\cancel{2}}\times\left(\frac{1}{64}\right)\left(\frac{1}{64}\right)+\cdots\right]$$

$$=4\left[1+\frac{1}{3}\left(\frac{1}{64}\right)-\left(\frac{1}{9}\right)\times\left(\frac{1}{64}\right)\times\left(\frac{1}{64}\right)+\cdots\right]$$

$$=4+\cancel{4}\times\frac{1}{3}\times\frac{1}{\cancel{64}_{16}}-\cancel{4}\times\frac{1}{9}\times\frac{1}{\cancel{64}_{16}}\times\frac{1}{64}+\cdots$$

$$=4+\frac{1}{48}-\frac{1}{9216}+\cdots$$

$\boxed{\text{Since } \frac{1}{9216}+\cdots \text{ is very small}}$

$$=4+0.02$$

$\sqrt[3]{65}=4.02$ *approximately*

Example5.25. Prove that $\sqrt[3]{x^3+7}-\sqrt[3]{x^3+4}$ ***is approximately equal to*** $\frac{1}{x^2}$ ***when*** x ***is large.***

$$\sqrt[3]{x^3+7}=(x^3+7)^{\frac{1}{3}}=\left[x^3\left(\frac{x^3}{x^3}+\frac{7}{x^3}\right)\right]^{\frac{1}{3}}=\left[x^3\left(1+\frac{7}{x^3}\right)\right]^{\frac{1}{3}}$$

$$=(x^3)^{\frac{1}{3}}\left(1+\frac{7}{x^3}\right)^{\frac{1}{3}}=x\left(1+\frac{7}{x^3}\right)^{\frac{1}{3}}$$

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots$$

$$x=\frac{7}{x^3},\quad n=\frac{1}{3}$$

$$\sqrt[3]{x^3+7}=x\left(1+\frac{1}{3}\left(\frac{7}{x^3}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{7}{x^3}\right)^2+\cdots\right)$$

$$= x\left(1+\frac{7}{3x^3}+\frac{\frac{1}{3}\left(\frac{-\cancel{1}2}{3}\right)}{1\times\cancel{2}}\times\frac{49}{x^6}+\cdots\right)$$

$$= x\left(1+\frac{7}{3x^3}-\frac{1}{9}\times\frac{49}{x^6}+\cdots\right)$$

$$= x\left(1+\frac{7}{3x^3}-\frac{49}{9x^6}+\cdots\right) = x+\frac{7}{3x^3}\times x-\frac{49}{9x^6}\times x+\cdots$$

$$\sqrt[3]{x^3+7} = x+\frac{7}{3x^2}-\frac{49}{9x^5}+\cdots \Longrightarrow \sqrt[3]{x^3+4} = (x^3+4)^{\frac{1}{3}}\ \ldots(1)$$

$$= \left[x^3\left(\frac{x^3}{x^3}+\frac{4}{x^3}\right)\right]^{\frac{1}{3}} = \left[x^3\left(1+\frac{4}{x^3}\right)\right]^{\frac{1}{3}} = (x^3)^{\frac{1}{3}}\left(1+\frac{4}{x^3}\right)^{\frac{1}{3}} = x\left(1+\frac{4}{x^3}\right)^{\frac{1}{3}}$$

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots$$

$$x=\frac{4}{x^3}, \qquad n=\frac{1}{3}$$

$$\sqrt[3]{x^3+4} = x\left(1+\frac{1}{3}\left(\frac{4}{x^3}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{4}{x^3}\right)^2+\cdots\right)$$

$$= x\left(1+\frac{4}{3x^3}+\frac{\frac{1}{3}\left(\frac{-\cancel{2}}{3}\right)}{1\times\cancel{2}}\times\frac{16}{x^6}+\cdots\right)$$

$$= x\left(1+\frac{4}{3x^3}+\frac{1}{3}\left(\frac{-1}{3}\right)\times\frac{16}{x^6}+\cdots\right) = x\left(1+\frac{4}{3x^3}-\frac{16}{9x^6}+\cdots\right)$$

$$= x+x\times\frac{4}{3x^3}-x\frac{16}{9x^6}+\cdots \Rightarrow \sqrt[3]{x^3+4} = x+\frac{4}{3x^2}-\frac{16}{9x^5}+\cdots\ \ldots(2)$$

Since x is large, $\frac{1}{x}$ is very small and $\Rightarrow$ hence higher powers are negligible.

$$\sqrt[3]{x^3+7} = x+\frac{7}{3x^2} \text{ and } \sqrt[3]{x^3+4} = x+\frac{4}{3x^2}$$

$$\sqrt[3]{x^3+7}-\sqrt[3]{x^3+4} = x+\frac{7}{3x^2}-\left(x+\frac{4}{3x^2}\right)$$

$$= \cancel{x} + \frac{7}{3x^2} - \cancel{x} - \frac{4}{3x^2}$$

$$= \frac{7}{3x^2} - \frac{4}{3x^2} = \frac{7-4}{3x^2} = \frac{3}{3x^2} = \frac{1}{x^2}$$

5.6.6 Exponential Series

$The\ series \sum_{n=0}^{\infty} \frac{x^n}{n!} is\ called\ an\ exponential\ series. It\ can\ be\ proved\ that\ this\ series\ converges\ for\ all\ values\ of\ x.$

$$For\ any\ real\ number\ x, \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x\ where\ e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$For\ all\ values\ of\ x,. By\ takin - x\ in\ place\ of\ x\ in\ (Theorem\ 1) we\ get$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

5.6.7 Logarithmic Series

$The\ series \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n}\ is\ called\ an\ logarithmic\ series.$

$This\ series\ converges\ for\ all\ values\ of\ x\ satisfying |x| < 1. This\ series\ converges\ when\ x = 1\ also.$

$For\ all\ values\ of\ x\ satisfying\ |x| < 1, the\ sum\ of\ the\ series\ is\ \log(1+x). Thus$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$For\ all\ values\ of\ x\ satisfying\ |x| < 1, By\ taking\ - x\ in\ the\ place$ of x

$$\log(1-x) = -x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \cdots$$

$For\ all\ values\ of\ x\ satisfying\ |x| < 1.$

$$Now\ \log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)\ . Using\ this\ we\ get$$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right]$$

Exercise 5.4

1. **Expand the following in ascending powers of x and find the condition on x for which the binomial expansion is valid.**

$(i)\ \dfrac{1}{5+x}$ $(ii)\ \dfrac{2}{(3+4x)^2}$ $(iii)\ (5+x^2)^{\frac{2}{3}}$ $(iv)\ (x+2)^{\frac{-2}{3}}$

$$(i)\ \frac{1}{5+x} = \frac{1}{5\left(\frac{5}{5}+\frac{x}{5}\right)} = \frac{1}{5\left(1+\frac{x}{5}\right)} = \frac{1}{5}\left(1+\frac{x}{5}\right)^{-1}$$

$$\boxed{(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots}$$

$$x = \frac{x}{5},\ n = 1\ \ If\ \left|\frac{x}{5}\right| < 1 \Rightarrow |x| < 5$$

$$= \frac{1}{5}\left[1 - (1)\left(\frac{x}{5}\right) + \frac{1(1+1)}{1\times 2}\left(\frac{x}{5}\right)^2 - \frac{1(1+1)(1+2)}{1\times 2\times 3}\left(\frac{x}{5}\right)^3 + \cdots\right]$$

$$= \frac{1}{5}\left[1 - \frac{x}{5} + \frac{2}{2}\left(\frac{x}{5}\right)^2 - \frac{\cancel{2}\times\cancel{3}}{1\times\cancel{2}\times\cancel{3}}\left(\frac{x}{5}\right)^3 + \cdots\right]$$

$$= \frac{1}{5}\left[1 - \frac{x}{5} + \left(\frac{x}{5}\right)^2 - \left(\frac{x}{5}\right)^3 + \cdots\right]$$

$$(ii)\ \frac{2}{(3+4x)^2} = \frac{2}{\left[3\left(\frac{3}{3}+\frac{4x}{3}\right)\right]^2} = \frac{2}{\left[3\left(1+\frac{4x}{3}\right)\right]^2} = \frac{2}{3^2\left(1+\frac{4}{3}x\right)^2}$$

$$= \frac{2}{9}\left(1+\frac{4}{3}x\right)^{-2}$$

$$\boxed{(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots}$$

$$x = \frac{4}{3}x,\quad n = 2$$

$$= \frac{2}{9}\left[1 - (2)\left(\frac{4}{3}x\right) + \frac{\cancel{2}(2+1)}{1\times\cancel{2}}\left(\frac{4x}{3}\right)^2 - \frac{2(2+1)(2+2)}{1\times 2\times 3}\left(\frac{4x}{3}\right)^3 + \cdots\right]$$

$$= \frac{2}{9}\left[1 - 2\left(\frac{4x}{3}\right) + 3\left(\frac{4x}{3}\right)^2 - \frac{\cancel{2}\times\cancel{3}\times 4}{1\times\cancel{2}\times\cancel{3}}\left(\frac{4x}{3}\right)^3 + \cdots\right]$$

$$= \frac{2}{9}\left[1 - 2\left(\frac{4x}{3}\right) + 3\left(\frac{4x}{3}\right)^2 - 4\left(\frac{4x}{3}\right)^3 + \cdots\right]$$

$$(iii)\ (5+x^2)^{\frac{2}{3}} = \left[5\left(\frac{5}{5}+\frac{x^2}{5}\right)\right]^{\frac{2}{3}} = \left[5\left(1+\frac{x^2}{5}\right)\right]^{\frac{2}{3}} = 5^{\frac{2}{3}}\left(1+\frac{x^2}{5}\right)^{\frac{2}{3}}$$

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

$$x=\frac{x^2}{5},\ n=\frac{2}{3}$$

$$If\ \left|\frac{x^2}{5}\right|<1 \Rightarrow \frac{x^2}{|5|}<1 \Rightarrow |x^2|<5$$

$$=5^{\frac{2}{3}}\left(1+\frac{2}{3}\left(\frac{x^2}{5}\right)+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{1\times 2}\left(\frac{x^2}{5}\right)^2+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{1\times 2\times 3}\left(\frac{x^2}{5}\right)^3+\cdots\right)$$

$$=5^{\frac{2}{3}}\left(1+\frac{2}{3}\left(\frac{x^2}{5}\right)+\frac{\frac{\cancel{2}}{3}\left(\frac{-1}{3}\right)}{1\times \cancel{2}}\left(\frac{x^2}{5}\right)^2+\frac{\frac{\cancel{2}}{3}\times\frac{-1}{3}\times\frac{-4}{3}}{1\times \cancel{2}\times 3}\left(\frac{x^2}{5}\right)^3+\cdots\right)$$

$$=5^{\frac{2}{3}}\left(1+\frac{2}{3}\left(\frac{x^2}{5}\right)-\frac{1}{9}\left(\frac{x^2}{5}\right)^2+\frac{4}{27}\times\frac{1}{3}\left(\frac{x^2}{5}\right)^3+\cdots\right]$$

$$=5^{\frac{2}{3}}\left(1+\frac{2}{3}\left(\frac{x^2}{5}\right)-\frac{1}{9}\left(\frac{x^2}{5}\right)^2+\frac{4}{81}\left(\frac{x^2}{5}\right)^3+\cdots\right] = (2+x)^{-\frac{2}{3}}$$

$$=\left[2\left(\frac{2}{2}+\frac{x}{2}\right)\right]^{-\frac{2}{3}} = \left[2\left(1+\frac{x}{2}\right)\right]^{-\frac{2}{3}} = 2^{\frac{-2}{3}}\left(1+\frac{x}{2}\right)^{-\frac{2}{3}} = \frac{1}{2^{\frac{2}{3}}}\left(1+\frac{x}{2}\right)^{-\frac{2}{3}}$$

$$(1+x)^{-n} = 1-nx+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3+\cdots$$

$$x=\frac{x}{2},\ n=\frac{2}{3}$$

$$=\frac{1}{2^{\frac{2}{3}}}\left(1-\frac{2}{3}\left(\frac{x}{2}\right)+\frac{\frac{2}{3}\left(\frac{2}{3}+1\right)}{2!}\left(\frac{x}{2}\right)^2-\frac{\frac{2}{3}\left(\frac{2}{3}+1\right)\left(\frac{2}{3}+2\right)}{1\times 2\times 3}\left(\frac{x^2}{5}\right)^3+\cdots\right)$$

$$\boldsymbol{(iv)\ (x+2)^{\frac{-2}{3}}}$$

$$= \frac{1}{2^{\frac{2}{3}}}\left(1 - \frac{2}{3}\left(\frac{x}{2}\right) + \frac{\frac{\cancel{2}}{3}\times\frac{5}{3}}{1\times\cancel{2}}\left(\frac{x}{2}\right)^2 - \frac{\frac{\cancel{2}}{3}\times\frac{5}{3}\times\frac{8}{3}}{1\times\cancel{2}\times 3}\left(\frac{x^2}{2}\right)^3 + \cdots\right)$$

$$= \frac{1}{2^{\frac{2}{3}}}\left[1 - \frac{\cancel{2}}{3}\left(\frac{x}{\cancel{2}}\right) + \frac{5}{9}\left(\frac{x}{2}\right)^2 - \frac{40}{9}\times\left(\frac{x^2}{2}\right)^3 + \cdots\right]$$

$$= \frac{1}{2^{\frac{2}{3}}}\left[1 - \frac{x}{3} + \frac{5}{9}\times\frac{x^2}{4} - \frac{40}{9}\times\frac{1}{3}\left(\frac{x^2}{2}\right)^3 + \cdots\right]$$

$$= \frac{1}{2^{\frac{2}{3}}}\left[1 - \frac{x}{3} + \frac{5x^2}{36} - \frac{40}{27}\left(\frac{x^6}{8}\right) + \cdots\right] = \frac{1}{2^{\frac{2}{3}}}\left[1 - \frac{x}{3} + \frac{5x^2}{36} - \frac{40x^6}{216} + \cdots\right]$$

2. Find $\sqrt[3]{1001}$ *approximately* (*two decimal places*).

$$\sqrt[3]{1001} = (1+1000)^{\frac{1}{3}}$$

$$= \left[1000\left(\frac{1}{1000} + \frac{1000}{1000}\right)\right]^{\frac{1}{3}} = \left[1000\left(\frac{1}{1000} + 1\right)\right]^{\frac{1}{3}} = (1000)^{\frac{1}{3}}\left(1 + \frac{1}{1000}\right)^{\frac{1}{3}}$$

$$= \sqrt[3]{1000}\left(1 + \frac{1}{1000}\right)^{\frac{1}{3}} = 10\left(1 + \frac{1}{1000}\right)^{\frac{1}{3}}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$x = \frac{1}{1000} \qquad n = \frac{1}{3}$$

$$= 10\left[1 + \frac{1}{3}\left(\frac{1}{1000}\right) + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}\left(\frac{1}{1000}\right)^2 + \cdots\right]$$

$$= 10\left[1 + \frac{1}{3}\left(\frac{1}{1000}\right) + \frac{\frac{1}{3}\left(\frac{-\cancel{2}}{3}\right)}{1\times\cancel{2}}\left(\frac{1}{1000}\right)^2 + \cdots\right]$$

$$= 10\left[1 + \frac{1}{3000} - \frac{1}{3}\times\frac{1}{1000000} + \cdots\right]$$

$$= 10\left(1 + \frac{1}{3000} - \frac{1}{9000000} + \cdots\right) \quad = 10\left(1 + 0.0003333\right)$$

$$= 10\left(1.00033333\right) = 10.00333$$

3. *Prove that $\sqrt[3]{x^3+6} - \sqrt[3]{x^3+3}$ is approximately equal to $\frac{1}{x^2}$ when x is large.*

$$\sqrt[3]{x^3+6} = (x^3+6)^{\frac{1}{3}} = \left[x^3\left(\frac{x^3}{x^3} + \frac{6}{x^3}\right)\right]^{\frac{1}{3}} = \left[x^3\left(1 + \frac{6}{x^3}\right)\right]^{\frac{1}{3}}$$

$$= (x^3)^{\frac{1}{3}}\left(1 + \frac{6}{x^3}\right)^{\frac{1}{3}} \quad = x\left(1 + \frac{6}{x^3}\right)^{\frac{1}{3}}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \quad x = \frac{6}{x^3}, \quad n = \frac{1}{3}$$

$$\sqrt[3]{x^3+6} = x\left(1 + \frac{1}{\cancel{3}}\left(\frac{\cancel{6}^{\,2}}{x^3}\right) + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}\left(\frac{6}{x^3}\right)^2 + \cdots\right)$$

$$= x\left(1 + \frac{2}{x^3} + \frac{\frac{1}{3}\left(\frac{-\cancel{2}}{3}\right)}{1 \times \cancel{2}} \times \frac{36}{x^6} + \cdots\right) = x\left(1 + \frac{2}{x^3} - \frac{1}{\cancel{9}} \times \frac{\cancel{36}^{\,4}}{x^6} + \cdots\right)$$

$$= x\left(1 + \frac{2}{x^3} - \frac{4}{x^6} + \cdots\right) \quad = x + x \times \frac{2}{x^3} - x \times \frac{4}{x^6} + \cdots$$

$$\sqrt[3]{x^3+6} = x + \frac{2}{x^2} - \frac{4}{x^5} + \cdots$$

$$\sqrt[3]{x^3+3} = (x^3+3)^{\frac{1}{3}} = \left[x^3\left(\frac{x^3}{x^3} + \frac{3}{x^3}\right)\right]^{\frac{1}{3}} = \left[x^3\left(1 + \frac{3}{x^3}\right)\right]^{\frac{1}{3}}$$

$$= (x^3)^{\frac{1}{3}}\left(1 + \frac{3}{x^3}\right)^{\frac{1}{3}} = x\left(1 + \frac{3}{x^3}\right)^{\frac{1}{3}}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$x = \frac{3}{x^3} \qquad n = \frac{1}{3}$$

$$\sqrt[3]{x^3+3} = x\left(1+\frac{1}{3}\left(\frac{3}{x^3}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{3}{x^3}\right)^2+\cdots\right)$$

$$= x\left(1+\frac{1}{x^3}+\frac{\frac{1}{3}\left(\frac{-\cancel{2}}{3}\right)}{1\times\cancel{2}}\times\frac{9}{x^6}+\cdots\right) \quad = x\left(1+\frac{1}{x^3}-\frac{1}{\cancel{9}}\left(\frac{\cancel{9}}{x^6}\right)+\cdots\right)$$

$$= x\left(1+\frac{1}{x^3}+\frac{1}{x^6}+\cdots\right)$$

$$\sqrt[3]{x^3+3} = x + x\times\frac{1}{x^3} - x\frac{1}{x^6}+\cdots$$

$$\sqrt[3]{x^3+3} = x+\frac{1}{x^2}-\frac{1}{x^5}+\cdots$$

$$\sqrt[3]{x^3+6}-\sqrt[3]{x^3+3}$$

$$= x+\frac{2}{x^2}-\frac{4}{x^5}+\cdots-\left(x+\frac{1}{x^2}-\frac{1}{x^5}+\cdots\right)$$

$$= \cancel{x}+\frac{2}{x^2}-\frac{4}{x^5}+\cdots-\cancel{x}-\frac{1}{x^2}+\frac{1}{x^5}-\cdots$$

$$= \frac{2}{x^2}-\frac{1}{x^2}-\frac{4}{x^5}+\frac{1}{x^5}+\cdots \quad \textit{Since } \frac{3}{x^5}+\cdots \textit{ is very small}$$

$$= \frac{1}{x^2}-\frac{3}{x^5}+\cdots = \frac{1}{x^2} \qquad \textit{approximately}$$

4. *Prove that $\dfrac{\sqrt{1-x}}{\sqrt{1+x}}$ is approximately equal to $1-x+\dfrac{x^2}{2}$ when x is very small.*

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} = \frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}} = (1-x)^{\frac{1}{2}}(1+x)^{\frac{-1}{2}}$$

$$= \left(1-\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\times 2}x^2+\cdots\right)\left(1-\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)}{}x^2+\cdots\right)$$

$$=\left(1-\frac{1}{2}x+\frac{\frac{1}{2}\times\frac{-1}{2}}{1\times 2}x^2+\cdots\right)\left(1-\frac{1}{2}x+\frac{\frac{1}{2}\times\frac{3}{2}}{1\times 2}x^2+\cdots\right)$$

$$=\left(1-\frac{x}{2}-\frac{\frac{1}{4}}{2}x^2+\cdots\right)\left(1-\frac{x}{2}+\frac{\frac{3}{4}}{2}x^2+\cdots\right)$$

$$=\left(1-\frac{x}{2}-\frac{1}{8}x^2+\cdots\right)\left(1-\frac{x}{2}+\frac{3}{8}x^2+\cdots\right)$$

$$=1-\frac{x}{2}+\frac{3}{8}x^2-\frac{x}{2}+\frac{x^2}{4}-\frac{1}{8}x^2+\cdots$$

$$=1-\frac{2x}{2}+\frac{3}{8}x^2+\frac{x^2}{4}-\frac{x^2}{8}+\cdots=1-x+\frac{3x^2+2x^2-x^2}{8}+\cdots$$

$$=1-x+\frac{4x^2}{8}+\cdots=1-x+\frac{x^2}{2}+\cdots$$

5. *Write the first 6 terms of the exponential series* $(i)\,e^{5x}\,(ii)\,e^{-2x}\,(iii)\,e^{\frac{1}{2}x}$

$$(i)\ e^{5x}=1+\frac{(5x)}{1!}+\frac{(5x)^2}{2!}+\frac{(5x)^3}{3!}+\frac{(5x)^4}{4!}+\frac{(5x)^5}{5!}$$

$$=1+5x+\frac{25x^2}{2}+\frac{125x^3}{6}+\frac{625x^4}{24}+\frac{625x^5}{24}$$

$$(ii)\,e^{-2x}=1-\frac{(2x)}{1!}+\frac{(2x)^2}{2!}-\frac{(2x)^3}{3!}+\frac{(2x)^4}{4!}-\frac{(2x)^5}{5!}$$

$$=1-2x+2x^2-\frac{4x^3}{3}+\frac{2x^4}{3}-\frac{4x^5}{15}$$

$$(iii)\ e^{\frac{1}{2}x}=1+\left(\frac{x}{2}\right)+\frac{\left(\frac{x}{2}\right)^2}{2!}+\frac{\left(\frac{x}{2}\right)^3}{3!}+\frac{\left(\frac{x}{2}\right)^4}{4!}+\frac{\left(\frac{x}{2}\right)^5}{5!}$$

$$=1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}$$

Logarithmic Series

$$log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

$$log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots$$

$$log\left(\frac{1+x}{1-x}\right)= 2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right]$$

6. *Write the first 6 terms of the logarithmic series* (*i*) $\log(1+4x)$ (*ii*) $\log(1-2x)$ (*iii*) $log\left(\frac{1+3x}{1-3x}\right)$ (*iv*) $log\left(\frac{1-2x}{1+2x}\right)$ *Find the interval on which expansions are valid.*

Valid if $|4x| < 1 \Longrightarrow |x| < \frac{1}{4}$

$$(i)\ \log(1+4x) = 4x - \frac{(4x)^2}{2} + \frac{(4x)^3}{3} - \frac{(4x)^4}{4} + \frac{(4x)^5}{5} - \frac{(4x)^6}{6}$$

$$= 4x - 8x^2 + \frac{64x^3}{3} - 64x^4 + \frac{256x^5}{5} - \frac{512x^6}{3}$$

$(ii)\ \log(1-2x)$ *Valid if* $|2x| < 1 \Longrightarrow |x| < \frac{1}{2}$

$$= -(2x) - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} - \frac{(2x)^4}{4} - \frac{(2x)^5}{5} - \frac{(2x)^6}{6}$$

$$= -2x - 2x^2 - \frac{8x^3}{3} - 4x^4 - \frac{32x^5}{5} - \frac{32x^6}{3}$$

$(iii)\ log\left(\frac{1+3x}{1-3x}\right)$ *Valid if* $|3x| < 1 \Longrightarrow |x| < \frac{1}{3}$ $\quad log\left(\frac{1+x}{1-x}\right)$

$$= 2\left((3x) + \frac{(3x)^3}{3} + \frac{(3x)^5}{5} + \frac{(3x)^7}{7} + \frac{(3x)^9}{9} + \frac{(3x)^{11}}{11}\right)$$

$(iv)\ \log\left(\frac{1-2x}{1+2x}\right)$ *Valid if* $|2x| < 1 \Longrightarrow |x| < \frac{1}{2}$ $\quad \boxed{2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right]}$

$$= log\left(\frac{1+2x}{1-2x}\right)^{-1} = -\,log\left(\frac{1+2x}{1-2x}\right)$$

$$= -2\left((2x) + \frac{(2x)^3}{3} + \frac{(2x)^5}{5} + \frac{(2x)^7}{7} + \frac{(2x)^9}{9} + \frac{(2x)^{11}}{11}\right)$$

7. *If* $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$, *then show that* $x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!}\ldots$

Given $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$y = -log(1-x)$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \ldots$$

$$y = \log(1-x)^{-1} \Longrightarrow e^y = (1-x)^{-1}$$

$$1 - x = e^{-y} \Longrightarrow 1 - x = e^{-y}$$

$$\cancel{1} - x = \cancel{1} - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \cdots$$

$$-x = -y + \frac{y^2}{2!} - \frac{y^3}{3!} + \cdots$$

$$x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} \ldots$$

8. *If $p-q$ is small compared to either p or q, then show that*

$$\sqrt[n]{\frac{p}{q}} = \frac{(n+1)p + (n-1)q}{(n-1)p + (n+1)q}. \text{ Hence find } \sqrt[8]{\frac{15}{16}}.$$

$$\frac{p}{q} = \frac{1 + \frac{p-q}{p+q}}{1 - \frac{p-q}{p+q}} = \frac{p+q+p-q}{p+q-p+q} = \frac{2p}{2q} = \frac{p}{q}$$

$$\left(\frac{p}{q}\right)^{\frac{1}{n}} = \left(\frac{1 + \frac{p-q}{p+q}}{1 - \frac{p-q}{p+q}}\right)^{\frac{1}{n}} = \frac{\left(1 + \frac{p-q}{p+q}\right)^{\frac{1}{n}}}{\left(1 - \frac{p-q}{p+q}\right)^{\frac{1}{n}}}$$

$$= \frac{1 + \frac{1}{n}\left(\frac{p-q}{p+q}\right)}{1 - \frac{1}{n}\left(\frac{p-q}{p+q}\right)} = \frac{1 + \frac{p-q}{np+nq}}{1 - \frac{p-q}{np+nq}} = \frac{\frac{np+nq+p-q}{np+nq}}{\frac{np+nq-p+q}{np+nq}} = \frac{np+nq+p-q}{np+nq-p+q}$$

$$= \frac{np+p+nq-q}{np-p+nq+q} = \frac{(n+1)p + (n-1)q}{(n-1)p + (n+1)q}$$

$$put\ n = 8, p = 15, q = 16$$

$$\sqrt[8]{\frac{15}{16}} = \frac{9(15) + 7(16)}{7(15) + 9(16)} = \frac{135 + 112}{105 + 144} = \frac{247}{249} = 0.9916$$

9. *Find the coefficient of x^4 in the expansion of* $\boldsymbol{\frac{3 - 4x + x^2}{e^{2x}}}$

$$\frac{3 - 4x + x^2}{e^{2x}} = (3 - 4x + x^2)e^{-2x}$$

$$= (3 - 4x + x^2)\left[1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!}\right]$$

$$= (3 - 4x + x^2)\left[1 - \frac{2x}{1!} + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \frac{16x^4}{4!}\right]$$

$$Coefficient\ of\ x^4 = \frac{3\times 16}{4!} + \frac{4\times 8}{3!} + \frac{4}{2!}$$

$$= \frac{\cancel{3}\times \cancel{16}^{\cancel{4}\,2}}{1\times\cancel{2}\times\cancel{3}\times\cancel{4}} + \frac{\cancel{4}^{2}\times 8}{1\times\cancel{2}\times 3} + \frac{\cancel{4}^{2}}{1\times\cancel{2}} \quad = 2 + \frac{16}{3} + 2 = \frac{28}{3}$$

10. Find the value of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{1}{9^{n-1}}+\frac{1}{9^{2n-1}}\right)$

$$\sum_{1}^{\infty}\frac{1}{2n-1}\left(\frac{1}{9^{n-1}}+\frac{1}{9^{2n-1}}\right) = \sum_{1}^{\infty}\frac{1}{2n-1}\left(\frac{1}{9^{n}\times 9^{-1}}+\frac{1}{9^{2n}\times 9^{-1}}\right)$$

$$= \sum_{1}^{\infty}\frac{1}{2n-1}\left(\frac{9}{9^{n}}+\frac{9}{9^{2n}}\right) = 9\sum_{1}^{\infty}\frac{1}{2n-1}\left(\frac{1}{9^{n}}+\frac{1}{9^{2n}}\right)$$

$$= 9\left[\frac{1}{9}+\frac{1}{9^2}+\frac{1}{3}\left(\frac{1}{9}\right)^2+\frac{1}{3}\left(\frac{1}{9}\right)^4+\frac{1}{5}\left(\frac{1}{9}\right)^3+\frac{1}{5}\left(\frac{1}{9}\right)^6+\frac{1}{7}\left(\frac{1}{9}\right)^4+\frac{1}{7}\left(\frac{1}{9}\right)^8\cdots\right]$$

$$= 9\left[\left(\frac{1}{9}\right)+\frac{1}{3}\left(\frac{1}{9}\right)^2+\frac{1}{5}\left(\frac{1}{9}\right)^3+\cdots\right]+9\left[\frac{1}{9^2}+\frac{1}{3}\left(\frac{1}{9}\right)^4+\frac{1}{5}\left(\frac{1}{9}\right)^6+\cdots\right]$$

$$= 9\left[\left(\frac{1}{3}\right)^2+\frac{1}{3}\left(\frac{1}{3}\right)^4+\frac{1}{5}\left(\frac{1}{3}\right)^6+\cdots\right]+9\left[\left(\frac{1}{9}\right)^2+\frac{1}{3}\left(\frac{1}{9}\right)^4+\frac{1}{5}\left(\frac{1}{9}\right)^6+\cdots\right]$$

$$= \cancel{9}^{3}\left(\frac{1}{\cancel{3}}\right)\left[\left(\frac{1}{3}\right)+\frac{\left(\frac{1}{3}\right)^3}{3}+\frac{\left(\frac{1}{3}\right)^5}{5}+\cdots\right]+9\left(\frac{1}{9}\right)\left[\left(\frac{1}{9}\right)+\frac{\left(\frac{1}{9}\right)^3}{3}+\frac{\left(\frac{1}{9}\right)^5}{5}+\cdots\right]$$

$$log\left(\frac{1+x}{1-x}\right) = 2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right] \Rightarrow \frac{1}{2}log\left(\frac{1+x}{1-x}\right) = x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots$$

$$= \left(\frac{3}{2}\right)\left[log\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)+\frac{1}{2}log\left(\frac{1+\frac{1}{9}}{1-\frac{1}{9}}\right)\right] = \left(\frac{3}{2}\right)\left[log\left(\frac{\frac{4}{3}}{\frac{2}{3}}\right)+\frac{1}{2}log\left(\frac{\frac{10}{9}}{\frac{8}{9}}\right)\right]$$

$$= \frac{3}{2}log(2)+\frac{1}{2}log\left(\frac{5}{4}\right) \quad = \frac{1}{2}\left[3log2+log\frac{5}{4}\right]$$

$$= \frac{1}{2}\left[log8+log\frac{5}{4}\right] = \frac{1}{2}\log\left(\cancel{8}^{2}\times\frac{5}{\cancel{4}}\right) \quad = \frac{1}{2}\log(10)$$

$$= log\sqrt{10}$$

$EXERCISE : 6.1$

Locus of a point

Definition 6.2 A point is an exact position or location on a plane surface

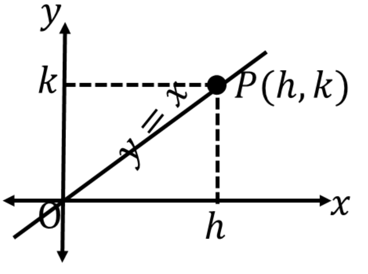
The path traced out by a moving point under certain conditions is called the locus of that point.

Alternatively, when a point moves in accordance with a geometrical law, its path is called locus. The plural of locus is loci.

Locus denotes $P(x, y)$

Example 6.1: Find the locus of a point which moves such that its distance from the x − axis is equal to the distance from the y − axis.

Let $P(h, k)$ *be the point on the locus*



Distance from x-axis = Distance from y-axis

$$h = k$$

$$x = h, y = k$$

Locus of (x, y) is $y = x$ is a straight line passes through the origin

Example 6.2: Find the path traced out by the point $\left(ct, \frac{c}{t}\right)$, *here $t \neq 0$ is the parameter and c is a constant*.

Let $P(h, k)$ *be the point on the locus*

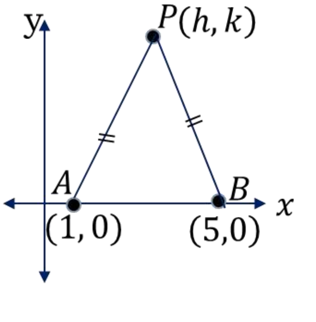
$$Given: h = ct \implies \frac{h}{c} = t \implies k = \frac{c}{t} \ldots (1)$$

$$subs\ t = \frac{h}{c}\ in\ eqn\ (1)\ k = \frac{c}{t} \implies k = \frac{c}{\frac{h}{c}} \implies k = c \times \frac{c}{h} \implies k = \frac{c^2}{h} \implies hk = c^2$$

$$Where\ x = h,\ y = k \implies The\ locus\ of\ the\ point\ P\ is xy = c^2$$

Example 6.3: Find the locus of a point P moves such that its distance from two fixed points A(1, 0) and B(5, 0), are always equal.

Let $P(h, k)$ *be the point on the locus*



$$Given: PA = PB \implies PA^2 = PB^2$$

Squaring on both sides

Distance between two points $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$\underset{x_1\ y_1}{A(1, 0)},\ \underset{x_2\ y_2}{P(h, k)}$$

$$PA = \sqrt{(h-1)^2 + (k-0)^2} \implies PA = \sqrt{(h-1)^2 + k^2}$$

$$PA^2 = (h-1)^2 + k^2$$

$$\underset{x_1\ y_1}{B(5, 0)},\ \underset{x_2\ y_2}{P(h, k)}$$

$PB = \sqrt{(h-5)^2 + (k-0)^2} \implies PB = \sqrt{(h-5)^2 + k^2} \implies$

$PA^2 = PB^2 \implies (h-1)^2 + \cancel{k^2} = (h-5)^2 + \cancel{k^2} \implies (h-1)^2 = (h-5)^2$

$h^2 - 2(1)h + 1^2 = h^2 - 2(5)h + 5^2$

$h^2 - 2h + 1 = h^2 - 10h + 25$

$\cancel{h^2} - 2h + 1 - \cancel{h^2} + 10h - 25 = 0 \implies 8h - 24 = 0$

$8h = 24$

$h = \frac{\cancel{24}^{3}}{\cancel{8}} \implies h = 3 \implies$ *Where* $x = h$

The locus of the point P is $x = 3$

Example 6.4: If θ is a parameter, find the equation of the locus of a moving point whose coordinates are $(a\sec\theta, b\tan\theta)$.

Let $P(x, y)$ be the point on the locus

$\boxed{1 + \tan^2\theta = \sec^2\theta}$

$\boxed{1 = \sec^2\theta - \tan^2\theta}$

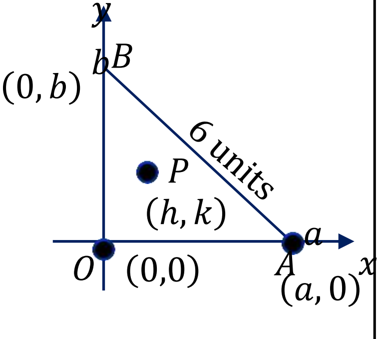
$x = a\sec\theta \implies \frac{x}{a} = \sec\theta \ldots (1)$

$y = b\tan\theta \implies \frac{y}{b} = \tan\theta \ldots (2)$

$\sec^2\theta - \tan^2\theta = 1$

$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \implies \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Example 6.5: A straight rod of the length 6 units, slides with its ends A and B always on the x and y axes respectively. If O is the origin, then find the locus of the centroid of ΔOAB.



Let the coordinates at O, A and B be

$O(0,0)$, $A(a, 0)$, $B(0, b)$

$\quad x_1 \; y_1 \quad x_2 \; y_2 \quad x_3 \; y_3$

Centroid of ΔOAB is (h, k)

$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = (h, k)$

$\left(\frac{0 + a + 0}{3}, \frac{0 + 0 + b}{3}\right) = (h, k) \implies \left(\frac{a}{3}, \frac{b}{3}\right) = (h, k)$

$\frac{a}{3} = h, \ \frac{b}{3} = k \implies a = 3h, \ b = 3k$

From right ΔOAB, $OA^2 + OB^2 = AB^2$

$a^2 + b^2 = 6^2 \implies (3h)^2 + (3k)^2 = 6^2 \implies 9h^2 + 9k^2 = 36$

$\div 9$

$h^2 + k^2 = 4$

Where $x = h, \ y = k$

The locus of the point P is $x^2 + y^2 = 4$

Example 6.6: If θ is a parameter, find the equation of the locus of a moving point whose coordinates are $(a(\theta - \sin\theta), a(1-\cos\theta))$.

Let $P(h,k)$ be the point on the locus

$$h = a(\theta - \sin\theta)$$

$$\frac{h}{a} = \theta - \sin\theta \implies \sin\theta = \theta - \frac{h}{a}$$

$$k = a(1-\cos\theta) \implies \frac{k}{a} = 1 - \cos\theta \implies \cos\theta = 1 - \frac{k}{a}$$

$$\cos\theta = \frac{a-k}{a} \implies \theta = \cos^{-1}\left(\frac{a-k}{a}\right)$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\theta - \frac{h}{a}\right)^2 + \left(\frac{a-k}{a}\right)^2 = 1 \implies \left(\theta - \frac{h}{a}\right)^2 + \frac{(a-k)^2}{a^2} = 1$$

$$\left(\theta - \frac{h}{a}\right)^2 = 1 - \frac{(a-k)^2}{a^2} \implies \left(\theta - \frac{h}{a}\right)^2 = \frac{a^2-(a-k)^2}{a^2}$$

$$\left(\theta - \frac{h}{a}\right)^2 = \frac{a^2-(a-k)^2}{a^2} \implies \left(\theta - \frac{h}{a}\right)^2 = \frac{a^2-(a^2-2ak+k^2)}{a^2}$$

$$\left(\theta - \frac{h}{a}\right)^2 = \frac{\cancel{a^2}-\cancel{a^2}+2ak-k^2}{a^2} \implies \left(\theta - \frac{h}{a}\right)^2 = \frac{2ak-k^2}{a^2}$$

$$\theta - \frac{h}{a} = \sqrt{\frac{2ak-k^2}{a^2}} \implies \theta - \frac{h}{a} = \frac{\sqrt{2ak-k^2}}{a} \implies \theta - \frac{\sqrt{2ak-k^2}}{a} = \frac{h}{a}$$

$$a\left(\theta - \frac{\sqrt{2ak-k^2}}{a}\right) = h \implies h = a\theta - a\frac{\sqrt{2ak-k^2}}{a}$$

$$h = a\theta - \sqrt{2ak-k^2}$$

$$\text{where } \theta = \cos^{-1}\left(\frac{a-k}{a}\right)$$

$$h = a\cos^{-1}\left(\frac{a-k}{a}\right) - \sqrt{2ak-k^2}$$

The locus of the point P is $x = a\cos^{-1}\left(\frac{a-y}{a}\right) - \sqrt{2ay-y^2}$

1. Find the locus of P, if for all values of α the co − ordinates of a moving point P is (i) $(9\cos\alpha, 9\sin\alpha)$

Let $P(h,k)$ be the point on the locus

$$h = 9\cos\alpha \implies \frac{h}{9} = \cos\alpha \implies \frac{h^2}{9^2} = \cos^2\alpha \ \ldots(1)$$

$$k = 9\sin\alpha \Rightarrow \frac{k}{9} = \sin\alpha \Rightarrow \frac{k^2}{9^2} = \sin^2\alpha \;\ldots(2)$$

Adding (1) *and* (2)

$$\frac{h^2}{9^2} + \frac{k^2}{9^2} = \cos^2\alpha + \sin^2\alpha \Rightarrow \frac{h^2 + k^2}{9^2} = 1 \Rightarrow h^2 + k^2 = 9^2$$

$$h^2 + k^2 = 81$$

$\therefore$ *The locus of* $P(x, y)$ *is* $x^2 + y^2 = 81$

(ii) $(\mathbf{9\cos\alpha\ , 6\sin\alpha})$

Let $P(x, y)$ *be the point on the locus*

$$h = 9\cos\alpha \Rightarrow \frac{h}{9} = \cos\alpha \Rightarrow \frac{h^2}{9^2} = \cos^2\alpha \;\ldots(1)$$

$$k = 6\sin\alpha \Rightarrow \frac{k}{6} = \sin\alpha \Rightarrow \frac{k^2}{6^2} = \sin^2\alpha \;\ldots(2)$$

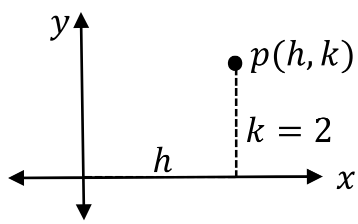
Adding (1) *and* (2)

$$\frac{h^2}{9^2} + \frac{k^2}{6^2} = \cos^2\alpha + \sin^2\alpha \Rightarrow \frac{h^2}{81} + \frac{k^2}{36} = 1$$

$\therefore$ *The locus of* $P(x, y)$ *is* $\frac{x^2}{81} + \frac{y^2}{36} = 1$

2. Find the locus of a point P that moves at a constant distant of

(i) *two units from the x − axis*

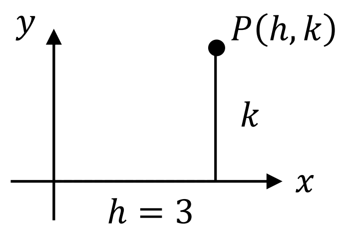


Let $p(h, k)$ *be the moving point*

$$k = 2$$
$$y = 2$$

Locus is $y - 2 = 0$

(ii) *three units from the y − axis.*



Let $P(h, k)$ be the given point

$$h = 3$$
$$x = 3$$

Locus is $x - 3 = 0$

3. If θ is a parameter, find the equation of the locus of a moving point, whose coordinates are $x = a\cos^3\theta, y = a\sin^3\theta$.

Let $P(x, y)$ *be the point on the locus*

$$x = a\cos^3\theta \Rightarrow \frac{x}{a} = \cos^3\theta \Rightarrow \left(\frac{x}{a}\right)^{\frac{1}{3}} = \cos\theta \Rightarrow \left(\frac{x}{a}\right)^{\frac{2}{3}} = \cos^2\theta \;\ldots(1)$$

$$y = a\sin^3\theta \Rightarrow \frac{y}{a} = \sin^3\theta \Rightarrow \left(\frac{y}{a}\right)^{\frac{1}{3}} = \sin\theta \Rightarrow \left(\frac{y}{a}\right)^{\frac{2}{3}} = \sin^2\theta \;\ldots(2)$$

$Adding\ (1) and\ (2)$

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{a}\right)^{\frac{2}{3}} = cos^2\theta + sin^2\theta \Rightarrow \frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{a^{\frac{2}{3}}} = 1$$

$$\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{a^{\frac{2}{3}}} = 1 \Rightarrow x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

4. *Find the value of k and b, if the points $P(-3,1)$ and $Q(2,b)$ lie on the locus of $x^2 - 5x + ky = 0$.*

$P(-3,1)$ *lies on the locus of* $x^2 - 5x + ky = 0$

$$\underset{x}{}\ \underset{y}{}$$

$$(-3)^2 - 5(-3) + k(1) = 0 \Rightarrow 9 + 15 + k = 0 \Rightarrow 24 + k = 0$$

$$k = -24$$

$Q(2,b)$ *lies on the locus of* $x^2 - 5x + ky = 0$

$$\underset{x}{}\ \underset{y}{}$$

$$(2)^2 - 5(2) - 24 \times (b) = 0 \Rightarrow 4 - 10 - 24b = 0$$

$$-6 = 24b \Rightarrow b = -\frac{\cancel{6}}{\cancel{24}_{4}}$$

$$b = -\frac{1}{4}$$

6. Find the equation of the locus of a point such that the sum of the squares of the distance from the points $(3,5), (1,-1)$ is equal to 20.

Let $p(h,k)$ *be the moving point*

$A(3,5)$ *and* $B(1,-1)$ *are given points*

Given: $PA^2 + PB^2 = 20$

$$\underset{x_1\ y_1}{A(3,5)}\ \underset{x_2\ y_2}{P(h,k)}$$

$$PA = \sqrt{(h-3)^2 + (k-5)^2} \Rightarrow \boxed{PA^2 = (h-3)^2 + (k-5)^2}$$

$$\underset{x_1\ y_1}{B(1,-1)},\ \underset{x_2\ y_2}{P(h,k)}$$

$$PB = \sqrt{(h-1)^2 + (k+1)^2} \Rightarrow \boxed{PB^2 = (h-1)^2 + (k+1)^2}$$

$$(h-3)^2 + (k-5)^2 + (h-1)^2 + (k+1)^2 = 20$$

$$h^2 - 2(3)h + 9 + k^2 - 2(5)k + 25 + h^2 - 2(1)h + 1$$
$$+ k^2 + 2(1)k + 1 = 20$$

$$h^2 - 6h + 9 + k^2 - 10k + 25 + h^2 - 2h + 1 + k^2 + 2k + 1 = 20$$

$$2h^2 + 2k^2 - 8h - 8k + 36 = 20$$

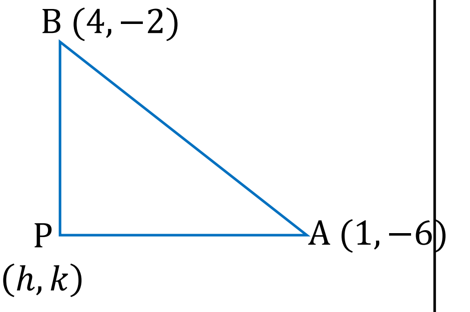
$$2h^2 + 2k^2 - 8h - 8k + 36 - 20 = 0 \Rightarrow 2h^2 + 2k^2 - 8h - 8k + 16 = 0$$

$$\div 2$$

$$h^2 + k^2 - 4h - 4k + 8 = 0$$

7. *Find the equation of the locus of the point P such that the line segment AB, joining the points A(1, −6) and B(4, −2), subtends a right angle at P*

Let p(h, k) be the point on the locus



$$PA \perp PB$$

$$\text{slope of } PA \times \text{Slope of } PB = -1$$

$$\underset{x_1\,,y_1}{P(h,k)}, \underset{x_2,y_2}{A(1,-6)}$$

$$\text{Slope of } PA = \frac{y_2-y_1}{x_2-x_1} = \frac{-6-k}{1-h} = \frac{-(k+6)}{1-h}$$

$$\underset{x_1\,,y_1}{P(h,k)}, \underset{x_2,y_2}{B(4,-2)}$$

$$\text{Slope of } PB = \frac{y_2-y_1}{x_2-x_1} = \frac{-2-k}{4-h} = \frac{-(k+2)}{4-h}$$

$$-\left(\frac{k+6}{1-h}\right) \times -\left(\frac{k+2}{4-\mathrm{h}}\right) = -1 \Rightarrow \left(\frac{k+6}{1-\mathrm{h}}\right) \times \left(\frac{k+2}{4-\mathrm{h}}\right) = -1$$

$$\frac{k^2+2k+6k+12}{h^2-4h-h+4} = -1 \Rightarrow \frac{k^2+8k+12}{h^2-5h+4} = -1$$

$$k^2+8k+12 = -1(h^2-5h+4) \Rightarrow k^2+8k+12 = -h^2+5h-4$$

$$h^2+k^2-5h+8k+12+4=0$$

$$h^2+k^2-5h+8k+16=0$$

$$\text{Locus of } (h,k) is\ \ x^2+y^2-5x+8y+16=0$$

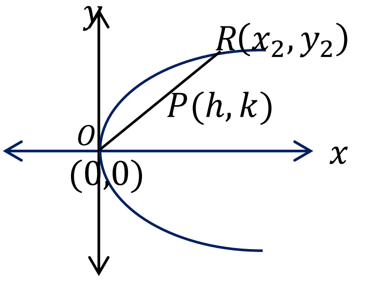
8. If O is origin and R is a variable point on $y^2 = 4x$, then find the equation of the locus of the mid − point of the line segment OR

Let $P(h,k)$ be the locus of the midpoint

$O\ (0,0)$ *and* $R(x_2, y_2)$

$$\text{Midpoint of } OR = P(h,k) \implies \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = (h,k)$$

$$\left(\frac{0+x_2}{2}, \frac{0+y_2}{2}\right) = (h,k) \implies \left(\frac{x_2}{2}, \frac{y_2}{2}\right) = (h,k)$$



$$\frac{x_2}{2} = h\ ,\ \ \frac{y_2}{2} = k \implies x_2 = 2h\ ,\ y_2 = 2k$$

$$\therefore R(2h, 2k)$$

$R(2h,2k)$ *lies on the parabola* $y^2 = 4x$

$$(2k)^2 = 4(2h) \implies \cancel{4}k^2 = \overset{2}{\cancel{8}}h \implies k^2 = 2h$$

$$h = x\ ,\ k = y$$

$$\text{Locus of } (x,y) \text{ is } y^2 = 2x$$

9. The coordinates of a moving point P are

$$\left(\frac{a}{2}(\operatorname{cosec}\theta + \sin\theta), \frac{b}{2}(\operatorname{cosec}\theta - \sin\theta)\right)$$

where θ is a variable parameter. show that the equation of the locus P is $b^2x^2 - a^2y^2 = a^2b^2$.

Let $x = \frac{a}{2}(\operatorname{cosec}\theta + \sin\theta) \implies \frac{2x}{a} = \operatorname{cosec}\theta + \sin\theta$

Squaring on both sides $\implies \left(\frac{2x}{a}\right)^2 = (\operatorname{cosec}\theta + \sin\theta)^2$ ......(1)

$(1) - (2)$

$$\left(\frac{2x}{a}\right)^2 - \left(\frac{2y}{b}\right)^2 = (\operatorname{cosec}\theta + \sin\theta)^2 - (\operatorname{cosec}\theta - \sin\theta)^2$$

Let $y = \frac{b}{2}(\operatorname{cosec}\theta - \sin\theta) \implies \frac{2y}{b} = \operatorname{cosec}\theta - \sin\theta$

Squaring on both sides $\implies \left(\frac{2y}{b}\right)^2 = (\operatorname{cosec}\theta - \sin\theta)^2$ ......(2)

$$\frac{4x^2}{a^2} - \frac{4y^2}{b^2} = (\operatorname{cosec}^2\theta + 2\operatorname{cosec}\theta\sin\theta + \sin^2\theta) - (\operatorname{cosec}^2\theta - 2\operatorname{cosec}\theta\sin\theta + \sin^2\theta)$$

$$4\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \cancel{\operatorname{cosec}^2\theta} + 2\operatorname{cosec}\theta\sin\theta + \cancel{\sin^2\theta} - \cancel{\operatorname{cosec}^2\theta} + 2\operatorname{cosec}\theta\sin\theta - \cancel{\sin^2\theta}$$

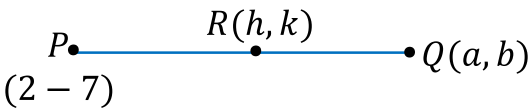
$$\cancel{4}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \cancel{4}\operatorname{cosec}\theta\sin\theta \implies \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{\sin\theta} \times \sin\theta$$

$$\frac{b^2x^2 - a^2y^2}{a^2b^2} = 1 \implies b^2x^2 - a^2y^2 = a^2b^2$$

10. If $P(2,-7)$ is a given point and Q is a point on $2x^2+9y^2=18$, then find the equations of the locus of the mid – point of PQ.

Let $P(2,-7)$ be the given points

let $Q(a,b)$ be a point on $2x^2+9y^2=18$



$$2a^2+9b^2=18 \ldots (1)$$

Let $R(h,k)$ be the mid – point of PQ

$$\underset{x_1\ \ y_1}{P(2,-7)} \text{ and } \underset{x_2\ \ y_2}{Q(a,b)}$$

$$Midpoint\ of\ PQ=(h,k) \Rightarrow \left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=(h,k)$$

$$\left(\frac{2+a}{2},\frac{-7+b}{2}\right)=(h,k) \Rightarrow h=\frac{2+a}{2},\ k=\frac{-7+b}{2}$$

$$2+a=2h,\ 2k=b-7$$

$$\boxed{a=2h-2} \quad \boxed{b=2k+7}$$

subs the values of a and b in eqn (1) $2a^2+9b^2=18$

where $a=2h-2,\ b=2k+7$

$$2(2h-2)^2+9(2k+7)^2=18$$

$$2(4h^2-8h+4)+9(4k^2+28k+49)=18$$

$$8h^2-16h+8+36k^2+252k+441=18$$

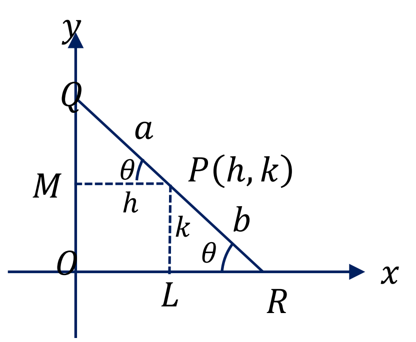
$$8h^2+36k^2-16h+252k+449-18=0$$

$$8h^2+36k^2-16h+252k+431=0$$

$\therefore$ *Locus of (h,k) is* $8x^2+36y^2-16x+252y+431=0$

11. If R is any point on the x – axis and Q is any point on the y – axis P is a variable point on RQ with $RP=b$, $PQ=a$. then find the equation of locus of P.

Let $P(h,k)$ be the point on PQ



$$\Delta\ PMQ,\ \cos\theta=\frac{h}{a} \Rightarrow \left(\frac{h}{a}\right)^2=\cos^2\theta \ \ldots (1)$$

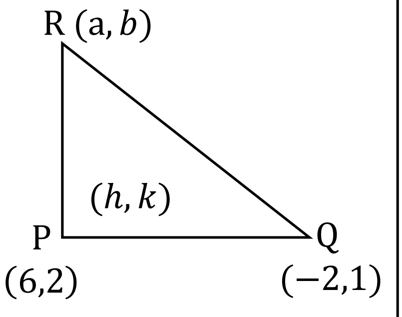
$$\Delta\ PLR,\ \sin\theta=\frac{k}{b} \Rightarrow \left(\frac{k}{b}\right)^2=\sin^2\theta \ \ldots (2)$$

Adding (1) and (2)

$$\left(\frac{h}{a}\right)^2+\left(\frac{k}{b}\right)^2=\cos^2\theta+\sin^2\theta \Rightarrow \frac{h^2}{a^2}+\frac{k^2}{b^2}=1$$

$\therefore$ *The required equation of the locus of P is* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

12. If the points $P(6,2)$ and $Q(-2,1)$ and are the vertices of a ΔPQR and R is the point on the locus $y = x^2 - 3x + 4$, then find the equation of the locus of centroid of ΔPQR



$$P(\underset{x_1}{6}, \underset{y_1}{2}),\ Q(\underset{x_2}{-2}, \underset{y_2}{1}),\ R(\underset{x_3}{a}, \underset{y_3}{b})$$

$$Centroid\ of\ \Delta\, PQR = (h,k)$$

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right) = (h,k)$$

$$\left(\frac{6-2+a}{3}, \frac{2+1+b}{3}\right) = (h,k)$$

$$\frac{4+a}{3} = h\ ,\ \frac{3+b}{3} = k \Longrightarrow\ 4+a = 3h,\ 3+b = 3k$$

$$a = 3h-4,\ \ b = 3k-3$$

The point $R(3h-4, 3k-3)$ *lies on* $y = x^2 - 3x + 4$

$$3k-3 = (3h-4)^2 - 3(3h-4) + 4$$

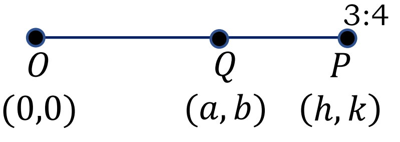
$$3k-3 = 9h^2 - 24h + 16 - 9h + 12 + 4$$

$$9h^2 - 33h + 32 = 3k - 3$$

$$9h^2 - 33h + 32 - 3k + 3 = 0\ \Longrightarrow\ 9h^2 - 33h - 3k + 35 = 0$$

Locus is $9x^2 - 33x - 3y + 35 = 0$

13. If Q is a point on the locus of $x^2 + y^2 + 4x - 3y + 7 = 0$, then find the equation of locus of p which divides line segment OQ externally in the ratio $3:4$, where O is origin



$$O(\underset{x_1\,y_1}{0,0})\ and\ Q(\underset{x_2\ y_2}{a, b})$$

The point $P(h,k)$ *divides* OQ *externally in the ratio* $\underset{l}{3} : \underset{m}{4}$

$$P(h,k) = \left[\frac{lx_2 - mx_1}{l-m}, \frac{ly_2 - my_1}{l-m}\right]$$

$$(h,k) = \left(\frac{3a-4(0)}{3-4}, \frac{3b-4(0)}{3-4}\right)\ \Longrightarrow\ (h,k) = \left(\frac{3a}{-1}, \frac{3b}{-1}\right)$$

$$h = -3a,\ k = -3b \Longrightarrow a = \frac{-h}{3},\ b = \frac{-k}{3}$$

The point Q is $\left(-\frac{h}{3}, -\frac{k}{3}\right)$

The point Q $\left(-\frac{h}{3}, -\frac{k}{3}\right)$ *lies on* $x^2 + y^2 + 4x - 3y + 7 = 0$

$$\left(\frac{-h}{3}\right)^2+\left(\frac{-k}{3}\right)^2+4\left(\frac{-h}{3}\right)-3\left(\frac{-k}{3}\right)+7=0$$

$$\boxed{\overset{3}{\cancel{9}}\times\frac{4h}{\cancel{3}}=12h}$$

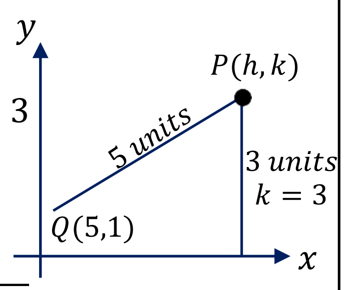
$$\frac{h^2}{9}+\frac{k^2}{9}-\frac{4h}{3}+k+7=0$$

$\times 9$

$$h^2+k^2-12h+9k+63=0$$

$$\text{Locus of } (h,k)\ is\ x^2+y^2-12x+9y+63=0$$

14. Find the point on the locus of point that are 3 units from x − axis and 5 units from the point $(5, 1)$.



Let P be (h,k) and $Q=(5,1)$

$P(h,k)$ is at a distance of 3 units from x − axis. $i.e\ k=3$

Given $PQ=5$

$$\underset{x_1\ \ y_1}{P(h,k)}\ and\ \underset{x_2\ \ y_2}{Q(5,1)}$$

$$PQ=\sqrt{(h-5)^2+(k-1)^2}\Longrightarrow 5=\sqrt{(h-5)^2+(k-1)^2}$$

squaring on both sides

$$25=(h-5)^2+(k-1)^2\Longrightarrow h^2-2(5)h+25+(3-1)^2=25$$

$$h^2-10h+25+(2)^2=25\Longrightarrow h^2-10h+25+4=25$$

$$h^2-10h+29=25\Longrightarrow h^2-10h+29-25=0$$

$$h^2-10h+4=0$$

$$a=1,b=-10,c=4$$

$$h=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\Longrightarrow h=\frac{-(-10)\pm\sqrt{(-10)^2-4\times1\times4}}{2\times1}$$

$$h=\frac{10\pm\sqrt{100-16}}{2}=\frac{10\pm\sqrt{84}}{2}=\frac{10\pm\sqrt{2\times2\times21}}{2}=\frac{10\pm2\sqrt{21}}{2}$$

$$h=\frac{\cancel{2}(5\pm\sqrt{21})}{\cancel{2}}\Longrightarrow\boxed{h=5\pm\sqrt{21}}$$

$\therefore$ *The points are* $\left(5+\sqrt{21},3\right),\left(5-\sqrt{21},3\right)$

15. The sum of the distance of a moving point $(4, 0)$ and $(-4, 0)$ is always 10 units. Find the equation of the locus of the moving point

Let $P(h,k)$ be the moving point

$$\boxed{Distance=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$$

$A(4,0), B(-4,0)$ are given points.

Given that $PA+PB=10$

$$\underset{x_1\ \ y_1}{A(4,0)}\ and\ \underset{x_2\ \ y_2}{P(h,k)}$$

$$\underset{x_1\ \ y_1}{B(-4,0)}\ and\ \underset{x_2\ \ y_2}{P(h,k)}$$

$$\sqrt{(h-4)^2+(k-0)^2}+\sqrt{(h+4)^2+(k-0)^2}=10$$

$$\sqrt{(h-4)^2+k^2}+\sqrt{(h+4)^2+k^2}=10$$

$$\sqrt{(h-4)^2+k^2} = 10 - \sqrt{(h+4)^2+k^2}$$

squaring on both sides

$$(h-4)^2+k^2 = \left(10-\sqrt{(h+4)^2+k^2}\right)^2$$

$$h^2-8h+16+k^2 = 100 + (h+4)^2 + k^2 - 20\sqrt{(h+4)^2+k^2}$$

$$20\sqrt{(h+4)^2+k^2} = 100 + \cancel{h^2} + 8h + \cancel{16} + \cancel{k^2} - \cancel{h^2} + 8h - \cancel{16} - \cancel{k^2}$$

$$20\sqrt{(h+4)^2+k^2} = 16h+100$$

$$\div 4$$

$$5\sqrt{(h+4)^2+k^2} = 4h+25$$

squaring on both sides

$$25\left[(h+4)^2+k^2\right] = (4h+25)^2$$

$$25\left[h^2+8h+16+k^2\right] = (4h)^2+2(4h)(25)+25^2$$

$$25h^2+2\cancel{0}0h+400+25k^2 = 16h^2+2\cancel{0}0h+625$$

$$25h^2+25k^2-16h^2 = 625-400$$

$$9h^2+25k^2 = 225 \qquad \Rightarrow \frac{h^2}{25}+\frac{k^2}{9} = 1$$

$$\div 225$$

$$h = x, k = y$$

$$\textit{Locus of } (x,y) \textit{ is } \frac{x^2}{25}+\frac{y^2}{9} = 1$$

STRAIGHT LINES

EXERCISE : 6.2

In general equation $ax + by + c = 0$ represents a line. (Straight line)

Given	***Slope***
Angle of inclination	$m = \tan\theta$
Two points (x_1, y_1) *and* (x_2, y_2)	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$
St. line $ax + by + c = 0$	$m = -\dfrac{a}{b}$

(i) **Slope and intercept form**: $y = mx + c$

(*ii*) ***Point – slope form***: $y - y_1 = m(x - x_1)$

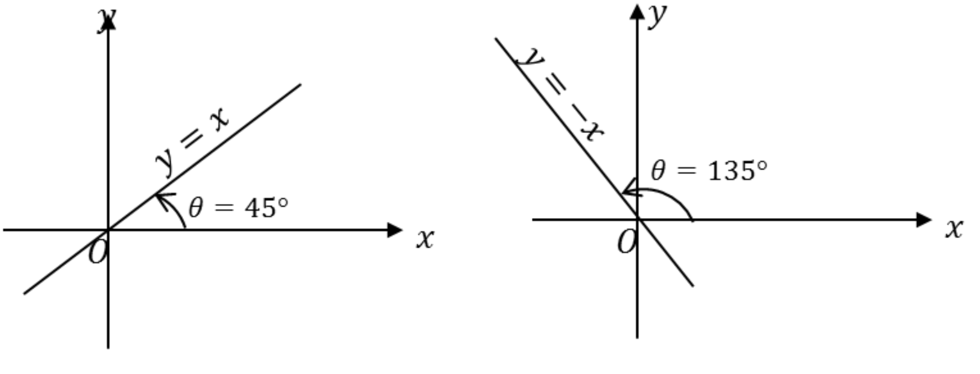
(*iii*) ***Two point form***: $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

(*iv*) ***Intercepts form***: $\dfrac{x}{a} + \dfrac{y}{b} = 1$

(*v*) *Normal form*: $x\cos\alpha + y\sin\alpha = p$

(*vi*) ***Parametric form***: $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$

Equation of Straight line

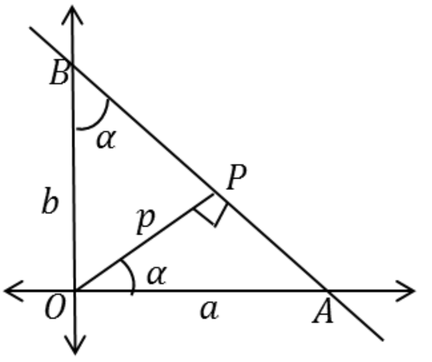


The line $y = x$ will bisect the angle between the coordinate axes

(v) ***Normal form: Let AB be the required line.***

Let OP be perpendicular from origin to this line.

Let $OP = p$ *and* $\angle POA = \alpha \Rightarrow \angle OBP = \alpha$



From ΔOAP, $\cos\alpha = \dfrac{OP}{OA} \Rightarrow OA = \dfrac{p}{\cos\alpha}$

$$a = \frac{p}{\cos\alpha}$$

From ΔOBP, $\sin\alpha = \dfrac{OP}{OB} \Rightarrow OB = \dfrac{p}{\sin\alpha}$

$$b = \frac{p}{\sin\alpha}$$

Equation of line is $\dfrac{x}{a} + \dfrac{y}{b} = 1 \Rightarrow \dfrac{x}{\frac{p}{\cos\alpha}} + \dfrac{y}{\frac{p}{\sin\alpha}} = 1$

$$\frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1 \quad \Rightarrow \quad \frac{x\cos\alpha + y\sin\alpha}{p} = 1$$

$x\cos\alpha + y\sin\alpha = p$ *This is called normal form.*

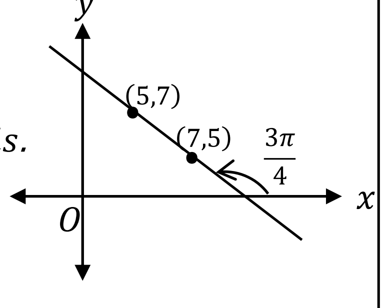
when p is the perpendicular from origin tothe angle made by this perpendicular with $x -$ *axis.*

Eg 6.7: Find the slope of the straight line passing through the points $(5,7)$ ***and*** $(7,5)$ ***also find the angle of inclination of the line with the*** $x -$ ***axis.***

Slope of a line joining two given points is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\underset{x_1,\,y_1}{(5,7)}, \underset{x_2,\,y_2}{(7,5)}$$

Slope of $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{5-7}{7-5} = \dfrac{-2}{2} = -1$



Let θ *be the angle of inclination of the with the axis.*

$$m = \tan\theta \Rightarrow \tan\theta = -1$$

$$\theta = 135^\circ \text{ or } \frac{3\pi}{4}$$

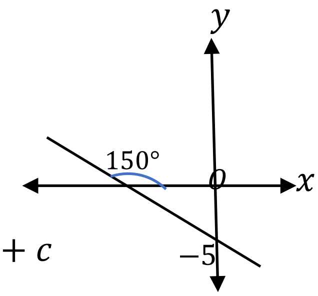
Slope and angle of the inclination of the line with $x -$ *axis are respectively*

$$m = -1 \text{ and } \theta = \frac{3\pi}{4}$$

Eg. 6.8: Find the equation of a straight line cutting an intercept of 5 from the negative direction of the y − axis and is incilined at an angle 150° to the x − axis.

Given that the negative y intercept is 5 i.e. $c = -5$ *and* $\theta = 150^\circ$

$$Slope\ m = \tan 150^\circ$$
$$= tan(180^\circ - 30^\circ) = -tan\,30^\circ$$
$$m = -\frac{1}{\sqrt{3}}$$



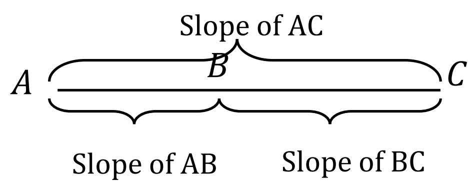
$Slope\ and\ intercept\ form\ of\ the\ equation\ is\ \ y = mx + c$

$$m = -\frac{1}{\sqrt{3}}\ and\ c = -5$$

$$y = -\frac{1}{\sqrt{3}}x - 5 \Rightarrow \sqrt{3}y = -x - 5\sqrt{3} \Rightarrow x + \sqrt{3y} + 5\sqrt{3} = 0$$
$$\times \sqrt{3}$$

$Eg. 6.9$: $\boldsymbol{Show\ the\ points\ \left(0, -\frac{3}{2}\right), (1, -1) and\ \left(2, -\frac{1}{2}\right) are\ collinear}$

$$A\left(\underset{x_1\ ,\ y_1}{0, -\frac{3}{2}}\right), B(\underset{x_2, y_2}{1, -1})$$



$$Slope\ of\ AB = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-1 + \frac{3}{2}}{1 - 0} = \frac{\frac{-2+3}{2}}{1} = \frac{1}{2}$$

$$\text{Slope of AB} = \frac{1}{2}$$

$$B(\underset{x_1\ ,\ y_1}{1, -1}), C\left(\underset{x_2, y_2}{2, -\frac{1}{2}}\right)$$

$$Slope\ of\ BC = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-\frac{1}{2} + 1}{2 - 1} = \frac{\frac{-1+2}{2}}{1} = \frac{1}{2}$$

$$\text{Slope of BC} = \frac{1}{2}$$

$Slope\ of\ AB\ = \ Slope\ of\ BC\ \ .Hence\ A, B, C\ are\ collinear$

$Eg.\ 6.10$: $\boldsymbol{The\ pamban\ sea\ bridge\ is\ a\ railway\ bridge\ of\ length\ about\ 2065m\ constructed\ on\ the\ PalkStrait, which\ connects\ the\ Island\ town\ of\ Rameswaram\ to\ Mandapam, the\ main\ land\ of\ India. The\ bridge\ is\ restricted\ to\ a\ uniform\ speed\ of\ only\ 12.5\ m/s. If\ a\ train\ of\ length\ 560m\ starts\ at\ the\ entry\ point\ of\ the\ bridge\ from\ mandapam,}$
$\boldsymbol{Then\ (i) find\ an\ equation\ of\ the\ motion\ the\ train. (ii) When\ does\ the\ engine\ touch\ island. (iii) When\ does\ the\ last\ coach\ cross\ the\ entry\ point\ of\ the\ bridge.\ \ (iv) What\ is\ the\ time\ taken\ by\ a\ train\ to\ cross\ the\ bridge.}$

Let the x − axis be the time in seconds the y − axis be distance in metres.

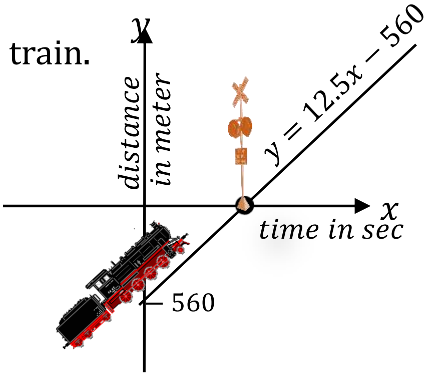
The length of the train $= 560m$

The uniform speed $12.5m/s$ is the slope of the train.

$$Total\ distance = y + 560$$

$$Speed = \frac{distance}{time} \Rightarrow 12.5 = \frac{y+560}{x}$$

$$12.5x = y + 560 \Rightarrow y = 12.5x - 560$$



(i) *The equation of the motion of the train,* $y = 12.5x - 560$

where $m = 12.5$ *and* $c = -560$,

(ii) *when does, the engine touch island* *At* $y = 2065$

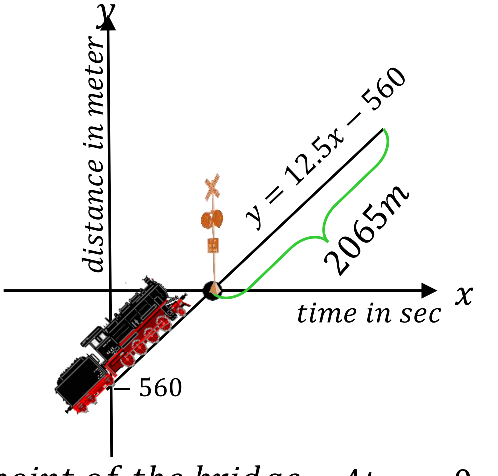
$$Speed = \frac{distance}{time} \Rightarrow 12.5 = \frac{y}{x}$$

$$y = 12.5x$$

$$Sub\ y = 2065$$

$$2065 = 12.5x \Rightarrow x = \frac{2065}{12.5}$$

$$x = \frac{20650}{125} \Rightarrow x = 165.2\ second$$



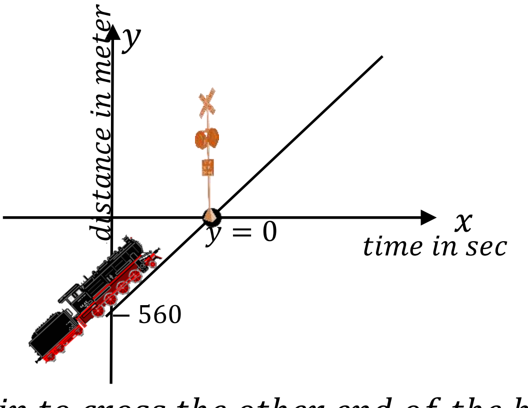
(iii) *when the last coach cross the entry point of the bridge,* *At* $y = 0$

$$sub\ y = 0\ in\ y = 12.5x - 560$$

$$0 = 12.5x - 560$$

$$560 = 12.5x \Rightarrow x = \frac{560}{12.5}$$

$$x = 44.8\ seconds.$$



(iv) *what is the time taken for the train to cross the other end of the bridge*

At $y = 2065$

$$sub\ y = 2065\ in\ y = 12.5x - 560$$

$$2065 = 12.5x - 560 \Rightarrow 2065 + 560 = 12.5x$$

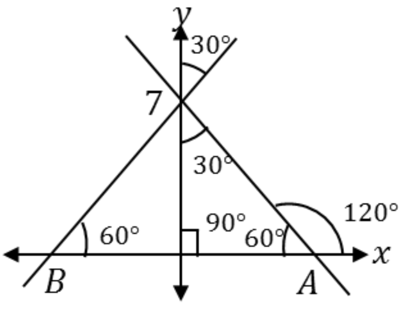
$$2625 = 12.5x \Rightarrow x = \frac{2625}{12.5}$$

$$x = 210\ seconds$$

Eg. 6.11: *Find the equation of the straight lines, making the y − intercept of 7 and angle* between the lines and the y−axis is 30°.

The two lines make the angles of 60° *and* 120° *with the axis*

Equation of lines are $y = m_1 x + c$ and $y = m_2 x + c$



$m_1 = \tan 60^\circ$

$m_1 = \sqrt{3}$

$m_2 = \tan 120^\circ$

$= \tan(180^\circ - 60^\circ)$

$= -\tan 60^\circ = -\sqrt{3}$

$y - intercept : c = 7$

$y = m_1 x + c, \qquad y = m_2 x + c$

$y = \sqrt{3}x + 7, \qquad y = -\sqrt{3}x + 7$

Eg. 6.12: *The seventh term of an arithmetic progression is 30 and tenth term is 21. (i) Find the first three terms of an A.P.*
(ii) which term of the A.P. is zero (if exists)
(iii) Find the relationship between slope of the straight line and common difference of A.P.

Let the x − axis be the number of the terms and the y − axis be the value of the terms.

$t_7 = 30$ *and* $t_{10} = 21$

Let $(x_1, y_1) = (7,30)$ and $(x_2, y_2) = (10,21)$

Equation of straight lines: $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

$$\frac{y-30}{21-30} = \frac{x-7}{10-7} \Rightarrow \frac{y-30}{\cancel{-9}\,-3} = \frac{x-7}{\cancel{3}} \Rightarrow y - 30 = -3(x-7)$$

$y - 30 = -3x + 21 \Rightarrow y = -3x + 21 + 30$

$y = -3x + 51$

when $x = 1$; $y = -3x + 51 = -3(1) + 51 = -3 + 51$

$y = 48$

when $x = 2$; $y = -3(2) + 51 = -6 + 51$

$y = 45$

when $x = 3$; $y = -3(3) + 51 = -9 + 51$

$y = 42$

The first three terms are 48, 45, 42.

(ii) which term of the A.P is zero when $y = 0$

$y = -3x + 51$

$$0 = -3x + 51 \Rightarrow 3x = 51 \Rightarrow x = \frac{51}{3}$$

$$x = 17$$

$17th\ \ term\ of\ A.P. is\ zero.$

(iii) *Find the relationship between slope of the straight line and common difference of A.P.*

$common\ difference\ of\ an\ A.P.\ d = t_2 - t_1$

$$d = 45 - 48 \Rightarrow d = -3$$

$Slope\ of\ the\ st.\ line - 3\ is\ equal\ to\ the\ common\ difference$

Eg. 6.13: The quantity demanded of a certain type of compact disk is 22,000 units when *a Unit price is Rs.8 the customer will not buy the disk at a unit price of Rs.30 or higher. On the Other side the manufacturer will not market any disk if the price is Rs.6 or lower. However, If the price Rs.14 the manufacture can supply 24000 units. Assume that the quantity Demanded and quantity supplied are linearly proportional to the price. Find (i) the demand equation (ii) supply equation (iii) the market equilibrium quantity and price. (iv) The quantity of demand and supply when the price is Rs.10.*

Let the x − axis represent the number of units in thousand and the y − axis represent the price in represent the price in rupees per unit.

(i) *For demand function,*

$let(x_1\ ,y_1) = (22\ ,8)\ and\ (x_2\ ,y_2) = (0\ ,30).$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \Rightarrow \frac{y - 8}{30 - 8} = \frac{x - 22}{0 - 22} \Rightarrow \frac{y - 8}{\cancel{22}} = \frac{x - 22}{\underset{-1}{-\cancel{22}}}$$

$$y - 8 = \frac{x - 22}{-1} \Rightarrow y - 8 = -x + 22 \Rightarrow y = -x + 22 + 8$$

$$y = -x + 30 \Rightarrow \therefore y_d = -x + 30$$

(ii) *For supply function*

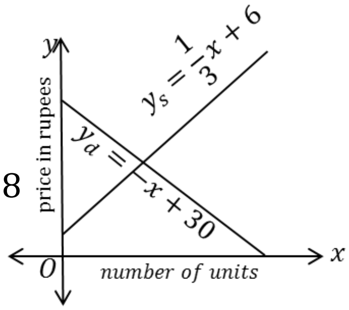
$Let(x_1\ ,y_1) = (0\ ,6)\ and\ (x_2\ ,y_2) = (24\ ,14)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \Rightarrow \frac{y - 6}{14 - 6} = \frac{x - 0}{24 - 0} \Rightarrow \frac{y - 6}{\cancel{8}} = \frac{x}{\cancel{24}_{3}}$$

$$y - 6 = \frac{1}{3}x \quad \Rightarrow y_s = \frac{1}{3}x + 6\ (Supply\ function)$$

(iii) At the market equilibrium the demand equals to supply,

$$y_d = y_s$$



$$-x + 30 = \frac{1}{3}x + 6 \implies -x - \frac{1}{3}x = -30 + 6$$

$$\frac{-3x - x}{3} = -24 \implies \frac{-\cancel{4}x}{3} = -\cancel{24}^{\,6} \implies \frac{x}{3} = 6 \implies x = 18$$

$$Sub\ x = 18\ in\ y_d = -x + 30$$

$$y_d = -18 + 30 \implies y = 12$$

Market equilibrium price is Rs. 12 *and no. of quantity is* $18{,}000$ *unit*

(iv) When the price $y = 10$, from the demand function $y_d = -x + 30$

$$10 = -x + 30 \implies x = 30 - 10$$

$$x = 20\ \ i.e., the\ demand\ is\ 20{,}000\ units.$$

From the supply function

$$y_s = \frac{1}{3}x + 6 \implies 10 = \frac{1}{3}x + 6 \implies 10 - 6 = \frac{1}{3}x$$

$$4 = \frac{1}{3}x \implies x = 12 \quad \therefore The\ supply\ is\ 12{,}000\ units.$$

Eg. 6.14: Find the equation of the straight line passing through $(-1, 1)$ and cutting off Equal intercepts, but opposite in signs with two coordinate axes.

Given : $x - intercepts = -(y - intercepts)$

$x - intercepts = a$ *and* $y - intercepts = -a$ *i.e* $a = a\,,\ b = -a$

Equation of the st. line : $\frac{x}{a} + \frac{y}{b} = 1$

$$\frac{x}{a} + \frac{y}{-a} = 1 \implies \frac{x}{a} - \frac{y}{a} = 1 \implies \frac{-1}{a} - \frac{1}{a} = 1 \implies \frac{-2}{a} = 1 \implies a = -2$$

It passes through $(-1,1)$

$$-\frac{x}{2} + \frac{y}{2} = 1 \implies \frac{-x + y}{2} = 1 \implies -x + y = 2$$

$$-x + y - 2 = 0 \implies x - y + 2 = 0$$

Eg. 6.15: A Straight line L with negative slope passes through the point $(9, 4)$ cuts the Positive coordinate axes at the points P and Q. As L varies, find the minimum value of $|OP| + |OQ|$, where O is the origin.

Let $m = -k$ *be the slope of the line L. since it passes through the point* $(9,4)$

The equation of the line L is $y - y_1 = m(x - x_1)$

$$y - 4 = -k(x - 9) \implies y - 4 = -kx + 9k$$

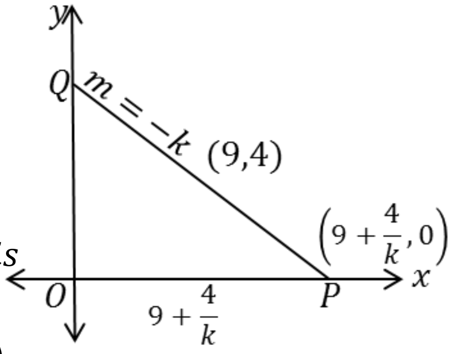
$$y = -kx + 9k + 4$$

The line intersect positive co − ordinates axis at P i.e., y = 0

$$0 = -kx + 9k + 4 \Longrightarrow kx = 9k + 4$$

$$x = \frac{9k}{k} + \frac{4}{k} \quad \Longrightarrow \quad x = 9 + \frac{4}{k}$$

$$\therefore P\left(9 + \frac{4}{k}, 0\right)$$



The line intersect positive co − ordinates axis at Q i.e., x = 0

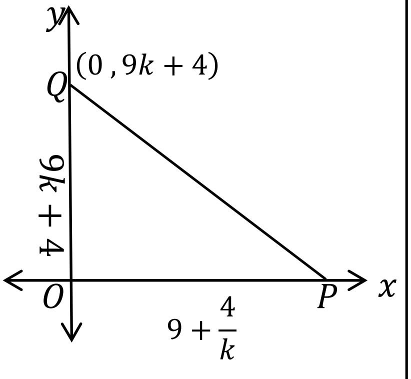
$$y = -kx + 9k + 4$$

$$y = -k(0) + 9k + 4$$

$$y = 9k + 4$$

$$\therefore Q(0, 9k + 4)$$

$$\boxed{\begin{array}{c} A.M \geq G.M \\ \dfrac{a+b}{2} \geq \sqrt{ab} \\ a + b \geq 2\sqrt{ab} \end{array}}$$



$$|OP| + |OQ| = \left|9 + \frac{4}{k}\right| + |4 + 9k|$$

$$= 9 + \frac{4}{k} + 4 + 9k = 13 + \left(\frac{4}{k} + 9k\right)$$

$$\geq 13 + 2\sqrt{\frac{4}{\cancel{k}} \times 9\cancel{k}}$$

$$|OP| + |OQ| \geq 13 + 2\sqrt{36}$$

Eg. 6.16: The length of the perpendicular drawn from the origin to a line is 12 and angle 150° with positive direction of the x-axis. Find the equation of the line.

Here, $p = 12$ and $\alpha = 150^\circ$, so the equation of the required line is of the form

$$x \cos\alpha + y \sin\alpha = p$$

$$x \cos 150^\circ + y \sin 150^\circ = 12$$

$$x\left(-\frac{\sqrt{3}}{2}\right) + y\left(\frac{1}{2}\right) = 12$$

$$-\frac{\sqrt{3}x}{2} + \frac{y}{2} = 12 \Longrightarrow \frac{-\sqrt{3}x + y}{2} = 12$$

$$-\sqrt{3}x + y = 24 \Longrightarrow \sqrt{3}x - y = -24$$

$$\times(-)$$

$$\sqrt{3}x - y + 24 = 0$$

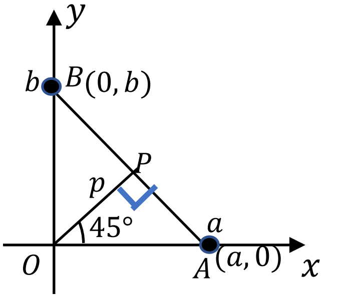
$$\boxed{\begin{aligned} \cos 150^\circ &= \cos(180^\circ - 30^\circ) \\ &= -\cos 30^\circ = -\frac{\sqrt{3}}{2} \\ \sin 150^\circ &= \sin(180^\circ - 30^\circ) \\ &= \sin 30^\circ = \frac{1}{2} \end{aligned}}$$

Eg. 6. 17: Area of the triangle formed by a line with the coordinate axes, is 36 square Units. Find the equation of the line if the perpendicular drawn from the origin to the line make an angles of 45° with positive the x-axis.

Let p be the length of the perpendicular drawn from the origin to the required line.

The perpendicular makes $\alpha = 45^\circ$ *with the* $x-axis$.

The equation of normal form, $x \cos\alpha + y \sin\alpha = p$



$$x \cos 45^\circ + y \sin 45^\circ = p$$

$$\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = p \Rightarrow \frac{x+y}{\sqrt{2}} = p \Rightarrow x + y = \sqrt{2}\,p$$

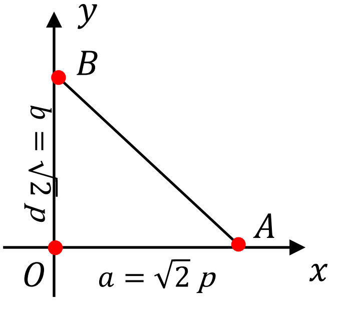
$A(a, 0)$ *lies on the line* $x + y = \sqrt{2}\,p$

$$a + 0 = \sqrt{2}\,p \Rightarrow a = \sqrt{2}\,p$$

$B(0, b)$ *lies on the line* $x + y = \sqrt{2}\,p$

$$0 + b = \sqrt{2}\,p \Rightarrow b = \sqrt{2}\,p$$

Area of the $\Delta OAB = 36$ *sq.units*



$$\frac{1}{2} \times Base \times Height = 36$$

$$\frac{1}{2} \times \sqrt{2}\,p \times \sqrt{2}\,p = 36 \Rightarrow \frac{1}{2} \times 2p^2 = 36$$

$$p^2 = 36$$

$$p = \sqrt{36} \Rightarrow p = \pm 6 \quad (\therefore p \text{ is positive})$$

$$sub\ p = 6\ in\ x + y = \sqrt{2}\,p$$

The equation of required line is $x + y = 6\sqrt{2}$

Eg. 6. 18: Find the equation of the lines make an angle 60° with positive x − axis and at A distance $5\sqrt{2}$ units measured from the point (4,7),along the line $x - y + 3 = 0$.

The angle of inclination of the line $x - y + 3 = 0$ *is* 45° *and a point on the line*

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

Point $(4, 7)$, $\theta = 45^\circ$ *and* $r = \pm 5\sqrt{2}$

$$\frac{x - x_1}{\cos 45^\circ} = \frac{y - y_1}{\sin 45^\circ} = \pm 5\sqrt{2} \Rightarrow \frac{x-4}{\frac{1}{\sqrt{2}}} = \frac{y-7}{\frac{1}{\sqrt{2}}} = \pm 5\sqrt{2}$$

$$\sqrt{2}(x - 4) = \sqrt{2}(y - 7) = \pm 5\sqrt{2}$$

$$\div \sqrt{2}$$

$$x-4= \ y-7=\pm 5$$
$$x-4 \ =\pm 5 \ , \ y-7=\pm 5$$

$$x-4=5 \ , \ \ y-7=5$$
$$x=5+4 \ , \ \ y=5+7$$
$$x=9 \ , \qquad y=12$$
$$\therefore (9\,,12)$$

$$x-4=-5 \ , \ \ y-7=-5$$
$$x=-5+4 \ , \ \ y=-5+7$$
$$x=-1 \ , \qquad y=2$$
$$\therefore (-1\,,2)$$

$$\theta=60^{\circ} \ \Longrightarrow m=\tan\theta$$
$$m=\tan 60^{\circ} \ \Longrightarrow m=\sqrt{3}$$

Equation of st.line having slope $m=\sqrt{3}$ *and point* $(\underset{x_1}{9}\,,\underset{y_1}{12})$ *is*

$$y-y_1=m(x-x_1)$$
$$y-12=\sqrt{3}(x-9)\Longrightarrow y-12= \ \sqrt{3}x-9\sqrt{3}$$
$$\sqrt{3}x-y+12-9\sqrt{3}=0$$

Equation of st.line having slope $m=\sqrt{3}$ *and point* $(\underset{x_1}{-1}\,,\underset{y_1}{2})$ *is*

$$y-y_1=m(x-x_1)$$
$$y-2=\sqrt{3}(x+1)\Longrightarrow y-2= \ \sqrt{3}x+\sqrt{3}$$
$$\sqrt{3}x-y+2+\sqrt{3}=0$$

Eg.6.19: Express the equation $\boldsymbol{\sqrt{3}\,x-y+4=0}$ ***in the following equivalent form: (i) Slope and intercept form, (ii) Intercept form, (iii) Normal form.***

Given : $\sqrt{3}x-y+4=0$

$$y=\sqrt{3}x+4$$

Compare with $y=mx+c$

$$slope=\sqrt{3} \ \ and \ y-intercept=4$$

(*ii*) *Intercept form* $\sqrt{3}x-y+4=0 \Longrightarrow \sqrt{3}x-y=-4$

$$\div -4$$

$$\frac{\sqrt{3}x}{-4}-\frac{y}{-4}=\frac{-4}{-4} \quad \Longrightarrow \quad \frac{-\sqrt{3}x}{4}+\frac{y}{4}=1$$
$$\frac{x}{\left(\frac{-4}{\sqrt{3}}\right)}+\frac{y}{4}=1$$

Comparing: $\dfrac{x}{a}+\dfrac{y}{b}=1$

$$a=-\frac{4}{\sqrt{3}}\ , \ b=4$$

(*iii*) *Normal form*: $\sqrt{3}x-y=-4$

$$-\sqrt{3}x+y=4$$

Here $A=-\sqrt{3}\ , \ B=1$

$$\sqrt{A^2+B^2} = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$-\sqrt{3}x + y = 4$$

$$\div 2$$

$$\frac{-\sqrt{3}}{2}x + \frac{y}{2} = 2$$

Comparing: $x\ \cos\alpha + y\ \sin\alpha = p$

$$\cos\alpha = -\frac{\sqrt{3}}{2}\ , \sin\alpha = \frac{1}{2}\ ,\ \ p = 2$$

$$\alpha = 150^{\circ} \implies \alpha = \frac{5\pi}{6}$$

$$x\ cos\frac{5\pi}{6} + y\ sin\frac{5\pi}{6} = 2$$

Eg.6.20: ***Rewrite*** $\boldsymbol{\sqrt{3}x + y + 4 = 0}$ ***in to normal form.***

$$\sqrt{3}x + y + 4\ = 0 \implies \sqrt{3}x + y\ = -4$$

$$-\sqrt{3}x - y\ = 4$$

$$A = -\sqrt{3}\quad and \quad B = -1$$

$$\sin\left(270^{\circ} - 60^{\circ}\right) = -\cos 60^{\circ}$$

$$\sin 210^{\circ} = -\frac{1}{2}$$

$$\sqrt{A^2+B^2} = \sqrt{(-\sqrt{3})^2 + (-1)^2}$$

$$= \sqrt{3+1}\ = \sqrt{4}\ = 2$$

$$\frac{-\sqrt{3}x}{2} - \frac{y}{2}\ = \frac{4}{2}$$

Compare: $x\ cos\alpha +\ y\ sin\alpha = p$

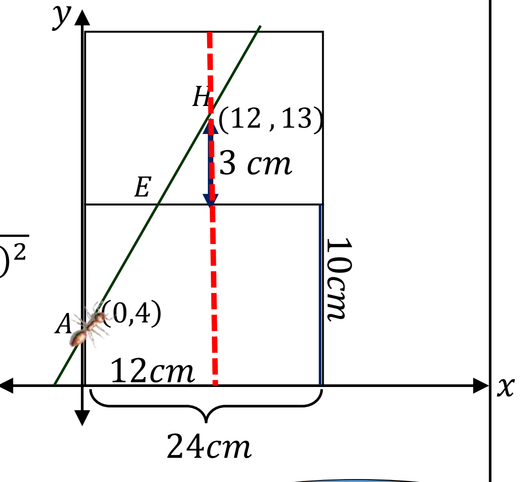
$$cos\alpha = \frac{-\sqrt{3}}{2}, sin\alpha = -\frac{1}{2}\ and\ p = 2$$

$$\alpha = 210^{\circ} = \frac{7\pi}{6}\quad \alpha\ lies\ in\ the\ III^{rd}\ quadrant$$

Normal form: $x\ cos\alpha + y\ sin\alpha = p$

$$x\ cos\frac{7\pi}{6}\ + y\sin\frac{7\pi}{6} = 2$$

Eg. 6.21: Consider a hollow cylindrical vessel, with circumference 24cm and height 10cm. An ant is located on the outside of vessel 4cm from the bottom. There is a drop of honey at The diagrammatically opposite inside of the vessel ,3cm from the top. (i) what is the shortest Distance the ant would need to craw to get the honey drop?(ii) equation of the path traced Out by the ant.(iii) where the ant enter in to the cylinder? Here is a picture that illustrates The position of the ant and the honey.



$(\boldsymbol{i})$ ***The shortest distance between***

$$A(\underset{x_1}{0},\underset{y_1}{4}) \text{ and } H(\underset{x_2}{12},\underset{y_2}{13})$$

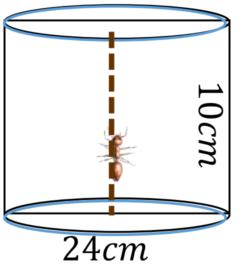
$$AH = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(12-0)^2 + (13-4)^2} = \sqrt{(12)^2 + (9)^2}$$

$$= \sqrt{144+81} = \sqrt{225}$$

$$AH = 15$$

$(\boldsymbol{ii})$ ***The equation of the path***



$$A(\underset{x_1}{0},\underset{y_1}{4}) \text{ and } H(\underset{x_2}{12},\underset{y_2}{13})$$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-4}{13-4} = \frac{x-0}{12-0}$$

$$\frac{y-4}{\cancel{9}\,3} = \frac{x}{\cancel{12}\,4} \Rightarrow \frac{y-4}{3} = \frac{x}{4} \Rightarrow 4(y-4) = 3x$$

$$4y - 16 = 3x \Rightarrow 4y = 3x + 16 \Rightarrow y = \frac{3}{4}x + \frac{16}{4}$$

$$\div 4$$

Equation of the path traced Out by the ant: $y = \frac{3}{4}x + 4$

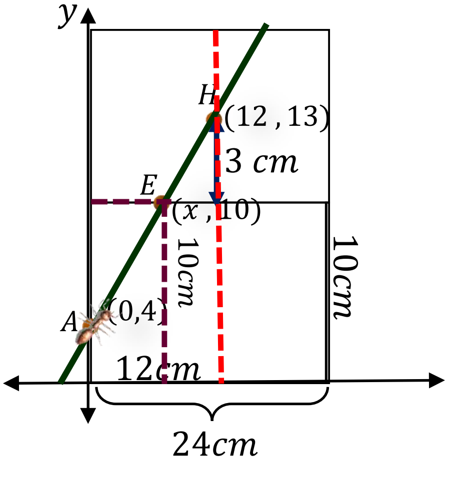
(iii) *At the entry* $E, y = 10$

$$y = \frac{3}{4}x + 4 \Rightarrow 10 = \frac{3}{4}x + 4$$

$$10 - 4 = \frac{3}{4}x \Rightarrow 6 = \frac{3}{4}x$$

$$x = \overset{2}{\cancel{6}} \times \frac{4}{\cancel{3}} \Rightarrow x = 8$$

$$\therefore E = (8, 10)$$



1. Find the equation of the line passing through the point $(1,1)$
(i) *With $y-$ intercept (-4)* (ii) *with slope 3* (iii) *and $(-2,3)$*
(iv) *and the perpendicular from the origin makes an angle* $60°$ with $x-$ axis.

(i) *Point is $(1,1)$ and y intercept is -4.*

$Equation$: $y = mx + c$

$y - intercept$: $c = -4$

$y = mx - 4$

It passes through the point $(1,1)$

$1 = m(1) - 4 \Rightarrow 1 = m - 4$

$m = 1 + 4 \Rightarrow m = 5$

$y = 5x - 4$

(ii) *The point $(1,1)$ and slope $m = 3$*

The equation of the line: $y - y_1 = m(x - x_1)$

$y - 1 = 3(x - 1) \Rightarrow y - 1 = 3x - 3 \Rightarrow y = 3x - 3 + 1$

$$\boxed{y = 3x - 2}$$

(iii) *The points are $(\underset{x_1}{1}, \underset{y_1}{1})$ and $(\underset{x_2}{-2}, \underset{y_2}{3})$*

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \Rightarrow \frac{y-1}{3-1} = \frac{x-1}{-2-1}$$

$$\frac{y-1}{2} = \frac{x-1}{-3} \Rightarrow -3y + 3 = 2x - 2$$

$$2x - 2 + 3y - 3 = 0 \Rightarrow 2x + 3y - 5 = 0$$

(iv) *The normal from is $x \cos\alpha + y \sin\alpha = p$*

Here $\alpha = 60^{\circ}$

$\therefore x \cos 60^{\circ} + y \sin 60^{\circ} = p$

$$x\frac{1}{2} + y\frac{\sqrt{3}}{2} = p$$

This passes through $(1,1)$

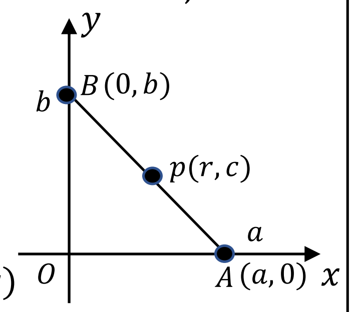
$$\frac{1}{2} + \frac{\sqrt{3}}{2} = p \Rightarrow p = \frac{\sqrt{3}+1}{2}$$

$$p = \frac{\sqrt{3}+1}{2} \; in \; \frac{x}{2} + \frac{\sqrt{3}y}{2} = p$$

$$\frac{x}{2} + \frac{y\sqrt{3}}{2} = \frac{\sqrt{3}+1}{2} \Rightarrow \frac{x + y\sqrt{3}}{\cancel{2}} = \frac{\sqrt{3}+1}{\cancel{2}}$$

$$x + y\sqrt{3} = \sqrt{3} + 1$$

2. If $P(r,c)$ is mid − point of a line segment between the axes, then show that $\frac{x}{r}+\frac{y}{c}=2$.



Let $A(a,0)$ and $B(0,b)$ be points on x and $y-$axis.

Midpoint of $A\,(a,0)$ and $B(0,b) = p(r,c)$

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = (r,c) \Rightarrow \left(\frac{a+0}{2}, \frac{0+b}{2}\right) = (r,c)$$

$$\left(\frac{a}{2}, \frac{b}{2}\right) = (r,c) \Rightarrow \frac{a}{2} = r,\ \frac{b}{2} = c$$

$$a = 2r,\ b = 2c$$

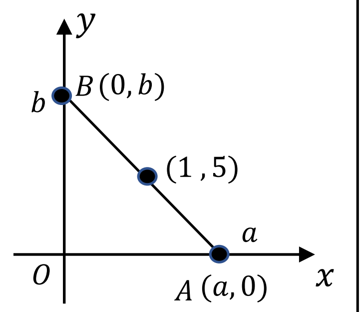
Equation of the line: $\frac{x}{a}+\frac{y}{b} = 1$

$$\frac{x}{2r}+\frac{y}{2c} = 1 \Rightarrow \frac{1}{2}\left(\frac{x}{r}+\frac{y}{c}\right) = 1 \Rightarrow \frac{x}{r}+\frac{y}{c} = 2$$

3. Find the equation of the line passing through the point $(1,5)$ and also divides the co-ordinate axes in the ratio 3:10.

Let the line divide the coordinate axes in the ratio $3:10$

$$a:b = 3:10 \Rightarrow a = 3k,\ b = 10k$$



Equation: $\frac{x}{a}+\frac{y}{b} = 1 \Rightarrow \frac{x}{3k}+\frac{y}{10k} = 1$

It passes through $(1,5)$

$$\frac{1}{3k}+\frac{5}{10k} = 1 \Rightarrow \frac{1}{3k}+\frac{1}{2k} = 1 \Rightarrow \frac{2+3}{6k} = 1$$

$$\frac{5}{6k} = 1 \Rightarrow 6k = 5 \Rightarrow k = \frac{5}{6}$$

Equation: $\frac{x}{3k}+\frac{y}{10k} = 1$ *where* $k = \frac{5}{6}$

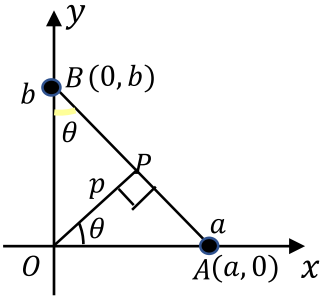
$$\frac{x}{\cancel{3}\left(\frac{5}{\cancel{6}_{2}}\right)} + \frac{y}{\cancel{10}^{5}\left(\frac{5}{\cancel{6}_{3}}\right)} = 1 \Rightarrow \frac{x}{\frac{5}{2}} + \frac{y}{\frac{25}{3}} = 1$$

$$\frac{2x}{5}+\frac{3y}{25} = 1 \Rightarrow \frac{10x+3y}{25} = 1$$

$$\boxed{10x+3y = 25}$$

4. If P is length of perpendicular from origin to the line whose intercepts on the *axes are a and b, then show that* $\frac{1}{p^2} = \frac{1}{a^2}+\frac{1}{b^2}$

In $\Delta\,OPA$, $\cos\theta = \frac{OP}{OA} \Rightarrow \cos\theta = \frac{p}{a}$



$$In\ \Delta\ OPB\ ,\ sin\theta = \frac{OP}{OB} \Rightarrow sin\theta = \frac{p}{b}$$

$$sin^2\theta + cos^2\theta = 1$$

$$\left(\frac{p}{b}\right)^2 + \left(\frac{p}{a}\right)^2 = 1 \Rightarrow \frac{p^2}{b^2} + \frac{p^2}{a^2} = 1$$

$$p^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = 1 \Rightarrow \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$$

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

5. The normal boiling point of water is 100°C or 212°F and the freezing point Of water is 0°C or 32°F. (i) find the linear relationship between C and F find (ii) the value of C for 98.6°F(iii) The value of F for 38 C.

$$Take\ the\ point\ (\underset{C_1\ ,\ F_1}{0\ ,32})\ and\ (\underset{C_2\ \ ,\ F_2}{100\ ,212})$$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{F-F_1}{F_2-F_1} = \frac{C-C_1}{C_2-C_1} \Rightarrow \frac{F-32}{212-32} = \frac{C-0}{100-0}$$

$$\frac{F-32}{180} = \frac{C}{100} \Rightarrow C = \frac{10\cancel{0}}{18\cancel{0}}(F-32) \Rightarrow \boxed{C = \frac{5}{9}(F-32)}$$

$$(or)$$

$$\frac{9}{5}C = F - 32 \Rightarrow \boxed{\mathrm{F} = \frac{9}{5}\mathrm{C} + 32}$$

$(i)\ when\ \ F = 98.6$

$$C = \frac{5}{9}(98.6-32) \Rightarrow C = \frac{5}{9}(66.6) \Rightarrow C = \frac{333}{9} \Rightarrow \boxed{C = 37^{\circ}}$$

$(ii)\ when\ C = 38^{\circ}C$

$$F = \frac{9}{5}C + 32 \Rightarrow F = \frac{9}{\cancel{5}} \times \overset{7.6}{\cancel{38}} + 32$$

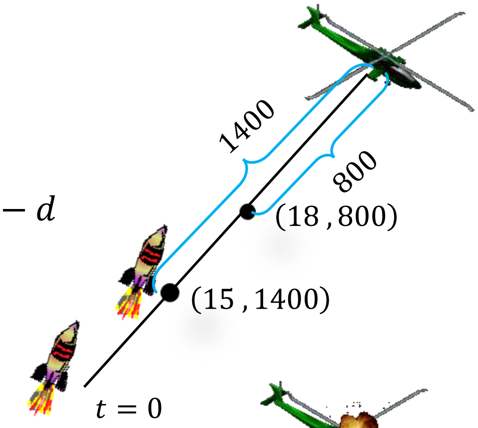
$$F = 68.4 + 32 \Rightarrow \boxed{F = 100.4^{\circ}F}$$

6. An object was launched was from a place P in constant speed to hit a target .At the 15th second it was 1400m away from the target and at the 18th second 800m away.Find (i) the distance between the place and the target (ii) the distance covered by it in 15 seconds (iii) time taken to hit the target .

$$Let\ the\ points\ be\ (\underset{t_1\ ,d_1}{15\ ,1400})\ and\ (\underset{t_2\ ,d_2}{18\ ,800})$$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{d-d_1}{d_2-d_1} = \frac{t-t_1}{t_2-t_1} \Rightarrow \frac{d-1400}{800-1400} = \frac{t-15}{18-15}$$

$$\frac{t-15}{\cancel{3}} = \frac{d-1400}{\underset{(-200)}{\cancel{-600}}} \Rightarrow t-15 = \frac{d-1400}{-200} \Rightarrow t-15 = \frac{1400-d}{200}$$



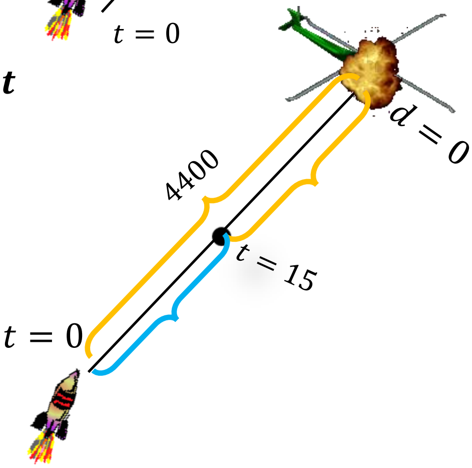
$$\boxed{t = \frac{1400-d}{200} + 15} \quad ......(1)$$

$$t-15 = \frac{1400-d}{200} \Rightarrow 200t - 3000 = 1400 - d$$

$$200t - 3000 - 1400 = -d$$

$$200t - 4400 = -d$$

$$d = 4400 - 200t \quad ...(2)$$



(i) Distance between the place and target

$t = 0$ in (2) $d = 4400 - 200t$

$$d = 4400 - 200(0)$$

$$\boxed{d = 4400}$$

(ii) Distance covered by it in 15 seconds

$$d = Total\ distances - 1400$$

$$d = 4400 - 1400$$

$$d = 3000$$

(iii) Time taken to hit the target

$$d = 0\ in\ (1) \quad t = \frac{1400-d}{200} + 15$$

$$t = \frac{1400-0}{200} + 15 \Rightarrow \quad t = \frac{1400}{200} + 15$$

$$t = 7 + 15 \Rightarrow \boxed{t = 22 seconds}$$

7. Population of a city in the years 2005 and 2010 are 1, 35000 and 1, 45, 000 respectively. Find the approximate population in the year 2015. (assuming that the growth of Population is constant)

The points are $(\underset{x_1}{2005}, \underset{y_1}{135000})$ and $(\underset{x_2}{2010}, \underset{y_2}{145000})$

$$\boxed{\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}}$$

$$\frac{y-135000}{145000-135000} = \frac{x-2005}{2010-2005} \Rightarrow \frac{y-135000}{\underset{2000}{\cancel{10,000}}} = \frac{x-2005}{\cancel{5}}$$

$$\frac{y-135000}{2000} = x - 2005$$

$when \ \ x = 2015$

$$\frac{y-135000}{2000} = 2015 - 2005 \Rightarrow \ \frac{y-135000}{2000} = 10$$

$$y - 135000 = 20000 \Rightarrow \ y = 20000 + 135000$$

$$\boxed{y = 1{,}55{,}000}$$

8. Find the equation of the line if the perpendicular drawn from the origin makes as angle 30° with x-axis and its length is 12.

$Given \ P = 12, \alpha = 30^\circ$

$The \ equation \ is \ (normal \ form) \ \ x \ \cos\alpha + y \ \sin\alpha = p$

$$x \ \cos 30^\circ + y \ \sin 30^\circ = 12$$

$$x\frac{\sqrt{3}}{2} + \frac{y}{2} = 12 \quad \Rightarrow \ \sqrt{3}\,x + y = 24$$

$\times 2$

9. Find the equation of the straight line passing through the point (8, 3) and having intercepts whose sum is 1.

Let x and y – intercepts of the straight line be a and b

$$a + b = 1$$

$$b = 1 - a$$

The equation of the straight line in intercepts form is $\frac{x}{a} + \frac{y}{b} = 1$

$$\frac{x}{a} + \frac{y}{1-a} = 1 \ldots (1)$$

Since this line passes through $(\underset{x}{8}, \underset{y}{3})$

$$\frac{8}{a} + \frac{3}{1-a} = 1 \Rightarrow \ \frac{8(1-a)+3a}{a(1-a)} = 1$$

$$8 - 8a + 3a = a(1-a) \ \Rightarrow \ 8 - 5a = a - a^2$$

$$8 - 5a - a + a^2 = 0 \ \Rightarrow \ a^2 - 6a + 8 = 0$$

$$(a-4)(a-2) = 0 \Rightarrow \ a - 4 = 0 \ and \ a - 2 = 0$$

$$a = 4 \ and \ a = 2$$

$+ \ -6 \quad \times 8$

$-4 \quad -2$

sub $a = 4$ in (1) $\frac{x}{a} + \frac{y}{1-a} = 1$

$$\frac{x}{4} + \frac{y}{1-4} = 1 \Rightarrow \frac{x}{4} + \frac{y}{-3} = 1$$

$$\frac{3x-4y}{12} = 1 \Rightarrow 3x - 4y = 12$$

$$3x - 4y - 12 = 0$$

sub $a = 2$ in (1) $\frac{x}{2} + \frac{y}{1-2} = 1$

$$\frac{x}{2} + \frac{y}{-1} = 1 \Rightarrow \frac{x}{2} - \frac{y}{1} = 1$$

$$\frac{x-2y}{2} = 1 \Rightarrow x - 2y = 2$$

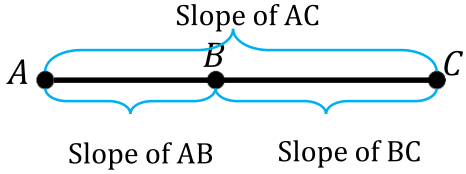
$$\boxed{x - 2y - 2 = 0}$$

10. Show that the points $(1,3)(2,1)$ and $\left(\frac{1}{2}, 4\right)$ are collinear by using (i)concept of *Slope* (ii) *using a straight line and other method*.

$$\underset{x_1\,,\,y_1}{A(1,3)}, \underset{x_2,\,y_2}{B(2,1)}$$

$$Slope\ of\ AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-3}{2-1} = \frac{-2}{1} = -2$$

Slope of AB $= -2$



$$\underset{x_1\,,\,y_1}{B(2,1)}, \underset{x_2,\,y_2}{C\left(\frac{1}{2}, 4\right)}$$

$$Slope\ of\ BC = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-1}{\frac{1}{2} - 2} = \frac{3}{\frac{1-4}{2}} = \frac{3}{\frac{-3}{2}}$$

$$= 3 \times -\frac{2}{3} = -2$$

Slope of BC $= -2$

Slope of AB $=$ Slope of BC $\quad$ *. Hence A, B, C are collinear*

(iii) Area of triangle

$$Let\ \underset{x_1\ \ y_1}{A(1,3)}, \underset{x_2\ \ y_2}{B(2,1)} \text{ and } \underset{x_3\ \ y_3}{C\left(\frac{1}{2}, 4\right)}$$

$$\text{The area of } \Delta ABC\ \Delta = \frac{1}{2}\left\{\begin{array}{cccc} 1 & 2 & \frac{1}{2} & 1 \\ 3 & 1 & 4 & 3 \end{array}\right\}$$

$$= \frac{1}{2}\left[1 + 8 + \frac{3}{2} - \{6 + \frac{1}{2} + 4\}\right]$$

$$= \frac{1}{2}\left\{9 + \frac{3}{2} - \left(10 + \frac{1}{2}\right)\right\} = \frac{1}{2}\left\{9 + \frac{3}{2} - 10 - \frac{1}{2}\right\}$$

$$= \frac{1}{2}\{9 + 1 - 10\} = \frac{1}{2}\{0\} = 0$$

11. A straight line is passing through the point $A(1, 2)$ with slope $\frac{5}{12}$ find points on the line which are 13 units away from A.

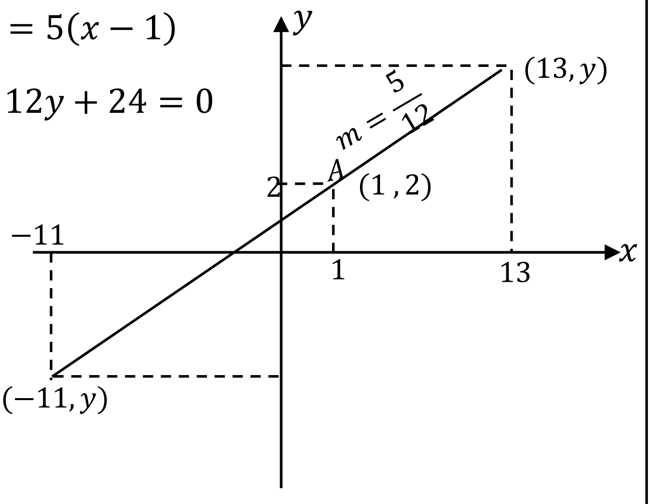
Equation straight line having slope $m = \frac{5}{12}$ and point $A(1,2)$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{5}{12}(x - 1) \Rightarrow 12(y - 2) = 5(x - 1)$$

$$12y - 24 = 5x - 5 \Rightarrow 5x - 5 - 12y + 24 = 0$$

$$5x - 12y + 19 = 0$$



$$when\ x = 13$$

$$5(13) - 12y = -19$$

$$65 - 12y = -19$$

$$-12y = -19 - 65$$

$$-12y = -84 \Rightarrow y = 7$$

$\therefore$ *The required point is* $(13,7)$

$$5x - 12y + 19 = 0$$

$$when\ x = -11$$

$$5(-11) - 12y = -19 \Rightarrow -55 - 12y = -19$$

$$-12y = -19 + 55 \Rightarrow -12y = 36$$

$$y = \frac{36}{-12} \Rightarrow y = -3$$

$\therefore$ *The required point is* $(-11, -3)$

12. A 150m long train is moving with constant velocity of 12.5m/s Find (i) the equation of the motion of the train.(ii) time taken to cross a pole. (iii) The time taken to cross the bridge of length 850m is?

(i) The equation of the motion of the train

length of the train $= 150m$ (Negative y - intercepts)

The uniform speed $= 12.5m/s$ *is the slope of the train.*

Total distance $= y + 150$

$$Speed = \frac{distance}{time} \Rightarrow 12.5 = \frac{y + 150}{x} \Rightarrow 12.5x = y + 150$$

$$12.5x = y + 150 \Rightarrow y = 12.5x - 150$$

Equation of a line: $y = mx + c$

$$m = 12.5,\ c = -150$$

$$y = 12.5x - 150$$

(ii) Time taken to cross a pole

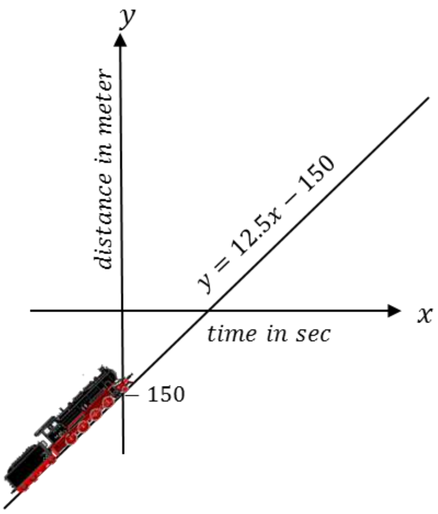
$$y = 12.5x - 150$$

$$put\ y = 0$$

$$0 = 12.5x - 150 \Rightarrow 150 = 12.5x \Rightarrow x = \frac{150}{12.5}$$

$$x = \frac{\cancel{1500}\ \cancel{300}\ \cancel{60}\ 12}{\cancel{125}\ \cancel{25}\ \cancel{5}}$$

$$\boxed{x = 12sec}$$



(iii) The time taken to cross the bridge of length 850m is

$$y = 12.5x - 150$$

$$put \quad y = 850$$

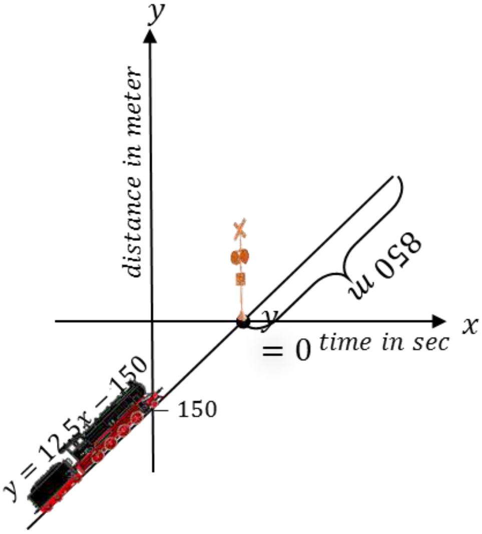
$$850 = 12.5x - 150$$

$$850 + 150 = 12.5x$$

$$1000 = 12.5x \Rightarrow x = \frac{1000}{12.5}$$

$$x = \frac{\cancel{10000}\ 2000}{\cancel{125}\ 25} \Rightarrow x = \frac{\cancel{2000}\ \cancel{400}\ 80}{\cancel{25}\ \cancel{5}}$$

$$\boxed{x = 80sec}$$



13. A spring was hung from a hook in the ceilling. A number of different weights were attached to the spring to make it stretch, and the total length of the spring was measured each time shown in the following table.

$weight, (kg)$	2	4	5	8
$Length, (cm)$	3	4	4.5	6

(i) Draw a graph sowing the results.

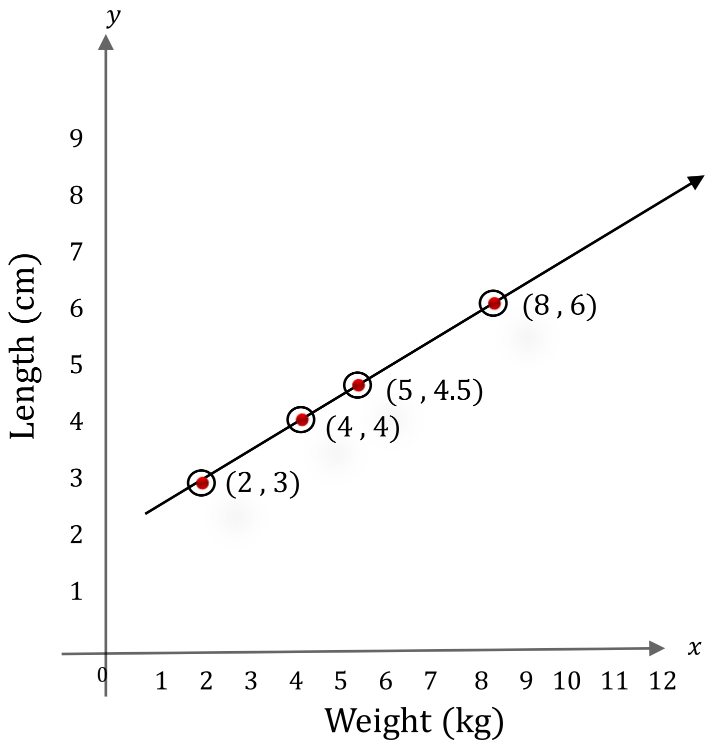
(ii) Find the equation relating of the length of the spring to the weight on it

(iii) What is the actual length of the spring.

(iv) If the spring has to stretch to 9 cm long, how much weight should be added?

(v) How long will the spring be when 6 kilograms of weight on it?

$weight, (x)$	2	4	5	8
$Length, (y)$	3	4	4.5	6



(ii) ***Find the equation relating the length of the spring to the weight on it***

$Two\ points\ \underset{x_1, y_1}{(2,3)}\ and\ \underset{x_2, y_2}{(4,4)}$

Equation of straight lines: $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

$$\frac{y-3}{4-3} = \frac{x-2}{4-2} \Rightarrow \frac{y-3}{1} = \frac{x-2}{2}$$

$$2(y-3) = x - 2 \Rightarrow 2y - 6 = x - 2$$

$$x - 2 - 2y + 6 = 0 \Rightarrow x - 2y + 4 = 0$$

(iii) *Actual length of the Spring*

$$put\ x = 0\ in\ x - 2y + 4 = 0$$

$$0 - 2y + 4 = 0 \Rightarrow -2y = -4$$

$$y = 2$$

$$\boxed{\therefore y = 2cm}$$

(iv) ***If the spring has to stretch to 9cm long. How much weight should be added?***

$$To\ find\ a\ value\ of:\ x = ?$$

$$put\ y = 9\ in\ x - 2y + 4 = 0$$

$$x - 2(9) + 4 = 0 \Rightarrow x - 18 + 4 = 0 \Rightarrow x - 14 = 0$$

$x = 14kg$ $\therefore 14kg$ must be added

(v) How long will the spring be when 6 kilograms of weight on it.

$To\ find\ a\ value\ of\ : y = ?$

$put\ x = 6\ in\ x - 2y + 4 = 0$

$6 - 2y + 4 = 0 \Longrightarrow 10 - 2y = 0$

$-2y = -10$

$y = 5\ cm$

$\boxed{length\ of\ the\ string = 5cm}$

14. ***A family is using liquefied petroleum gas (LPG) of weight 14.2 kg form Consumption. (full weight 29.5kg includes the empty cylinders are weight of 15.3kg) If it is use with constant rate then it lasts for 24 days. Then the new cylinder is replaced (i) Find the equation relating the quantity of gas in the cylinder to the days.***
(ii) Draw the graph for first 96 days.

Let x represent the number of days, y represent the weight of gas.

x	0	24
y	14.2	0

(i) $\therefore$ Equation is $\dfrac{y-y_1}{y_2-y_1}=\dfrac{x-x_1}{x_2-x_1} \Rightarrow \dfrac{y-14.2}{0-14.2}=\dfrac{x-0}{24-0}$

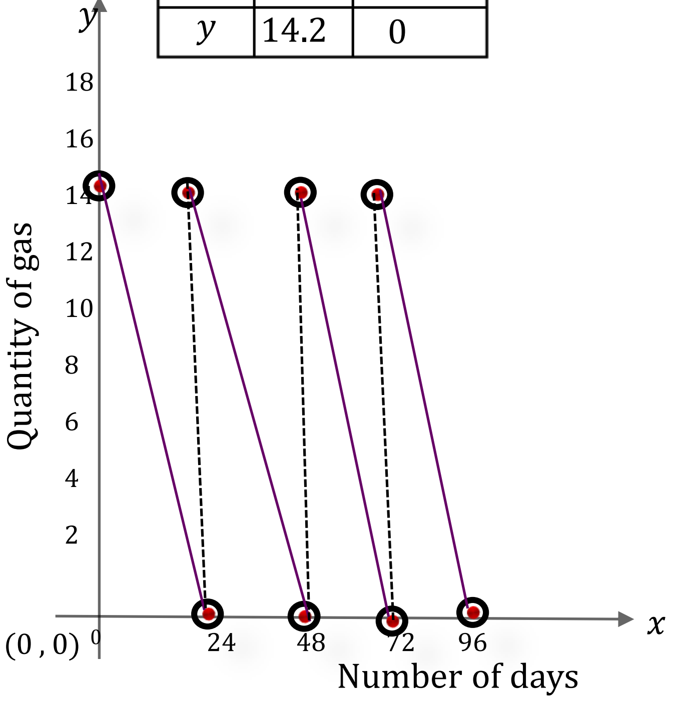
$$24(y-14.2)=-14.2(x) \Rightarrow y-14.2=-\frac{14.2}{24}x$$

$$y=-\frac{14.2}{24}x+14.2 \Rightarrow y=-\frac{\overset{71}{\cancel{142}}}{\underset{120}{\cancel{240}}}x+14.2$$

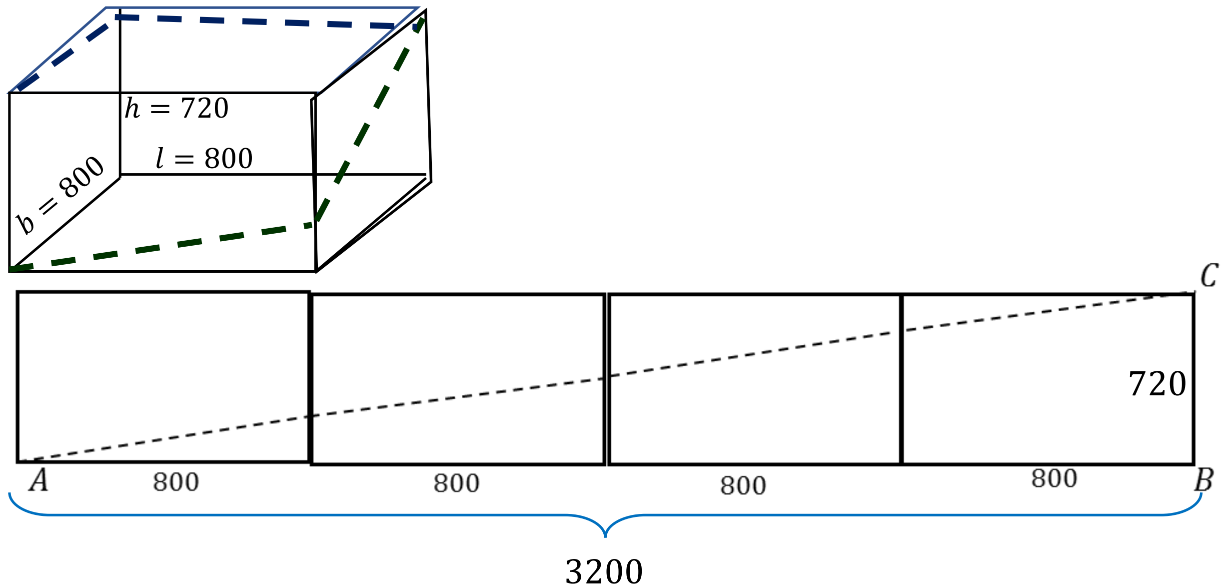
$$y=-\frac{71}{120}x+14.2,\ 0\le x\le 24$$

(ii) ***Draw the graph for first 96 days***

x	0	24
y	14.2	0



15. ***In a shopping mall there is a hall of cuboid shape with dimension $800 \times 800 \times 720$ units, which needs to be added the facility of an escalator in the Path as shown by the dotted line in the figure. Find (i) the minimum total length of the escalator. (ii) the heights at which the escalator changes its direction. (iii) the slope of the escalator at the turning points.***



(i) the minimum total length of the escalator

$AC^2 = AB^2 + BC^2$

$= (3200)^2 + (720)^2 = 10240000 + 518400$

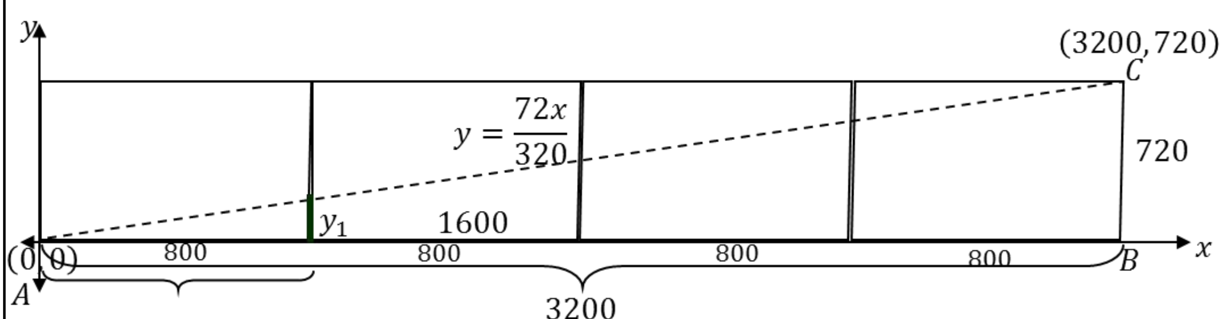
$AC^2 = 10758400 \implies AC = \sqrt{10758400}$

$AC = \sqrt{4 \times 4 \times 4 \times 4 \times 5 \times 5 \times 41 \times 41}$

$AC = 4 \times 4 \times 5 \times 41$

$AC = 3280$ *units*

4	10758400
4	2689600
4	672400
4	168100
5	42025
5	8405
41	1681
	41



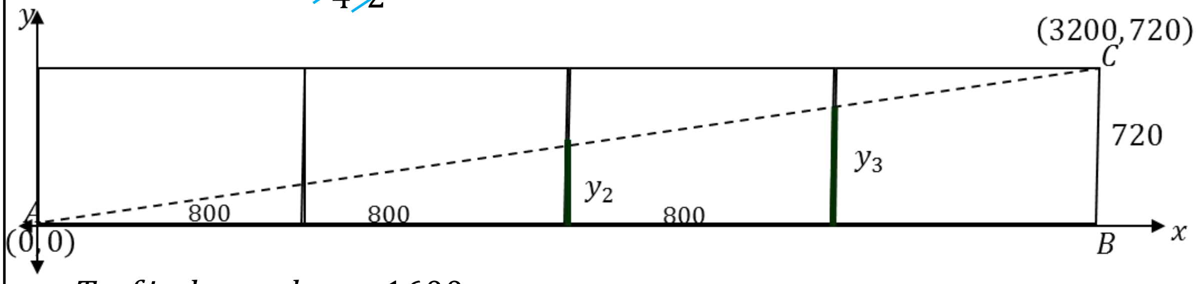
$\underset{(x_1, y_1)}{(0, 0)}$ *and* $\underset{(x_2, y_2)}{(3200, 720)}$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \implies \frac{y - 0}{720 - 0} = \frac{x - 0}{3200 - 0} \implies \frac{y}{720} = \frac{x}{3200}$$

$$y = \frac{x \times 72\cancel{0}}{320\cancel{0}} \implies y = \frac{72x}{320}$$

$To\ find\ \ y_1\ sub\ x = 800$

$$y_1 = \frac{\cancel{72}^{36}(\cancel{800}^{\cancel{10}\,5})}{\cancel{320}_{\cancel{4}\,\cancel{2}}} \Longrightarrow y_1 = 180\ units$$



$To\ find\ y_2\ sub\ x = 1600$

$$y_2 = \frac{\cancel{72}(\cancel{1600}^{\cancel{10}\,5})}{\cancel{320}_{\cancel{2}}} \Longrightarrow y_2 = 360\ units$$

$To\ find\ y_3\ sub\ x = 2400$

$$y_3 = \frac{\cancel{72}^{36}(\cancel{2400}^{\cancel{30}\ 15})}{\cancel{320}_{\cancel{4}\ \cancel{2}}} \Longrightarrow y_3 = 540\ units$$

(iii) the slope of the escalator at the turning points.

$$y = \frac{72x}{320}$$

$$y = mx + c$$

$$m = \frac{72}{320} = \frac{36}{160} = \frac{18}{80} = \frac{9}{40}$$

$EXERCISE : 6.3$

$6.4.4$ *Distance formulas*

(i) *two points*, (ii) *a point to a line*, (iii) *Two parallel lines*.

(i) *The distance between two points* (x_1, y_1) *and* (x_2, y_2) *is*

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(ii) *The length of the perpendicular from the point* (x_1, y_1) *to the line*

$ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

(iii) *The length of the perpendicular from origin to* $ax + by + c = 0$ *is*

$$\left|\frac{c}{\sqrt{a^2 + b^2}}\right|$$

(iv) *The distance between two parallel lines* $a_1x + b_1y + c_1 = 0$ *and* $a_1x + b_1y + c_2 = 0$

$$D = \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$$

(v) *The foot of the perpendicular from* (x_1, y_1) *to the line* $ax + by + c = 0$

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{(a_1x + b_1y + c)}{a^2 + b^2}$$

Eg. 6.22: Find the equation of a parallel line and a perpendicular line passing through the point $(1, 2)$ to the line $3x + 4y = 7$.

Parallel line to $3x + 4y = 7$ *is of the form* $3x + 4y = 3x_1 + 4y_1$

Let (x_1, y_1) be $(1, 2)$

$$3x + 4y = 3(1) + 4(2)$$
$$3x + 4y = 11$$

Perpendicular line to $3x + 4y = 7$ *is of the form*

$$4x - 3y = 4x_1 - 3y_1$$

Here $(x_1, y_1) = (1, 2)$

$$4x - 3y = 4(1) - 3(2)$$
$$4x - 3y = -2$$

The parallel and perpendicular lines are respectively

$$3x + 4y = 11, \; 4x - 3y = -2$$

Eg. 6.23: *Find the distance* (i) *between two points* $(5, 4)$ *and* $(2, 0)$

(ii) *from a point* $(1, 2)$ *to the line* $5x + 12y - 3 = 0$

(iii) *between two parallel lines* $3x + 4y = 12$ *and* $6x + 8y + 1 = 0$.

(i) *Distance between two points* $(x_1, y_1) = (5,4)$ *and* $(x_2, y_2) = (2,0)$ *is*

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(2-5)^2 + (0-4)^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25}$$

$$\boxed{D = 5}$$

(ii) *Distance from a point* $(1,2)$ *to the line* $5x + 12y - 3 = 0$

Distance from a points (x_1, y_1) *to the line* $ax + by + c = 0$ *is* $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$

$(x_1, y_1) = (1,2)$ $a = 5$, $b = 12$, $c = -3$

$$D = \left|\frac{5(1) + 12(2) - 3}{\sqrt{5^2 + 12^2}}\right| = \left|\frac{5 + 24 - 3}{\sqrt{25+144}}\right| = \left|\frac{26}{\sqrt{169}}\right| = \frac{26}{13}$$

$$\boxed{D = 2}$$

(iii) *Distance between two parallel lines*

$3x + 4y = 12$ *and* $6x + 8y + 1 = 0$ *is*

Given lines : $3x + 4y - 12 = 0$ *and* $6x + 8y + 1 = 0$

$\div 2$

$$3x + 4y + \frac{1}{2} = 0$$

$$a = 3,\ b = 4,\ c_1 = -12,\ c_2 = \frac{1}{2}$$

$$D = \left|\frac{c_1 - c_2}{\sqrt{a^2+b^2}}\right| = \left|\frac{-12 - \frac{1}{2}}{\sqrt{3^2+4^2}}\right| = \left|\frac{\frac{-24-1}{2}}{\sqrt{9+16}}\right|$$

$$= \left|\frac{\frac{-25}{2}}{\sqrt{25}}\right| = \left|\frac{\frac{-25}{2}}{5}\right| = \left|\frac{-25}{2} \times \frac{1}{5}\right| = \frac{5}{2} = 2.5 \text{ units}$$

Eg. 6.24: Find the nearest point on the line $2x + y = 5$ from the origin.

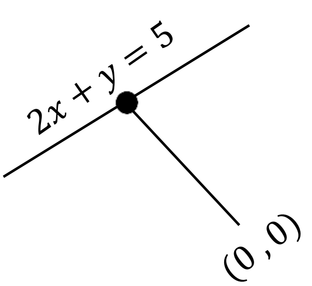
$2x + y = 5 \ldots (1)$

The line perpendicular to the given line, through the origin $(\underset{x_1}{0}, \underset{y_1}{0})$

$$x - 2y = x_1 - 2y_1$$

$$x - 2y = 0 - 2(0)$$

$$x - 2y = 0 \ldots (2)$$



Solve (1) *and* (2)

$(1) \times 2 \implies 4x + \cancel{2y} = 10$

$(2) \qquad x - \cancel{2y} = 0$

$$5x = 10 \implies x = \frac{10}{5} \implies \boxed{x = 2}$$

$sub\ x = 2\ in\ (1)\ 2x + y = 5$

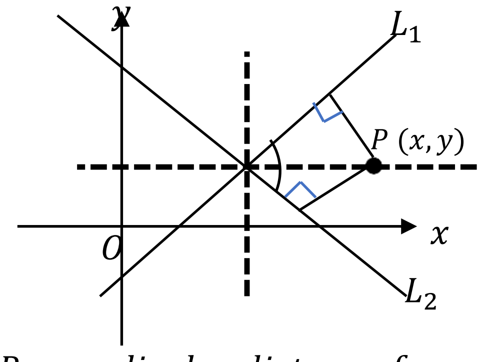
$$2(2) + y = 5 \Rightarrow 4 + y = 5$$

$$y = 5 - 4 \Rightarrow y = 1$$

$Hence\ the\ nearest\ point\ on\ the\ line\ 2x + y = 5\ from\ the\ origin\ is\ (2,1)$

What is an Angle Bisector?

An angle bisector has equal perpendicular distance from the two lines.



$Perpendicular\ distance\ from\ a\ points\ (x\ , y)\ to\ the\ line\ L_1 =$

$Perpendicular\ distance\ from\ a\ points\ (x\ , y)\ to\ the\ line\ L_2$

$\boldsymbol{Eg. 6.25}$: $\boldsymbol{Find\ the\ equation\ of\ the\ bisector\ of\ the\ acute\ angle\ between\ the\ line\ 3x + 4y + 2 = 0\ and\ 5x + 12y - 5 = 0}$

$$3x + 4y + 2 = 0\ and\ 5x + 12y - 5 = 0$$

$$\left|\frac{3x + 4y + 2}{\sqrt{3^2 + 4^2}}\right| = \left|\frac{5x + 12y - 5}{\sqrt{5^2 + 12^2}}\right| \Rightarrow \frac{3x + 4y + 2}{\sqrt{9 + 16}} = \pm\frac{5x + 12y - 5}{\sqrt{25 + 144}}$$

$$\frac{3x + 4y + 2}{\sqrt{25}} = \pm\frac{5x + 12y - 5}{\sqrt{169}} \Rightarrow \frac{3x + 4y + 2}{5} = \pm\frac{5x + 12y - 5}{13}$$

$$\frac{3x + 4y + 2}{5} = -\frac{5x + 12y - 5}{13} \Rightarrow 13(3x + 4y + 2) = -5(5x + 12y - 5)$$

$The\ equation\ of\ bisector\ of\ the\ acute\ angle\ between\ the\ lines$

$$39x + 52y + 26 = -25x - 60y + 25$$

$$39x + 52y + 26 + 25x + 60y - 25 = 0$$

$$\boxed{64x + 112y + 1 = 0}$$

$\boldsymbol{Eg. 6.26}$: $\boldsymbol{Find\ the\ points\ on\ the\ line\ x + y = 5, that\ lie\ at\ a\ distance\ 2\ units\ from\ the\ line\ 4x + 3y - 12 = 0.}$

$$x + y = 5$$

$$Let\ x = t$$

$$t + y = 5 \Rightarrow y = 5 - t$$

$Any\ point\ on\ the\ line\ x + y = 5\ is (t\ , 5 - t)$

$The\ distance\ from\ (t\ , 5 - t) to\ the\ line\ 4x + 3y - 12 = 0\ is\ given\ by\ 2\ units.$

$a = 4\,, b = 3\,, c = -12\,, x_1 = t, y_1 = 5 - t$

$$D = \left|\frac{\mathrm{a}x_1 + \mathrm{b}y_1 + \mathrm{c}}{\sqrt{a^2 + b^2}}\right| \Rightarrow 2 = \left|\frac{4(t) + 3(5 - t) - 12}{\sqrt{4^2 + 3^2}}\right|$$

$$\left|\frac{4\mathrm{t} + 15 - 3t - 12}{\sqrt{16 + 9}}\right| = 2 \Rightarrow \left|\frac{\mathrm{t} + 3}{\sqrt{25}}\right| = 2 \Rightarrow \left|\frac{\mathrm{t} + 3}{5}\right| = 2$$

$$\frac{t + 3}{5} = \pm 2 \Rightarrow t + 3 = \pm 10 \Rightarrow t + 3 = 10\,,\ t + 3 = -10$$

$t = 10 - 3\,,\ t = -10 - 3$

$t = 7, \qquad t = -13$

The points are $(t\,, 5 - t)$

$when\ \ t = 7, then\ (7\,, 5 - 7) = (7\,, -2)$

$when\ \ t = -13, then\ (-13\,, 5 + 13) = (-13\,, 18)$

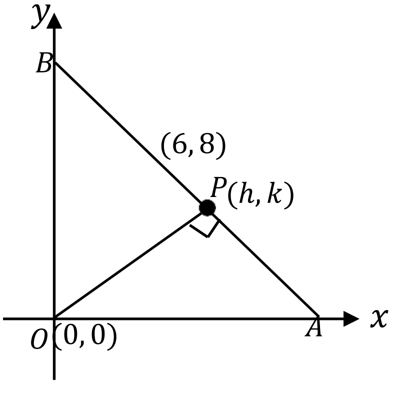
$\boldsymbol{Eg.6.27}$: $\boldsymbol{A\ Straight\ line\ passes\ through\ a\ fixed\ point\ (6\,, 8)\,find\ the\ locus\ of\ the\ foot\ of\ the\ perpendicular\ drawn\ to\ it\ from\ the\ origin\ O.}$

$Let\ P(h, k)\ be\ a\ point\ on\ the\ reqired\ locus.$

$The\ slope\ PQ\ of\ the\ line\ joining\ (\underset{x_1\ y_1}{0, 0})\ and\ (\underset{x_2\ y_2}{h, k})\ is$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_1 = \frac{k - 0}{h - 0} = \frac{k}{h} \Rightarrow m_1 = \frac{k}{h}$$



$Slope\ \ AB\ of\ the\ line\ joining\ (\underset{x_1\ y_1}{6, 8})\ and\ (\underset{x_2\ y_2}{h, k})$

$$m_2 = \frac{k - 8}{h - 6}$$

$$m_1 \times m_2 = -1 \quad \because OP \perp AB$$

$$\frac{k}{h} \times \frac{k - 8}{h - 6} = -1 \Rightarrow \frac{k^2 - 8k}{h^2 - 6h} = -1$$

$$k^2 - 8k = -(h^2 - 6h) \Rightarrow k^2 - 8k = -h^2 + 6h$$

$$h^2 + k^2 - 6h - 8k = 0$$

Locus of $P(h, k)$ is $x^2 + y^2 - 6x - 8y = 0$

$\mathbf{6.4.5}$ $\boldsymbol{Family\ of\ lines}$

Let $ax + by + c = 0$ and $a_1x + b_1y + c_1 = 0$ be two given lines. Then the equation of the line through the point of intersection of these lines is

$$\boxed{ax + by + c + \lambda\,(a_1x + b_1y + c_1) = 0}$$

Eg. 6.28: Find the equation of the straight lines in the family of the lines $y = mx + 2$, for which m and the $x-$coordinate of the point of intersection of the lines with $2x + 3y = 10$ are integers.

Given lines : $y = mx + 2$

$$mx - y = -2 \ldots (1) \text{ and } 2x + 3y = 10 \ldots (2)$$

Solve (1) *and* (2)

$$(1) \times 3 \Rightarrow 3mx - \cancel{3y} = -6$$

$$(2) \Rightarrow \quad 2x + \cancel{3y} = 10$$

$$3mx + 2x = 4 \Rightarrow x(3m + 2) = 4$$

$x = \dfrac{4}{3m + 2}$ *since x and m must be an integer*

[$3m + 2$ *must be equal to* $\pm 1, \pm 2, \pm 4$]

$$3m + 2 = \pm 1 \Rightarrow 3m = \pm 1 - 2 \Rightarrow 3m = 1 - 2, -1 - 2$$

$$3m = -1, -3 \Rightarrow m = -\frac{1}{3}, -\frac{3}{3} \Rightarrow m = -\frac{1}{3}, -1$$

$$\therefore m = -1$$

$$3m + 2 = \pm 2 \Rightarrow 3m = \pm 2 - 2 \Rightarrow 3m = 2 - 2, -2 - 2$$

$$3m = 0, -4 \Rightarrow m = \frac{0}{3}, -\frac{4}{3} \Rightarrow m = 0, -\frac{4}{3}$$

$$\therefore m = 0$$

$$3m + 2 = \pm 4 \Rightarrow 3m = \pm 4 - 2 \Rightarrow 3m = 4 - 2, -4 - 2$$

$$3m = 2, -6 \Rightarrow m = \frac{2}{3}, -\frac{6}{3} \Rightarrow m = \frac{2}{3}, -2$$

$$\boxed{\therefore m = -2}$$

$$y = mx + 2$$

when $m = -1 \Rightarrow y = -x + 2 \Rightarrow x + y - 2 = 0$

when $m = 0 \Rightarrow y = 0 + 2 \Rightarrow y - 2 = 0$

when $m = -2 \Rightarrow y = -2x + 2 \Rightarrow 2x + y - 2 = 0$

Eg. 6.30: *Suppose the Government has decided to erect a new Electrical power transmission substation to provide better power supply to two villages namely A and B. the substation has to be on the line l. the distance of villages A and B from the foot of the perpendicular P and Q on the line l are 3 km and 5km respectively and the distance between p and Q is 6km. (i) what is the smallest length of cable required to connect the power station to two villages. (using the knowledge in conjuction with the principle of reflection allows for approach to solve this problem*

Take conveniently PQ as $x - axis$, PA as $y - axis$ and P is origin (instead of vonventional origin O). Therefore, the coordinates are $p(0,0)A(0,3)$and $B(6,5)$

If the image of A about the $x - axis$, $\bar{A}$, then $\bar{A}$ is $(0, -3)$.

The required R is the point of intersection of the line $\bar{A}B$ and $x - axis$.
AR and BR are the path of the cable(road)

The shortest length of the cable is $AR + BR = BR + R\bar{A} = B\bar{A}$

$$B\bar{A} = \sqrt{(6-0)^2 + (5+3)^2} = 10\ km$$

Equation of the line $\bar{A}B$ is $y - (-3) = \dfrac{5-(-3)}{6-c} - (x - 0)$

$$4x - 3y = 9$$

$$\text{when } y = 0,\ R\ is \left(\frac{9}{4}, 0\right)$$

That is the substation should be located at a distance of 2.25 km from P.

The equation of AR is $4x + 3y = 9$

The equation of the cable lines (roads)of RA and RB are

$$4x - 3y = 9 \text{ and } 4x + 3y = 9$$

Eg. 6.29*: *Find the equation of the line through the intersection of the line $3x + 2y + 5 = 0$and $3x - 4y + 6 = 0$ and the point$(1, 1)$.

The family of equation of straight lines through the point of intersection of the line is of the form

$$(a_1x + b_1y + c_1) + \lambda\,(a_2x + b_2y + c_2) = 0$$

$$(3x + 2y + 5) + \lambda\,(3x - 4y + 6) = 0$$

it passes through the point $(1,1)$

$$\{3 + 2(1) + 5\} + \lambda\,\{3(1) - 4(1) + 6\} = 0$$

$$10 + \lambda(3 - 4 + 6) = 0 \Rightarrow 10 + \lambda(5) = 0$$

$$5\lambda = -10 \Rightarrow \lambda = -2$$

$$sub\ \lambda = -2\ in\ (3x + 2y + 5) + \lambda\,(3x - 4y + 6) = 0$$

$$(3x + 2y + 5) - 2\,(3x - 4y + 6) = 0$$

$$3x + 2y + 5 - 6x + 8y - 12 = 0$$

$$-3x + 10y - 7 = 0 \Rightarrow 3x - 10y + 7 = 0$$

Eg. 6.31: ***A car rental firm has charges Rs. 25 with 1.8 free kilometers, and Rs. 12 for every additional kilometer . Find the equation relating the cost y to the number of kilometers x. Also find the cost to travel 15 kilometers.***

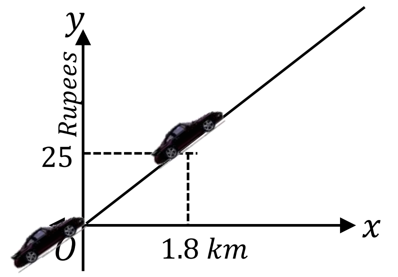
Given that up to 1.8 *kilometers the fixed rent is Rs.* 25.

The equation is $y = 25, 0 \le x \le 1.8 \ldots (1)$

After 1.8 *kilometers the rent is Rs.* 12 *for every additional kilometer.*

$$y = 25 + 12(x - 1.8), x > 1.8$$

The combined equation of (1) *and* (2)



$$y = \begin{cases} 25, \ 0 \le x \le 1.8 \\ 25 + 12(x - 1.8), x > 1.8 \end{cases}$$

when $x = 15$, *from* (2) $y = 25 + 12(x - 1.8)$

$$y = 25 + 12(15 - 1.8) = 25 + 12(13.2)$$
$$= 25 + 158.4 = 183.4$$

cost to travel 15 *kiolmeter is Rs.* 183.40.

1. *Show that the lines are* $3x + 2y + 9 = 0$ *and* $12x + 8y - 15 = 0$ *are parallel lines.*

The slope of $3x + 2y + 9 = 0$ *is* $m_1 = -\frac{3}{2}$

slope of $12x + 8y - 15 = 0$ *is* $m_2 = -\frac{12}{8} \Longrightarrow m_2 = \frac{-3}{2}$

$$\therefore m_1 = m_2$$

$\therefore$ *Given lines are parallel*

2. *Find the equation of the straight line parallel to* $5x - 4y + 3 = 0$ *and having* $x -$ *intercept* 3

Any line parallel to this $5x - 4y + 3 = 0$ *is* $5x - 4y = 5x_1 - 4y_1$

it passes through $(3, 0)$

$$\therefore 5x - 4y = 5(3) - 4(0)$$
$$5x - 4y = 15$$

$\therefore$ *The required equation is* $5x - 4y - 15 = 0$

3. *Find the distance between the line* $4x + 3y + 4 = 0$, *and a point* $(i)(-2, 4), (ii)(7, -3)$.

(i) *Perpendicular distance from* $(-2,4)$ *to the line* $4x + 3y + 4 = 0$ *is*

Distance from a points (x_1, y_1) *to the line* $ax + by + c = 0$ *is* $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

$$a = 4, \ b = 3, \ c = 4$$
$$(x_1, y_1) = (-2, 4)$$

$$= \left|\frac{4(-2) + 3(4) + 4}{\sqrt{4^2 + 3^2}}\right| = \left|\frac{-8 + 12 + 4}{\sqrt{16 + 9}}\right| = \left|\frac{8}{\sqrt{25}}\right| = \frac{8}{5}$$

Distance $= \frac{8}{5}$

(ii) Perpendicular distance from $(7,-3)$ to the line $4x+3y+4=0$

$$a=4\ ,\ b=3\ ,\ c=4$$
$$(x_1\ ,y_1)=(7,-3)$$

$$=\left|\frac{4(7)+3(-3)+4}{\sqrt{4^2+3^2}}\right|=\left|\frac{28-9+4}{\sqrt{4^2+3^2}}\right|=\left|\frac{23}{\sqrt{25}}\right|=\frac{23}{5}$$

$$Distance=\frac{23}{5}$$

4. Write the equation of the lines through the point $(1,-1)$ (i) parallel to $x+3y-4=0$ (ii) Perpendicular to $3x+4y=6$

(i) Given line is $x+3y-4=0$

Parallel to the line is $x+3y=x_1+3y_1$

This passes through $(1,-1)$

$$x+3y=1+3(-1)\implies x+3y=1-3$$
$$x+3y=-2$$

$\therefore$ Required line is $x+3y+2=0$

(ii) Given line is $3x+4y=6$

$$4x-3y=4x_1-3y_1$$

Here $(x_1,y_1)=(1,-1)$

$$4x-3y=4(1)-3(-1)$$
$$4x-3y=4+3\implies 4x-3y=7$$

$\therefore$ Required line is $4x-3y-7=0$

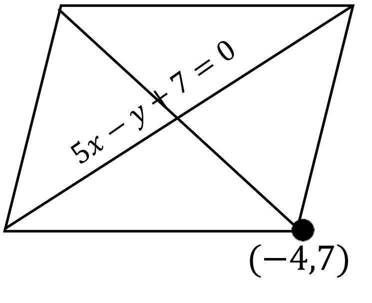
5. If $(-4,7)$ is one vertex of a rhombus and if the equation of one diagonal is $5x-y+7=0$, then find the equation of another diagonal.

Since the diagonal of the rhombus are perpendicular

To find perpendicular to the line $5x-y+7=0$ passing through $(-4,7)$

$$x+5y=x_1+5y_1$$

This passes through $(-4,7)$
(x_1,y_1)



$$x+5y=-4+5(7)$$
$$x+5y=35-4$$
$$x+5y=31$$

$\boxed{\therefore \text{Required line is } x+5y-31=0}$

6. Find the equation of the line passing through the point of intersection lines $4x-y+3=0$ and $5x+2y+7=0$, and (i) through the point $(-1,2)$ (ii) parallel to $x-y+5=0$ (iii) perpendicular to $x-2y+1=0$.

Given : $4x - y = -3 \ldots (1)$

$5x + 2y = -7 \ldots (2)$

To find point of intersection solve (1) *and* (2)

$(1) \times 2 \Rightarrow 8x - \cancel{2y} = -6$

$(2) \Rightarrow 5x + \cancel{2y} = -7$

$\cancel{13}x = -\cancel{13}\ -1 \Rightarrow x = -1$

Sub $x = -1$ in (1) $4x - y = -3$

$4(-1) - y = -3 \Rightarrow -4 - y = -3$

$-y = -3 + 4 \Rightarrow -y = 1 \Rightarrow y = -1$

$\therefore$ *The point of intersection is* $(-1, -1)$

(i) Equation of a straight line passing through the points $(\underset{x_1}{-1}, \underset{y_1}{-1})$ and $(\underset{x_2}{-1}, \underset{y_2}{2})$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \Rightarrow \frac{y + 1}{2 + 1} = \frac{x + 1}{-1 + 1}$$

$$\frac{y + 1}{3} = \frac{x + 1}{0} \Rightarrow 3x + 3 = 0$$

$\div 3$

$\boxed{x + 1 = 0}$

(ii) parallel to $x - y + 5 = 0$

Parallel to the line is $x - y = x_1 - y_1$

This passes through $\underset{(x_1, y_1)}{(-1, -1)}$

$x - y = -1 + 1 \Rightarrow x - y = 0$

(iii) perpendicular to $x - 2y + 1 = 0$.

perpendicular to the line $x - 2y + 1 = 0$

$2x + y = 2x_1 + y_1$

This passes through $\underset{(x_1, y_1)}{(-1, -1)}$

$2x + y = 2(-1) + (-1)$

$2x + y = -2 - 1 \Rightarrow 2x + y = -3$

$\boxed{2x + y + 3 = 0}$

7. Find the equations of two straight lines which are parallel to the line $12x + 5y + 2 = 0$ and at a unit distance from the point $(1, -1)$

Given line is $12x + 5y + 2 = 0$

Any line parallel to this $12x + 5y + k = 0$

Perpendicular distance from $(1, -1)$ *to the line* $12x + 5y + k = 0$

$(1, -1) = (x_1, y_1)$

$a = 12\ ,\ b = 5\ ,\ c = k$

$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right| = 1 \Rightarrow \frac{12(1) + 5(-1) + k}{\sqrt{12^2 + 5^2}} = \pm 1$$

$$\frac{12 - 5 + k}{\sqrt{144 + 25}} = \pm 1 \Rightarrow \frac{7 + k}{\sqrt{169}} = \pm 1 \Rightarrow \frac{7 + k}{13} = \pm 1$$

$7 + k = \pm 13 \Rightarrow 7 + k = 13, 7 + k = -13$

$k = 13 - 7, k = -13 - 7$

$k = 6, k = -20$

$\therefore$ *The required lines are* $12x + 5y + 6 = 0$ *and* $12x + 5y - 20 = 0$

8. *Find the equations of straight lines which are perpendicular to the line* $3x + 4y - 6 = 0$ *and are at a distance of* 4 *units from* $(2, 1)$

Given line is $3x + 4y - 6 = 0$

Any line perpendicular to this is $4x - 3y + k = 0$

Perpendicular distance from $(2, 1)$ *to the line* $4x - 3y + k = 0$ *is*

$(2, 1) = (x_1, y_1)$

$a = 4\ ,\ b = -3\ ,\ c = k$

$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right| = 4 \Rightarrow \frac{4 \times 2 + (-3)(1) + k}{\sqrt{4^2 + (-3)^2}} = \pm 4$$

$$\frac{8 - 3 + k}{\sqrt{16 + 9}} = \pm 4 \Rightarrow \frac{5 + k}{\sqrt{25}} = \pm 4 \Rightarrow \frac{5 + k}{5} = \pm 4$$

$5 + k = \pm 20 \Rightarrow 5 + k = 20, 5 + k = -20$

$k = 20 - 5,\ k = -20 - 5$

$k = 15, k = -25$

$\therefore$ *Required equation are* $4x - 3y + 15 = 0$ *and* $4x - 3y - 25 = 0$

9. *Find the equation of a straight line parallel to* $2x + 3y = 10$ *and which is such that the sum of its intercepts on the axes is* 15

Given line is $2x + 3y - 10 = 0$

Any line parallel to this is $2x + 3y + k = 0$

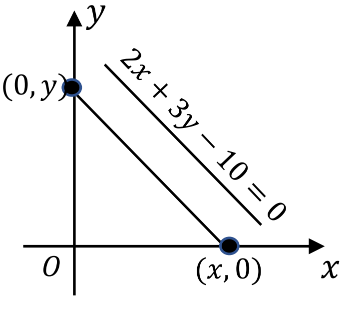
$x -$ *intercept of the line i.e* $y = 0$

$2x + 3(0) + k = 0 \Rightarrow 2x + k = 0$

$2x = -k \Rightarrow x = -\frac{k}{2}$

y − intercept of the line *i.e* $x = 0$

$2(0) + 3y + k = 0 \Rightarrow 3y + k = 0$



$$3y = -k \Rightarrow y = -\frac{k}{3}$$

Sum of the intercept $= 15$

$$-\frac{k}{2} + \left(-\frac{k}{3}\right) = 15 \Rightarrow -\frac{k}{2} - \frac{k}{3} = 15$$

$$\frac{-3k - 2k}{6} = 15 \Rightarrow \frac{-\cancel{5}k}{6} = \overset{3}{\cancel{15}} \Rightarrow \frac{-k}{6} = 3$$

$$-k = 18 \Rightarrow \boxed{k = -18}$$

$\therefore$ *Required equation is* $2x + 3y - 18 = 0$

10. *Find the length of the perpendicular and the co − ordinates of the foot of the perpendicular from* $(-10, -2)$ *to the line* $x + y - 2 = 0$.

The perpendicular distance from (x_1, y_1) *to the straight line ax + by + c = 0 is*

$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

Given : *point* $(-10, -2)$ *and st. line* $x + y - 2 = 0$

$x_1 = -10, y_1 = -2$ *and* $a = 1, b = 1, c = -2$

$$= \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right| = \left|\frac{1(-10) + 1(-2) + (-2)}{\sqrt{1^2 + 1^2}}\right|$$

$$= \left|\frac{-10 - 2 - 2}{\sqrt{1 + 1}}\right| = \left|\frac{-14}{\sqrt{2}}\right| = \frac{14}{\sqrt{2}} = \frac{14 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\overset{7}{\cancel{14}} \times \sqrt{2}}{\cancel{2}} = 7\sqrt{2} \text{ units}$$

The co − ordinates of the foot of perpendicular is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

$$\frac{x + 10}{1} = \frac{y + 2}{1} = -\frac{(-10 - 2 - 2)}{2}$$

$$x + 10 = y + 2 = -\frac{(-\overset{7}{\cancel{14}})}{\cancel{2}}$$

$$x + 10 = y + 2 = 7 \Rightarrow x + 10 = 7, \; y + 2 = 7$$

$$x = 7 - 10, \; y = 7 - 2$$

$$\therefore \quad x = -3 \; ; \quad y = 5$$

$\therefore$ *Required point is* $(-3, 5)$

11. *If* p_1 *and* p_2 *are the lengths of the perpendicular from the origin to the straight lines* $x \sec\theta + y \operatorname{cosec}\theta = 2a$ *and* $x\cos\theta - y\sin\theta = a\cos 2\theta$*, then prove that* $p_1^2 + p_2^2 = a^2$

$P_1 =$ *Perpendicular distance from origin* $\underset{x_1}{(0}, \underset{y_1}{0)}$ *to the line*

$x\ sec\ \theta + y\ cosec\ \theta - 2a = 0$

$a = sec\theta\ ,\ b = cosec\theta\ ,\ c = -2a$

$$P_1 = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right| \Rightarrow P_1 = \left|\frac{\sec\theta \times 0 + cosec\ \theta \times 0 - 2a}{\sqrt{sec^2\theta + cosec^2\theta}}\right|$$

$$P_1 = \left|\frac{-2a}{\sqrt{sec^2\theta + cosec^2\theta}}\right| \Rightarrow P_1 = \frac{2a}{\sqrt{sec^2\theta + cosec^2\theta}}$$

$$P_1^2 = \frac{4a^2}{sec^2\theta + cosec^2\theta}$$

$P_2 = Perpendicular\ distance\ from\ origin\ \underset{x_1\ \ y_1}{(0\ ,0)}\ to\ the\ line$

$x\cos\theta - y\sin\theta - a\cos 2\theta = 0$

$a = \cos\theta\ ,\ b = -\sin\theta\ ,\ c = -a\cos 2\theta$

$$P_2 = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right| \Rightarrow P_2 = \left|\frac{cos\ \theta \times 0 - sin\ \theta \times 0 - a\ cos\ 2\theta}{\sqrt{cos^2\theta + (-\sin\theta)^2}}\right|$$

$$P_2 = \left|\frac{0 - 0 - a\cos 2\theta}{\sqrt{cos^2\theta + sin^2\theta}}\right| \Rightarrow P_2 = \frac{a\cos 2\theta}{\sqrt{1}}$$

$$P_2 = a\cos 2\theta \Rightarrow \boxed{P_2^2 = a^2cos^2 2\theta}$$

$To\ prove:\ P_1^2 + P_2^2 = a^2$

$$P_1^2 + P_2^2 = \frac{4a^2}{sec^2\theta + cosec^2\theta} + a^2cos^2 2\theta$$

$$= \frac{4a^2}{\frac{1}{cos^2\theta} + \frac{1}{sin^2\theta}} + a^2cos^2 2\theta$$

$$= \frac{4a^2}{\frac{sin^2\theta + cos^2\theta}{sin^2\theta cos^2\theta}} + a^2cos^2 2\theta = \frac{4a^2}{\frac{1}{sin^2cos^2\theta}} + a^2cos^2 2\theta$$

$$= 4a^2sin^2\ \theta cos^2\theta + a^2\ cos^2 2\theta = a^2(4sin^2\theta cos^2\theta + cos^2 2\theta)$$

$$= a^2[(2sin\theta cos\theta)^2 + cos^2 2\theta] = a^2[\ (sin2\theta)^2 + cos^2 2\theta]$$

$$= a^2[sin^2 2\theta + cos^2 2\theta] = a^2(1)$$

$\boxed{sin2A = 2sinAcosA}$

$\boxed{p_1^2 + p_2^2 = a^2}$

12. *Find the distance between the parallel lines*

(i) $12x + 5y = 7$ *and* $12x + 5y + 7 = 0$

$12x + 5y - 7 = 0 \ldots (1)\ and\ 12x + 5y + 7 = 0\ \ldots (2)$

Distance between parallel lines $= \left|\dfrac{c_1 - c_2}{\sqrt{a^2+b^2}}\right|$

$Here\ a = 12, b = 5\ and\ c_1 = -7, c_2 = 7$

$$Distance\ between\ parallel\ lines = \left|\frac{-7-7}{\sqrt{12^2+5^2}}\right|$$

$$= \left|\frac{-14}{\sqrt{144+25}}\right| = \left|\frac{-14}{\sqrt{169}}\right| = \frac{14}{\sqrt{13\times 13}} \quad = \frac{14}{13}\,units$$

$(ii)\ \mathbf{3x - 4y + 5 = 0\ and\ 6x - 8y - 15 = 0}$

$3x - 4y + 5 = 0 \ldots (1)\ and\ 6x - 8y - 15 = 0$
$\div 2$

$$3x - 4y - \frac{15}{2} = 0\ \ldots (2)$$

Distance between parallel lines $= \left|\dfrac{c_1 - c_2}{\sqrt{a^2+b^2}}\right|$

$Here\ a = 3, b = 4\ \ and\ c_1 = 5, c_2 = -\dfrac{15}{2}$

$$Distance\ between\ parallel\ lines\ = \left|\frac{5+\frac{15}{2}}{\sqrt{3^2+\ 4^2}}\right|$$

$$= \left|\frac{\frac{10+15}{2}}{\sqrt{9+16}}\right| = \left|\frac{\frac{25}{2}}{\sqrt{25}}\right| = \frac{\frac{25}{2}}{5} = \frac{25}{2}\times\frac{1}{5} = \frac{5}{2}\,units$$

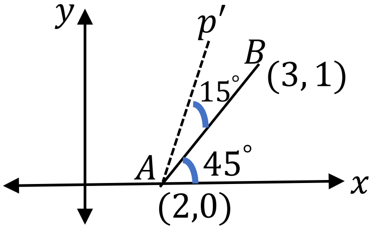
13. *Find the family of straight lines (i)Perpendicular (ii)parallel to* $\mathbf{3x + 4y - 12 = 0}$.

Given line is $3x + 4y - 12 = 0$

(i)*Any line parallel to* $3x + 4y - 12 = 0$ *is* $\ 3x + 4y + k = 0,\ k \in R$

(ii)*Any line perpendicular to* $3x + 4y - 12 = 0$ *is* $4x - 3y + k' = 0\ , k' \in R$

14. *If the line joining two points* $\mathbf{A(2,0)}$ *and* $\mathbf{B(3,1)}$ *is rotated AB about A in anticlockwise direction through an angle of* $\mathbf{15^\circ}$*, then find the equation of the line in new position.*



$A(\underset{x_1}{2},\underset{y_1}{0})\ and\ \ B(\underset{x_2}{3},\underset{y_2}{1})$

$$Slope\ of\ AB\ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-0}{3-2} = \frac{1}{1}$$

$$m = 1$$

$$\therefore \tan\theta = 1 \Longrightarrow \theta = 45^\circ$$

Let AP' *be the line obtained by rotating AB about A through angle* 15°

$$\therefore \angle P'AX = 45^\circ + 15^\circ = 60^\circ$$

$$m = \tan 60^{\circ} \Longrightarrow m = \sqrt{3}$$

The line pp′ having slope $m = \sqrt{3}$ *and point* $(\underset{x_1}{2}, \underset{y_1}{0})$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \sqrt{3}(x - 2) \Longrightarrow y = \sqrt{3}(x - 2)$$

$$y = \sqrt{3}x - 2\sqrt{3} \Longrightarrow \sqrt{3}x - y - 2\sqrt{3} = 0$$

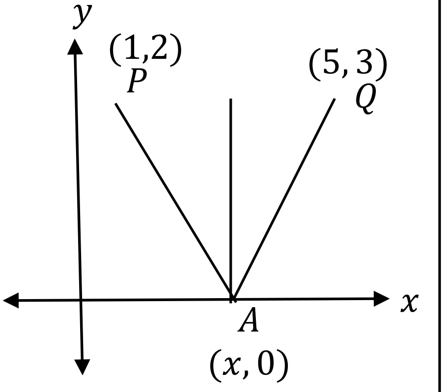
15. *A ray of light coming from the point* $(1, 2)$ *is reflected at a point A on the x − axis and it passes through the point* $(5, 3)$*. Find the co − oridinates point of A*

$$P(\underset{x_1}{1}, \underset{y_1}{2}),\ A(\underset{x_2}{x}, \underset{y_2}{0})$$

$$\text{Slope of } AP = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 2}{x - 1} \Longrightarrow \text{Slope of } AP = \frac{-2}{x - 1}$$

$$Q(\underset{x_1}{5}, \underset{y_1}{3}),\ A(\underset{x_2}{x}, \underset{y_2}{0})$$

$$\text{Slope of } AQ = \frac{0 - 3}{x - 5} \Longrightarrow \text{Slope of } AQ = \frac{-3}{x - 5}$$



Slopes are equal but opposite in sign

$$\frac{-2}{x - 1} = -\left(\frac{-3}{x - 5}\right) \Longrightarrow \frac{-2}{x - 1} = \frac{3}{x - 5}$$

$$-2(x - 5) = 3(x - 1) \Longrightarrow -2x + 10 = 3x - 3$$

$$10 + 3 = 3x + 2x \Longrightarrow 5x = 13 \Longrightarrow x = \frac{13}{5}$$

$$\therefore \text{The point } A \text{ is } \left(\frac{13}{5}, 0\right)$$

16. *A line is drawn perpendicular to* $5x = y + 7$*. Find the equation of the line if the area of the triangle formed by this line with co − ordinate axes is* 10 *sq. units.*

Given line is $5x = y + 7 \Longrightarrow 5x - y - 7 = 0$

Any line perpendicular to this $x + 5y + k = 0$

The line intersect x − axis i.e, $y = 0$

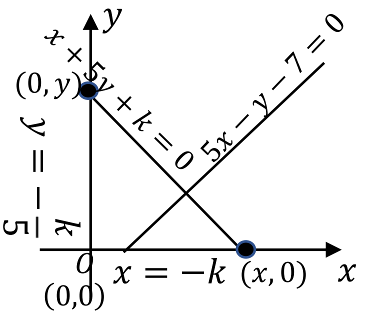
$$x + 5(0) + k = 0 \Longrightarrow x + k = 0$$

$$x = -k$$

The line intersect y − axis i.e, $x = 0$

$$0 + 5y + k = 0$$

$$5y + k = 0 \Longrightarrow 5y = -k \Longrightarrow y = -\frac{k}{5}$$



Area of the triangle $= 10$ *sq. units*

$$\frac{1}{2} \times \text{Base} \times \text{Height} = 10 \Longrightarrow \frac{1}{2}(-k)\left(\frac{-k}{5}\right) = 10$$

$$\frac{k^2}{10} = 10 \implies k^2 = 100 \implies k = \sqrt{100} \implies k = \pm 10$$

$The\ required\ lines\ are\ x + 5y \pm 10 = 0$

17. *Find the image of the point* $(-2, 3)$ *about the line* $x + 2y - 9 = 0$

$Let\ (h, k)\ be\ the\ image\ of\ (-2, 3)$

$Equation\ of\ line\ AB\ is\ x + 2y - 9 = 0 \ldots (1)$

$since\ PP'\ is\ perpendicular\ to\ AB$

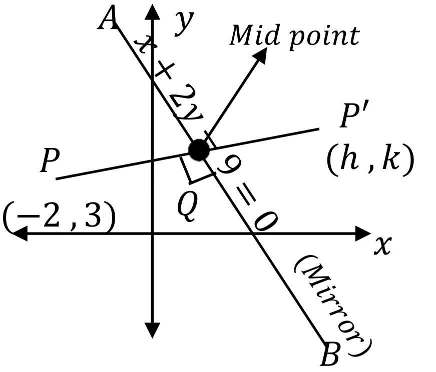
$Equation\ PP'\ is\ of\ the\ form: 2x - y + k = 0$

$it\ passes\ through\ the\ point\ (-2, 3)$

$$2(-2) - 3 + k = 0 \implies -4 - 3 + k = 0$$

$$-7 + k = 0 \implies k = 7$$

$Equation\ PP'\ is\ 2x - y + 7 = 0 \ \ldots (2)$



$To\ find\ point\ of\ intersection\ of\ AB\ and\ PP'\ solve\ (1)\ and\ (2)$

$$(1) \implies \quad x + \cancel{2y} - 9 = 0$$

$$(2) \times 2 \implies 4x - \cancel{2y} + 14 = 0$$

$$5x + 5 = 0 \implies \cancel{5x} = \overset{-1}{\cancel{-5}}$$

$$x = -1$$

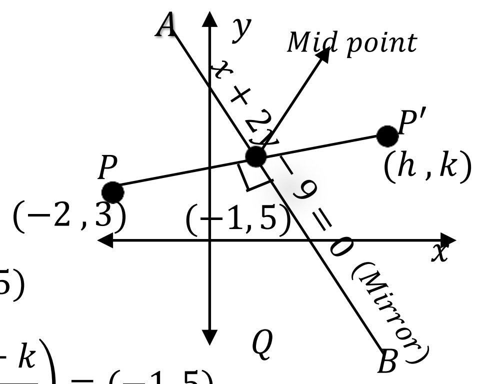
Sub $x = -1$ in (1) $x + 2y - 9 = 0$

$-1 + 2y - 9 = 0 \Rightarrow -10 + 2y = 0$

$\cancel{2}y = \overset{5}{\cancel{10}} \Rightarrow y = 5$

$y = 5$

$\therefore$ *The point of intersection is* $Q\ (-1, 5)$



Midpoint of $P(\underset{x_1,\,y_1}{-2\,,3})$ *and* $P'(\underset{x_2,\,y_2}{h\,,k}) = Q(-1, 5)$

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = (-1, 5) \Rightarrow \left(\frac{-2 + h}{2}, \frac{3 + k}{2}\right) = (-1, 5)$$

$$\frac{-2 + h}{2} = -1, \frac{3 + k}{2} = 5 \Rightarrow -2 + h = -2, 3 + k = 10$$

$$h = -2 + 2, k = 10 - 3 \Rightarrow h = 0, k = 7$$

$(0\,, 7)$ *is image of* $(-2\,, 3)$

18. *A photocopy store charges Rs. 1.50 per copy for the first 10 copies and Rs. 1.00 per copy after the* 10^{th} *copy. Let x be the number of copies, and let y be the total cost of photocopying.*

(*i*) *Draw graph of the cost as x goes from 0 to 50 copies.*

(*ii*) *Find the cost of making 40 copies*

No. of copies	0	1	2	4	10	20	30	40	50
Cost	0	1.5	3	6	15	25	35	45	55

Let y be the total cost and x be the no. of copy

$$y = (1.5)x \quad 0 \le x \le 10$$

Let y be the cost after 10 *copies*

$$y = 1.5 \times 10 + (x - 10) \times 1$$

$$= 15 + x - 10 = x + 5$$

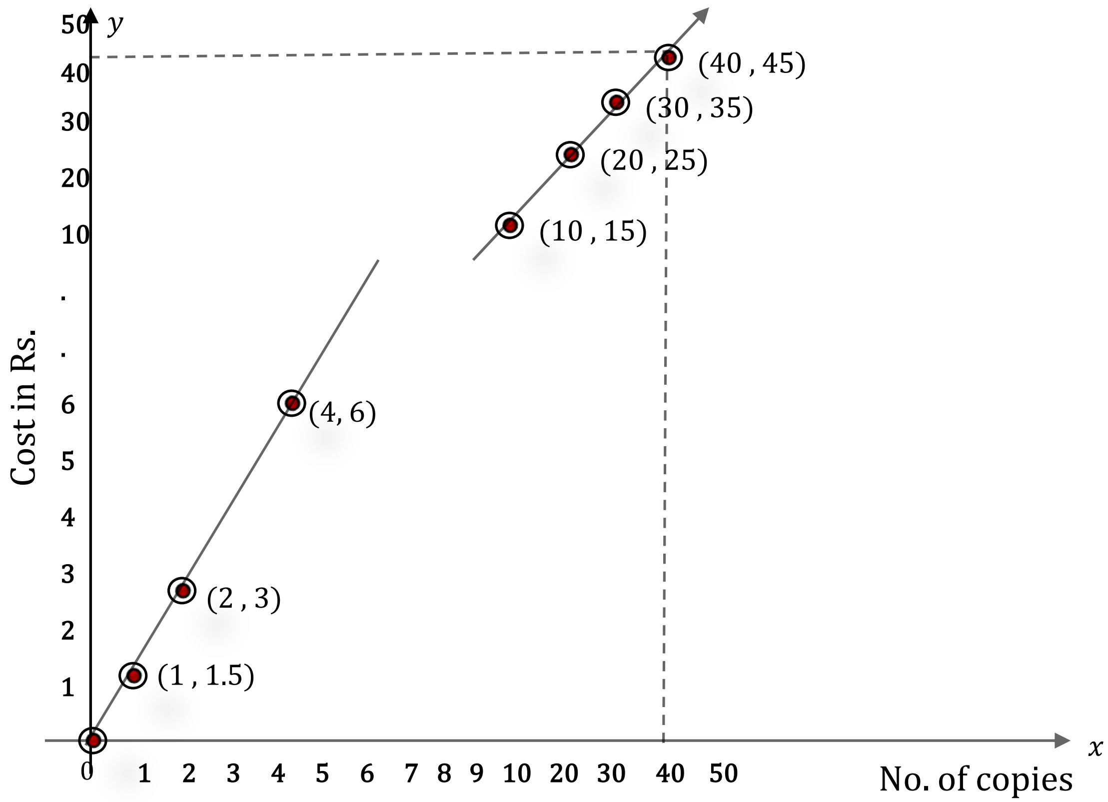
$$y = x + 5\,(10 \le x \le 50)$$

To find Cost of 40 *copies*:

$$y = x + 5 \quad when\ x = 40$$

$$y = 40 + 5 = 45$$

The points (0,0), (1,1.5) , (2, 3) , (4, 6) (10,15), (20,25), (30,35), (40,45), (50,55)



19. *Find atleast two equation of the straight lines in the family of the line $y = 5x + b$, for which b and the $x-$coordinate of the point of intersection of the lines with $3x - 4y = 6$ are integers*

Given lines $\quad y = 5x + b$

$$5x - y = -b \ldots (1) \text{ and } \quad 3x - 4y = 6 \ldots (2)$$

Solve (1) *and* (2)

$$(1) \times 4 \Rightarrow \quad 20x - \cancel{4y} = -4b$$

$$(2) \Rightarrow \quad \underset{(-)}{3x} \; \underset{(+)}{-\cancel{4y}} = \underset{(-)}{6}$$

$$17x = -4b - 6 \Rightarrow x = \frac{-4b-6}{17}$$

Given that x and b must be an integer

$$x = \frac{-4b-6}{17} \quad [-4b - 6 \text{ must be equal to } \pm 17, \pm 34, \ldots]$$

$$-4b - 6 = \pm 17 \Rightarrow -4b = \pm 17 + 6 \Rightarrow -4b = 17 + 6, -17 + 6$$

$$-4b = 23, -11 \Rightarrow b = -\frac{23}{4}, \frac{11}{4} \text{ is not an integer}$$

$$-4b - 6 = \pm 34 \Rightarrow -4b = \pm 34 + 6 \Rightarrow -4b = 34 + 6, -34 + 6$$

$$-4b = 40, -28 \Rightarrow b = \frac{40}{-4}, \frac{-28}{-4} \Rightarrow b = -10, 7$$

$$y = 5x + b$$

when $b = -10 \Rightarrow y = 5x - 10$

$when\ b = 7 \quad \Longrightarrow \quad y = 5x + 7$

20. $\boldsymbol{Find\ all\ the\ equation\ of\ the\ straight\ lines\ in\ the\ family\ of\ the\ lines\ y = mx - 3, for\ which\ m\ and\ the\ x - coordinate\ of\ the\ point\ of\ intersection\ of\ the\ lines\ with\ x - y = 6\ are\ integers.}$

$Given\ lines: \ y = mx - 3$

$mx - y = 3 \ldots (1)\ and\ x - y = 6 \ldots (2)$

$Solve\ (1)\ and\ (2)$

$$\begin{array}{rl} (1) \Longrightarrow & mx - \cancel{y} = 3 \\ & (-) \quad (+) \quad (-) \\ (2) \Longrightarrow & x - \cancel{y} = 6 \\ \hline & mx - x = -3 \end{array} \quad \Longrightarrow \quad x(m-1) = -3$$

$x = \dfrac{-3}{m-1}$ $since\ x\ and\ m\ must\ be\ an\ integer$

$[m - 1\ must\ be\ equal\ to \pm 1, \pm 3]$

$m - 1 = \pm 1 \Longrightarrow m = \pm 1 + 1 \Longrightarrow m = 1 + 1, -1 + 1$

$m = 2, 0$

$m - 1 = \pm 3 \Longrightarrow m = \pm 3 + 1 \Longrightarrow m = 3 + 1, -3 + 1$

$m = 4, -2$

$y = mx - 3$

$when\ m = 2 \Longrightarrow y = 2x - 3 \Longrightarrow 2x - y - 3 = 0$

$when\ m = 0 \Longrightarrow y = 0 - 3 \Longrightarrow y + 3 = 0$

$when\ m = 4 \Longrightarrow y = 4x - 3 \Longrightarrow 4x - y - 3 = 0$

$when\ m = -2 \Longrightarrow y = -2x - 3 \Longrightarrow 2x + y + 3 = 0$

EXERCISE : 6.4

Eg.6.33: Separate the equations $5x^2+6xy+y^2=0$.

$$+ \qquad \times$$
$$6 \qquad 5$$
$$\frac{1\cancel{5xy}}{\cancel{5x^2}}_{x} \qquad \frac{1\cancel{xy}}{5\cancel{x^2}}_{x}$$

$$5x^2+6xy+y^2=0$$
$$(5x+y)(x+y)=0$$
So that the lines are $5x+y=0$, *and* $x+y=0$

Eg.6.34: If exists, find the straight lines by separating the equation $2x^2+2xy+y^2=0$

Since the given equation is a homogeneous equation, divide the given equation by x^2

$$2x^2+2xy+y^2=0$$
$$\div x^2$$
$$\frac{2x^2}{x^2}+\frac{2xy}{x^2}+\frac{y^2}{x^2}=0 \Rightarrow 2+\frac{2y}{x}+\left(\frac{y}{x}\right)^2=0$$
$$sub\ \frac{y}{x}=m$$
$$m^2+2m+2=0$$
$$a=1, b=2, c=2$$
$$\Delta=\sqrt{b^2-4ac} \Rightarrow \Delta=\sqrt{2^2-4(1)(2)}$$
$$\Delta=\sqrt{4-8} \Rightarrow \Delta=\sqrt{-4}<0(imaginary)$$

The values of m *(slopes) are not real (complex number),*

$\therefore$ *No line will exist with the joint equation* $2x^2+2xy+y^2=0$

We sometimes say that this equation represents imaginary lines.

Note that in the entire plane, only $(0,0)$ *satisfies this equation.*

Eg.6.35: Find the equation of the pair through the origin and perpendicular to the pair of line $ax^2+2hxy+by^2=0$

Given that $ax^2+2hxy+by^2=0$

$$by^2+2hxy+ax^2=0$$
$$\div b$$
$$y^2+\frac{2h}{b}xy+\frac{a}{b}x^2=0$$
$$m_1+m_2=-\frac{2h}{b} \text{ and } m_1m_2=\frac{a}{b} \ \ldots(1)$$

Let m_1 *and* m_2 *be the slopes of these two lines.*

$$y-m_1x=0 \text{ and } y-m_2x=0\ldots(1)$$

The lines perpendicular to these two lines

$$\text{Slopes of perpendicular}=-\frac{1}{m}$$
$$m_1=-\frac{1}{m_1} \text{ and } m_2=-\frac{1}{m_2} \text{ in } (1)$$

$$y+\frac{1}{m_1}x=0 \quad and \quad y+\frac{1}{m_2}x=0$$

$$\frac{m_1\,y+x}{m_1}=0 \Rightarrow m_1\,y+x=0 \quad and \quad \frac{m_2y+x}{m_2}=0 \Rightarrow m_2\,y+x=0$$

The combined equation is $(m_1y+x)(m_2y+x)=0$

$$m_1m_2y^2+m_1xy+m_2xy+x^2=0$$

$$m_1m_2y^2+(m_1+m_2)xy+x^2=0$$

$$where\ m_1+m_2=-\frac{2h}{b} \quad and\ m_1m_2=\frac{a}{b}$$

$$\frac{a}{b}y^2-\frac{2h}{b}xy+x^2=0$$

$$\times b$$

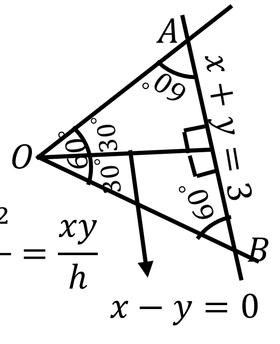
$$ay^2-2hxy+bx^2=0$$

The required equation is $ay^2-2hxy+bx^2=0$

Eg. 6.36: Show that the straight lines $x^2-4xy+y^2=0$ $x+y=3$ form an equilateral triangle

Let the line $x+y=3$ *intersects the pair of line* $x^2-4xy+y^2=0$ *at A and B.*

The angle between the lines $x^2-4xy+y^2=0$ *is*



$$tan\theta=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right| \Rightarrow tan\theta=\frac{2\sqrt{4-1}}{2}=\sqrt{3}$$

$$\theta=\ tan^{-1}\sqrt{3}=60^\circ$$

The angle bisectors of the angle AOB are given by $\dfrac{x^2-y^2}{a-b}=\dfrac{xy}{h}$

$$where\ a=1, b=1$$

$$\frac{x^2-y^2}{1-1}=\frac{xy}{h} \Rightarrow \frac{x^2-y^2}{0}=\frac{xy}{h}$$

$$x^2-y^2=0 \Rightarrow x+y=0 \ \ and\ \ x-y=0$$

The angle bisector $x-y=0$ *is perpendicular to the given line through AB*

$$x+y=3 \Rightarrow \Delta OAB\ is\ isosceles.$$

$$\angle ABO=\angle BAO=60^\circ$$

$\therefore$ *The given lines form an equilateral triangle.*

Eg. 6.37: If the pair of lines represented by $x^2-2cxy-y^2=0$ and $x^2-2dxy-y^2=0$ be such that each pair bisects the angle between the other pair, prove that $cd=-1$

Given that the pair of straight lines,

$$x^2-2cxy-y^2=0 \ldots(1)$$

$$x^2-2dxy-y^2=0 \ldots(2)$$

The equation of the angle bisectors of equation (1) is $\dfrac{x^2-y^2}{a-b}=\dfrac{xy}{h}$

$$where\ a=1, b=-1, 2h=-2c$$

$$h = -c$$

$$\frac{x^2 - y^2}{2} = \frac{xy}{-c} \implies -cx^2 + cy^2 = 2xy \implies cx^2 + 2xy - cy^2 = 0$$

$$cx^2 + 2xy - cy^2 = 0$$

$$\div c$$

$$x^2 + \frac{2}{c}xy - y^2 = 0 \ldots (3)$$

Equation (2) and (3) are same

Equate the coefficient of xy terms: $-2d = \frac{2}{c} \implies -d = \frac{1}{c}$

$$\boxed{cd = -1}$$

Eg. 6.38. If the equation $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represent a pair of straight lines, find

(i) the value of λ and the separate equation of the lines

(ii) point of intersection of the lines

(iii) angle between the lines.

$$\lambda x^2 - 10\,xy + 12y^2 + 5x - 16y - 3 = 0$$

Comparing with $ax^2 + 2hxy + by^2 + 2g\,x + 2fy + c = 0$

$$a = \lambda;\ 2h = -10,\ b = 12,\ 2g = 5,\quad 2f = -16,\ c = -3$$

$$h = -5 \qquad g = \frac{5}{2} \qquad f = -8$$

The condition to represent a pair of straight lines is
$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\lambda(12)(-3) + 2(-8)\left(\frac{5}{2}\right)(-5) - \lambda(-8)^2 - 12\left(\frac{5}{2}\right)^2 - (-3)(-5)^2 = 0$$

$$-36\lambda + 200 - 64\lambda \cancel{-75} \cancel{+75} = 0$$

$$-100\lambda = -200 \implies \lambda = 2$$

The pair of straight lines is $2x^2 - 10\,xy + 12y^2 + 5x - 16y - 3 = 0$

Consider quadratic equation :

$$2x^2 - 10xy - 12y^2 = (x - 2y)(2x - 6y)$$

$$(x - 2y)(2x - 6y) = 2x^2 - 10xy - 12y^2$$

$+$ $\times$
-10 -24
-6
$\frac{\cancel{2xy}}{\cancel{2x^2}}$ $\frac{-\cancel{12xy}}{\cancel{2x^2}}$
x x

$$(x - 2y + l)(2x - 6y + m) = 2x^2 - 10xy - 12y^2 + 5x - 16y - 3$$

Equating x − terms : $2lx + mx = 5x$

Equating y − terms : $-6ly - 2my = -16y$

Equating coefficient of x : $2l + m = 5 \ldots (1)$

Equating coefficient of y : $-6l - 2m = -16 \ldots (2)$

Solve (1) *and* (2)

$$(1)\times 2 \Longrightarrow \quad 4l+\cancel{2m}=10$$
$$(2) \qquad \Longrightarrow -6l-\cancel{2m}=-16$$
$$-2l=-6$$
$$\boxed{l=3}$$

Sub $l=3$ *in* (1)

$$2l+m=5$$
$$3(3)+4m=5$$
$$9+4m=5 \Longrightarrow 4m=5-9$$
$$4m=-4$$
$$\boxed{m=-1}$$

$\therefore$ *The separate equations are* $x-2y+3=0$ *and* $2x-6y-1=0$

(*ii*) *point of intersection of the lines is given by solving the two equation of the lines,*

$$\text{or use the formula } \left(\frac{hf-bg}{ab-h^2}, \frac{hg-af}{ab-h^2}\right)$$

$$a=2,\ b=12,\ h=-5,\ g=\frac{5}{2},\ f=-8,\ c=-3$$

$$=\left[\frac{(-5)(-8)-12\left(\frac{5}{2}\right)}{2(12)-(-5)^2}, \frac{(-5)\left(\frac{5}{2}\right)-2(-8)}{2(12)-(-5)^2}\right]=\left[\frac{40-6(5)}{24-25}, \frac{\frac{-25}{2}+16}{24-25}\right]$$

$$=\left[\frac{40-30}{-1}, \frac{\frac{-25+32}{2}}{-1}\right]$$

$$(x,y)=\left(-10,-\frac{7}{2}\right)$$

(*iii*) *Angle between the lines is given by* $\tan\theta=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$

$$=\frac{2\sqrt{25-24}}{2+12}=\frac{1}{7}$$

$$\boxed{\theta=\tan^{-1}\left(\frac{1}{7}\right)}$$

Eg. 6.39: A student when walks from his house, at an average speed of 6 kmph, reaches his school ten minutes before the school starts. When his average speed is 4 kmph, he reaches his school five minutes late. If he starts to school every day at 8.00 A.M, then find
(i) the distance between his house and the school
(ii) the minimum average speed to reach the school on time and time taken to reach the school
(iii) the time the school gate closes
(iv) the pair of straight lines of his path of walk.

Let $x-$*axis be the time in hours and* $y-$*axis be the distance in kilometer.*

Let x *be the time and* y *be the distance*

$y = Speed \times Time$

$y = 6\left(x - \frac{10}{60}\right)$ $\boxed{10mins = \frac{10}{60}hrs}$

$$Speed = \frac{Distance}{Time}$$

$$Distance = Speed \times Time$$

$$y = 6\left(\frac{60x - 10}{60}\right) \Rightarrow y = 60\left(\frac{6x - 1}{60}\right)$$

$$y = 6x - 1 \Rightarrow 6x - y = 1 \ldots(1)$$

$$y = 4\left(x + \frac{5}{60}\right) \Rightarrow y = 4\left(\frac{60x + 5}{60}\right) \Rightarrow y = 4 \times 5\left(\frac{12x + 1}{60}\right)$$

$$y = 20\left(\frac{12x + 1}{60}\right) \Rightarrow y = \frac{12x + 1}{3} \Rightarrow 3y = 12x + 1$$

$$12x - 3y = -1 \ldots(2)$$

Solving (1) *and* (2)

$$(1) \times 3 \Rightarrow \quad 18x - \cancel{3y} = 3$$

$$(2) \quad \Rightarrow \quad \underset{(-)}{12x} \underset{(+)}{- \cancel{3y}} = \underset{(+)}{-1}$$

$$6x = 4 \Rightarrow x = \frac{4}{6}$$

$$x = \frac{2}{3}$$

$sub\ x = \frac{2}{3}\ in\ (1)\ \ 6x - y = 1$

$$6\left(\frac{2}{3}\right) - y = 1 \Rightarrow 4 - y = 1 \Rightarrow -y = 1 - 4 \Rightarrow -y = -3$$

$$y = 3$$

$$(x, y) = \left(\frac{2}{3}, 3\right)$$

$$x = \frac{2}{3}hour \Rightarrow x = \frac{2}{3} \times 60mins$$

$$x = 40minutes, \quad y = 3km$$

(i) *The distance between his house and the school* $= 3km$

(ii) *the minimum average speed to reach the school on time*

$$Average\ speed = \frac{Distance}{Time} = \frac{3}{\frac{2}{3}} = 3 \times \frac{3}{2} = \frac{9}{2} = 4.5km/h$$

Time taken : $40minutes$

(iii) *The school gate closes at* 8.40 *AM*

(iv) *The pair of straight lines of his path of walk to school is*

$$(6x - y - 1)(12x - 3y + 1) = 0$$

$$72x^2 - 30xy + 3y^2 - 6x + 2y - 1 = 0$$

Eg. 6.40: *If one of the straight lines of $ax^2 + 2hxy + by^2 = 0$ is perpendicular to $px + qy = 0$, then show that $ap^2 + 2hpq + bp^2 = 0$*

Let 'm_1' and 'm_2' be the slopes of pair of straight lines.

$$m_1 + m_2 = -\frac{2h}{b}, \quad m_1 m_2 = \frac{a}{b}$$

Slope of $px + qy = 0$ *is* $m = -\frac{p}{q}$

Since one of the straight lines of $ax^2 + 2hxy + by^2 = 0$ *is perpendicular to* $px + qy = 0$,

$$mm_1 = -1 \quad or \quad mm_2 = -1$$

$$(mm_1 + 1)(mm_2 + 1) = 0$$

$$m^2 m_1 m_2 + mm_1 + mm_2 + 1 = 0 \implies (m_1 m_2)m^2 + m(m_1 + m_2) + 1 = 0$$

$$(m_1 m_2)m^2 + m(m_1 + m_2) + 1 = 0$$

$$\left(\frac{a}{b}\right)\left(-\frac{p}{q}\right)^2 + \left(-\frac{p}{q}\right)\left(-\frac{2h}{b}\right) + 1 = 0$$

$$\left(\frac{a}{b}\right)\left(\frac{p^2}{q^2}\right) + \left(\frac{2hp}{bq}\right) + 1 = 0$$

$$\frac{ap^2}{bq^2} + \frac{2hp}{bq} + 1 = 0 \implies \frac{ap^2 + 2hpq + bq^2}{bq^2} = 0$$

$$ap^2 + 2hpq + bq^2 = 0$$

Eg 6.41: *Show that the straight lines joining the origin to the points of intersection of $3x - 2y + 2 = 0$ and $3x^2 + 5xy - 2y^2 + 4x + 5y = 0$ are at right angles.*

The straight lines joining the origin and the points of intersection of given equationis a second degree homogeneous equation.

Following steps show, the way of homogenizing the $3x^2 + 5xy - 2y^2 + 4x + 5y = 0$ *with* $3x - 2y + 2 = 0$

$$\boxed{\begin{gathered} 3x - 2y = -2 \\ \div -2 \\ \frac{3x - 2y}{-2} = 1 \end{gathered}}$$

$$3x^2 + 5xy - 2y^2 + (4x + 5y)(1) = 0 \ and \ \frac{(3x - 2y)}{-2} = 1$$

$$3x^2 + 5xy - 2y^2 + (4x + 5y)\left(\frac{3x - 2y}{-2}\right) = 0$$

$$\times -2$$

$$(-2)(3x^2 + 5xy - 2y^2) + (4x + 5y)(3x - 2y) = 0$$

$$-6x^2 - 10xy + 4y^2 + 12x^2 - 8xy + 15xy - 10y^2 = 0$$

$$6x^2 - 3xy - 6y^2 = 0$$

$$\div 3$$

$$2x^2 + xy - 2y^2 = 0$$

$$a = 6, b = -6 \implies a + b = 0$$

$\therefore$ *The lines are at right angles.*

1. *Find the combined equation of the straight lines whose separate equations are* $x-2y-3=0$ *and* $x+y+5=0$.

The separate equations are $x-2y-3=0, x+y+5=0$

$\therefore$ *Combined equation is* $(x-2y-3)(x-y+5)=0$

$$x^2-xy+5x-2xy+2y^2-10y-3x+3y-15=0$$

$$x^2-3xy-2y^2+2x-7y-15=0$$

2. *Show that* $4x^2+4xy+y^2-6x-3y-4=0$ *represent a pair of parallel lines*.

$$4x^2+4xy+y^2-6x-3y-4=0$$

Comparing with $ax^2+2hxy+by^2+2gx+2fy+c=0$

$$a=4; 2h=4, b=1, 2g=-6, 2f=-3, c=-4$$

$$h=2 \qquad g=-3 \quad , f=-\frac{3}{2}$$

$$h^2-ab=2^2-(4)(1)=4-4=0$$

$h^2-ab=0 \Longrightarrow$ *The given equation represents a pair of parallel lines.*

3. *Show that* $2x^2+3xy-2y^2+3x+y+1=0$ *represent a pair of perpendicular lines*.

$$2x^2+3xy-2y^2+3x+y+1=0$$

Comparing with $ax^2+2hxy+by^2+2gx+2fy+c=0$

$$a=2;\ 2h=3,\ b=-2, 2g=3,\ 2f=1,\ c=1$$

$$h=\frac{3}{2} \qquad g=\frac{3}{2} \qquad f=\frac{1}{2}$$

$$a+b=2+(-2)=0$$

$\therefore$ *It represents a pair of perpendicular lines.*

4. *Show that the equation* $2x^2-xy-3y^2-6x+19y-20=0$ *represent a pair of intersecting lines. Show further that the angle between them is* $tan^{-1}(5)$.

$$2x^2-1xy-3y^2-6x+19y-20=0$$

Comparing with $ax^2+2hxy+by^2+2gx+2fy+c=0$

$$a=2;\ 2h=-1, b=-3, 2g=-6, 2f=19, c=-20$$

$$h=-\frac{1}{2} \qquad g=-3 \qquad f=\frac{19}{2}$$

The condition to represent a pair of straight lines :
$abc+2fgh-af^2-bg^2-ch^2=0$

$$(2)(-3)(-20)+2\left(\frac{19}{2}\right)(-3)\left(\frac{-1}{2}\right)-(2)\left(\frac{19}{2}\right)^2-(-3)(3)^2-(-20)\left(\frac{-1}{2}\right)^2=0$$

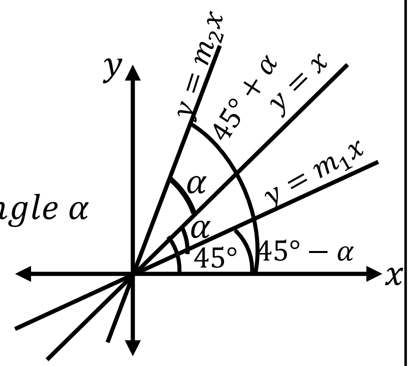
$120+\frac{57}{2}-\frac{361}{2}+27+5=0 \therefore$ *It represent a pair of straight lines.*

$$Angle\ between\ the\ lines\ is\ given\ by\ \ tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$= \left|\frac{2\sqrt{\frac{1}{4}+6}}{2-3}\right| = \left|\frac{2\sqrt{\frac{1+24}{4}}}{-1}\right| = \left|\frac{2\sqrt{\frac{25}{4}}}{-1}\right| = \left|\frac{2\times\frac{5}{2}}{-1}\right| = |-5|$$

$\theta = \tan^{-1} 5$ *is the acute angle between the lines.*

5. *Prove that the equation to the straight lines through the origin, each of which makes an angle α with the straight line $y = x$ is $x^2 - 2xy\ \sec 2\alpha + y^2 = 0$.*



Given line $y = x$

compare : $y = mx$

$$m = 1 \Rightarrow tan\theta = 1$$

$$\theta = 45^\circ$$

Let the lines be $y = m_1x$ *and* $y = m_2x$ *makes an angle* α *with the line* $y = x$

Here $m_1 = tan(45° - \alpha)$ *and* $m_2 = tan(45° + \alpha)$

$$m_1 = \frac{tan45° - tan\alpha}{1 + tan45° tan\alpha} \Rightarrow m_1 = \frac{1 - tan\alpha}{1 + tan\alpha}$$

$$m_2 = \frac{tan45° + tan\alpha}{1 - tan45° tan\alpha} \Rightarrow m_2 = \frac{1 + tan\alpha}{1 - tan\alpha}$$

$$Product\ of\ the\ slopes:\ m_1 \times m_2 = \frac{1 - \cancel{tan\alpha}}{1 + \cancel{tan\alpha}} \times \frac{1 + \cancel{tan\alpha}}{1 - \cancel{tan\alpha}}$$

$$Product\ of\ the\ slopes:\ m_1 \times m_2 = 1$$

$$Sum\ of\ the\ slopes:\ m_1 + m_2 = \frac{1 - tan\alpha}{1 + tan\alpha} + \frac{1 + tan\alpha}{1 - tan\alpha}$$

$$m_1 + m_2 = \frac{(1 - tan\alpha)^2 + (1 + tan\alpha)^2}{(1 + tan\alpha)(1 - tan\alpha)}$$

$$m_1 + m_2 = \frac{1 - 2(1)(tan\alpha) + tan^2\alpha + 1 + 2(1)(tan\alpha) + tan^2\alpha}{1^2 - tan^2\alpha}$$

$$m_1 + m_2 = \frac{1 + \cancel{2tan\alpha} + tan^2\alpha + 1 - \cancel{2tan\alpha} + tan^2\alpha}{1^2 - tan^2\alpha}$$

$$m_1 + m_2 = \frac{2 + 2tan^2\alpha}{1 - tan^2\alpha} \Rightarrow m_1 + m_2 = \frac{2(1 + tan^2\alpha)}{1 - tan^2\alpha}$$

$$m_1 + m_2 = 2\left(\frac{1}{cos2\alpha}\right) \Rightarrow m_1 + m_2 = 2sec2\alpha$$

$$\boxed{cos\,2A = \frac{1 - tan^2 A}{1 + tan^2 A} \qquad \frac{1}{cos\,2A} = \frac{1 + tan^2 A}{1 - tan^2 A}}$$

$y = m_1x \implies m_1x - y = 0$

$y = m_2x \implies m_2x - y = 0$

The combined equation is $(m_1x - y)(m_2x - y) = 0$

$m_1m_2x^2 - m_1xy - m_2xy + y^2 = 0$

$m_1m_2x^2 - xy\,(m_1 + m_2) + y^2 = 0$

where $m_1m_2 = 1$ *and* $m_1 + m_2 = 2sec2\alpha$

$x^2 - xy\,(2sec2\alpha) + y^2 = 0 \implies x^2 - \ 2xy\, sec2\alpha + y^2 = 0$

6. *Find the equation of the pair of straight lines passing through the point* $(1, 3)$ *and perpendicular to the lines* $2x - 3y + 1 = 0$ *and* $5x + y - 3 = 0$.

The straight line perpendicular to $2x - 3y + 1 = 0$ is of the form

$3x + 2y + k = 0$

it passes through the point $(1\,,3)$

$3 + 6 + k = 0$

$9 + k = 0 \implies k = -9$

$\therefore$ $3x + 2y - 9 = 0$ is the straight line perpendicular to $2x - 3y + 1 = 0$

The straight line perpendicular to $5x + y - 3 = 0$ is of the form

$x - 5y + k = 0$

it passes through the point $(1\,,3)$

$1 - 15 + k = 0 \implies k = 14$

$\therefore$ $x - 5y + 14 = 0$ is the straight line perpendicular to $5x + y - 3 = 0$

Combined equation is $(3x + 2y - 9)\,(x - 5y + 14) = 0$

$3x^2 - 15xy + 42x + 2xy - 10y^2 + 28y - 9x + 45y - 126 = 0$

$3x^2 - 13xy - 10y^2 + 33x + 73y - 126 = 0$

7. *Find the separate equation of the following pair of straight lines*

(*i*) $3x^2 + 2xy - y^2 = 0$.

(*ii*) $6(x-1)^2 + 5(x-1)(y-2) - 4(y-2)^2 = 0$

(*iii*) $2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$

(*i*) $3x^2 + 2xy - y^2 = 0$

$3x^2 + 3xy - xy - y^2 = 0 \implies 3x(x + y) - y(x + y) = 0$

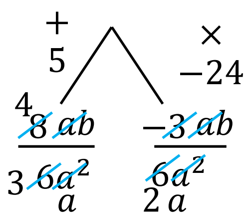
$(x + y)(3x - y) = 0$

The separate equation is $x + y = 0$ *and* $3x - y = 0$

(*ii*) $6(x-1)^2 + 5(x-1)(y-2) - 4(y-2)^2 = 0$

Let $a = x - 1, b = y - 2$

$6a^2 + 5ab - 4b^2 = 0$



$(3a + 4b)(2a - b) = 0$

$[3(x-1) + 4(y-2)][2(x-1) - (y-2)] = 0$

$(3x - 3 + 4y - 8)\,(2x - 2 - y + 2) = 0$

$$(3x+4y-1)(2x-y)=0$$

Separate equations are $3x+4y-1=0$ *and* $2x-y=0$

(iii) $2x^2-xy-3y^2-6x+19y-20=0$

Let us take : $2x^2-xy-3y^2$

$+ \quad \times$

$-1 \quad 6$

$-3xy \quad 2xy$

$$2x^2-xy-3y^2=2x^2-3xy+2xy-3y^2$$
$$=x(2x-3y)+y(2x-3y)$$
$$2x^2-xy-3y^2=(2x-3y)(x+y)$$

$$2x^2-xy-3y^2-6x+19y-20=(2x-3y+l)(x+y+m)$$

Equating x terms: $lx+2mx=-6x$

Equating y terms: $ly-3my=19y$

Equating coefficient of x : $l+2m=-6\ \dots(1)$

Equating coefficient of y : $l-3m=19\ \dots(2)$

Solve (1) and (2)

$$\cancel{l}+2m=-6$$
$$(-)\ (+)\qquad(-)$$
$$\cancel{l}-3m=19$$

$$5m=-25\Rightarrow m=-5$$

Sub $m=-5$ *in* (1) $l+2m=-6$

$$l+2(-5)=-6\Rightarrow l-10=-6$$

$$l=-6+10\Rightarrow\boxed{l=4}$$

$\therefore$ *The separate equations are* $2x-3y+4=0$ *and* $x+y-5=0.$

8. *The slope of one of the straight lines $ax^2+2hxy+by^2=0$ is twice that of the other, Show that $8h^2=9ab$.*

Let 'm_1' and 'm_2' be the slopes of pair of straight lines.

$$m_1m_2=\frac{a}{b}\ \text{ and }\ m_1+m_2=-\frac{2h}{b}$$

Given : $m_1=2m_2$

$$2m_2+m_2=-\frac{2h}{b}\Rightarrow 3m_2=-\frac{2h}{b}\Rightarrow m_2=-\frac{2h}{b}\times\frac{1}{3}\Rightarrow m_2=-\frac{2h}{3b}$$

$$m_1m_2=\frac{a}{b}\Rightarrow 2m_2\times m_2=\frac{a}{b}\Rightarrow 2{m_2}^2=\frac{a}{b}\ \ Sub\ m_2=-\frac{2h}{3b}$$

$$2\left(-\frac{2h}{3b}\right)^2=\frac{a}{b}\Rightarrow 2\times\frac{4h^2}{9\cancel{b^2}_{\,b}}=\frac{a}{\cancel{b}}$$

$$\frac{8h^2}{9b}=a\ \Rightarrow\ 8h^2=9ab$$

9. *The slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is thrice that of the other, Show that $3h^2 = 4ab$.*

Let 'm_1' and 'm_2' be the slopes of pair of straight lines.

$$m_1 m_2 = \frac{a}{b} \quad and \quad m_1 + m_2 = -\frac{2h}{b}$$

$Given : m_1 = 3m_2$

$$3m_2 + m_2 = -\frac{2h}{b} \Rightarrow 4m_2 = -\frac{2h}{b} \Rightarrow m_2 = -\frac{2h}{b} \times \frac{1}{4} \Rightarrow m_2 = -\frac{h}{2b}$$

$$m_1 m_2 = \frac{a}{b} \Rightarrow 3m_2 \times m_2 = \frac{a}{b} \Rightarrow 3{m_2}^2 = \frac{a}{b} \; Sub\; m_2 = -\frac{h}{2b}$$

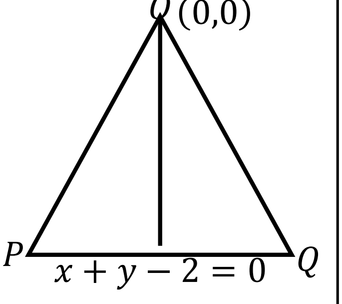
$$3\left(-\frac{h}{2b}\right)^2 = \frac{a}{b} \Rightarrow 3 \times \frac{h^2}{2\cancel{b^2}_{b}} = \frac{a}{\cancel{b}}$$

$$\frac{3h^2}{4b} = a \Rightarrow 3h^2 = 4ab$$

10. *A ΔOPQ is formed by the pair of straight lines $x^2 - 4xy + y^2 = 0$ and the line PQ. The equation of PQ is $x + y - 2 = 0$ Find the equation of the median of the triangle ΔOPQ drawn from the origin o.*

Equation of pair of st. line of OP & OQ: $x^2 - 4xy + y^2 = 0 \ldots (1)$

Equation of st. line PQ: $x + y - 2 = 0 \ldots (2)$



From $(2)\; x = 2 - y$

Sub $x = 2 - y$ *in* $(1)\; x^2 - 4xy + y^2 = 0$

$$(2-y)^2 - 4(2-y)y + y^2 = 0$$

$$4 - 4y + y^2 - 8y + 4y^2 + y^2 = 0$$

$$6y^2 - 12y + 4 = 0 \Rightarrow 3y^2 - 6y + 2 = 0$$

$\div 2$ $\qquad a = 3, b = -6, c = 2$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow y = \frac{6 \pm \sqrt{36 - 24}}{6} = \frac{6 \pm \sqrt{12}}{6}$$

$$y = \frac{6 \pm 2\sqrt{3}}{6} \Rightarrow y = \frac{6}{6} \pm \frac{2\sqrt{3}}{6} \Rightarrow y = \left(1 \pm \frac{\sqrt{3}}{3}\right)$$

$$y = 1 \pm \frac{\sqrt{3}}{\sqrt{3} \times \sqrt{3}} \Rightarrow y = 1 \pm \frac{1}{\sqrt{3}}$$

$$sub\; y = 1 \pm \frac{1}{\sqrt{3}} \; in\; (2)\; x = 2 - y$$

$$x = 2 - \left(1 \pm \frac{1}{\sqrt{3}}\right) \Rightarrow x = 2 - 1 \pm \frac{1}{\sqrt{3}}$$

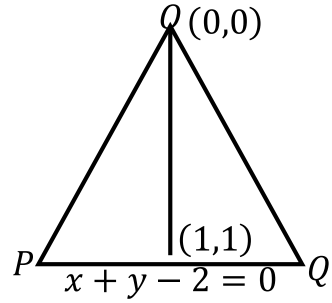
$$x = 1 \pm \frac{1}{\sqrt{3}}$$

$$P \text{ is } \left(1+\frac{1}{\sqrt{3}}, 1-\frac{1}{\sqrt{3}}\right) \text{ and } Q \text{ is } \left(1-\frac{1}{\sqrt{3}}, 1+\frac{1}{\sqrt{3}}\right)$$

$$\text{Midpoint of } PQ = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$\text{Midpoint of } PQ = \left(\frac{\left(1+\frac{1}{\sqrt{3}}+1-\frac{1}{\sqrt{3}}\right)}{2}, \frac{\left(1-\frac{1}{\sqrt{3}}+1+\frac{1}{\sqrt{3}}\right)}{2}\right)$$

$$= \left[\frac{2}{2}, \frac{2}{2}\right] = (1,1)$$



$$\text{Equation of line joining } \underset{x_1\ y_1}{(0,0)} \text{ and } \underset{x_2\ y_2}{(1,1)}$$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-0}{1-0} = \frac{x-0}{1-0}$$

$$\frac{y}{1} = \frac{x}{1} \Rightarrow y = x$$

$x - y = 0$ *is the equation of medium of* ΔPOQ *from the origin*

11. *Find p and q, if the following equation represents a pair of perpendicular lines.* $\boldsymbol{6x^2+5xy-py^2+7x+qy-5=0}$.

$$6x^2+5xy-py^2+7x+qy-5=0.$$

since it represent a pair of perpendicular lines.

$$\text{coefficient of } x^2 + \text{coefficient of } y^2 = 0$$

$$6-p=0 \Rightarrow p=6$$

$$6x^2+5xy-6y^2+7x+qy-5=0$$

Comparing with $ax^2+2hxy+by^2+2gx+2fy+c=0$

$$a=6;\quad 2h=5,\quad b=-6, 2g=7,\quad 2f=q,\quad c=-5$$

$$h=\frac{5}{2} \qquad g=\frac{7}{2} \qquad f=\frac{q}{2}$$

The condition to represent a pair of straight lines is
$abc+2fgh-af^2-bg^2-ch^2=0$

$$(6)(-6)(-5)+2\left(\frac{q}{2}\right)\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)-(6)\left(\frac{q}{2}\right)^2-(-6)\left(\frac{7}{2}\right)^2-(-5)\left(\frac{5}{2}\right)^2=0$$

$$180+\frac{35q}{4}-\frac{6q^2}{4}+6\left(\frac{49}{4}\right)+5\left(\frac{25}{4}\right)=0$$

$\times 4$

$$720+35q-6q^2+294+125=0$$

$$-6q^2+35q+1139=0$$

$$6q^2-35q-1139=0$$

$+$ -35 $\times$ 6834

-102 67

$$6q^2-102q+67q-1139=0 \Rightarrow 6q(q-17)+67(q-17)=0$$

$(q-17)(6q+67)=0 \Rightarrow q-17=0,\ 6q+67=0$

$\boxed{q=17,}$ $6q=-67$

$$\boxed{q=-\frac{67}{6}}$$

12. *Find k such that the equation $12x^2+7xy-12y^2-x+7y+k=0$ represents a pair of straight lines. Find whether these lines are parallel or intersecting.*

$$12x^2+7xy-12y^2-x+7y+k=0$$

Comparing with $ax^2+2hxy+by^2+2gx+2fy+c=0$

$$a=12;\ 2h=7,\ b=-12,\ 2g=-1,\ 2f=7,\ c=k$$

$$h=\frac{7}{2} \qquad g=\frac{-1}{2} \quad f=\frac{7}{2}$$

The condition to represent a pair of straight lines is
$\text{abc}+2\text{fgh}-\text{af}^2-\text{bg}^2-\text{ch}^2=0$

$$12\times-12\times k+\cancel{2}\times\frac{7}{\cancel{2}}\times-\frac{1}{2}\times\frac{7}{2}-12\left(\frac{7}{2}\right)^2+12\left(-\frac{1}{2}\right)^2-k\times\left(\frac{7}{2}\right)^2=0$$

$$-144k-\frac{49}{4}-12\times\frac{49}{4}+12\times\frac{1}{4}-k\times\frac{49}{4}=0$$

$$-144k-\frac{49}{4}-\frac{588}{4}+\frac{12}{4}-\frac{49k}{4}=0$$

$$-144k-\frac{49}{4}-\frac{588}{4}+\frac{12}{4}-\frac{49k}{4}=0 \ [\times \text{ by 4 on both side}]$$

$$-576k-49-588+12-49k=0 \Rightarrow -625k-625=0$$

$$-625k=625 \Rightarrow -k=1 \Rightarrow \boxed{k=-1}$$

The pair of straight lines is $12x^2+7xy-12y^2-x+7y-1=0$

$$\textit{coefficient of } x^2+\textit{coefficient of } y^2=0$$

The lines are perpendicular to each other or intersecting..

13. *For what value of k does the equation $12x^2+2kxy+2y^2+11x-5y+2=0$ represent two straight lines.*

$$12x^2+2kxy+2y^2+11x-5y+2=0$$

Comparing with $ax^2+2hxy+by^2+2gx+2fy+c=0$

$$a=12;\ 2h=2k, b=2,\ 2g=11,\ 2f=-5, c=2$$

$$h=k \qquad g=\frac{11}{2} \qquad f=\frac{-5}{2}$$

The condition to represent a pair of straight lines is
$\text{abc}+2\text{fgh}-\text{af}^2-\text{bg}^2-\text{ch}^2=0$

$$(12)(2)(2) + 2\left(\frac{-5}{2}\right)\left(\frac{11}{2}\right)k - (12)\left(\frac{-5}{2}\right)^2 - (2)\left(\frac{11}{2}\right)^2 - 2k^2 = 0$$

$$48 - \frac{55k}{2} - \overset{3}{\cancel{12}}\left(\frac{25}{\cancel{4}}\right) - 2\left(\frac{121}{4}\right) - 2k^2 = 0$$

$$48 - \frac{55k}{2} - 75 - \frac{121}{2} - 2k^2 = 0 \implies 96 - 55k - 150 - 121 - 4k^2 = 0$$

$$-4k^2 - 55k - 175 = 0 \implies 4k^2 + 55k + 175 = 0$$

$$4k^2 + 35k + 20k + 175 = 0 \implies k(4k + 35) + 5(4k + 35) = 0$$

$$(k + 5)(4k + 35) = 0 \implies k + 5 = 0,\ 4k + 35 = 0$$

$$\boxed{k = -5} \qquad 4k = -35$$

$$\boxed{k = -\frac{35}{4}}$$

$+$ 55 $\quad \times$ 700

35 $\quad$ 20

14. Show that $9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$ represents a pair of parallel lines and find the distance between them.

$$9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$$

Comparing with $\quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$$a = 9; 2h = -24, b = 16, 2g = -12, 2f = 16, c = -12$$

$$h = -12 \qquad g = -6 \qquad f = 8$$

$+$ -24 $\quad \times$ 144

$\frac{\overset{4}{-\cancel{12xy}}}{\underset{3}{\cancel{9x^2}}} = \frac{4y}{3x}$... $\quad \frac{-\cancel{12xy}}{\cancel{9x^2}}$

$$h^2 - ab = (-12)^2 - (9)(16) = 144 - 144 = 0$$

$h^2 - ab = 0$ ∴ the given equation represents a pair of parallel lines.

Consider quadratic equation : $9x^2 - 24xy + 16y^2 = (3x - 4y)(3x - 4y)$

$$(3x - 4y)(3x - 4y) = 9x^2 - 24xy + 16y^2$$

$$(3x - 4y + l)(3x - 4y + m) = 9x^2 - 24xy + 16y^2 - 12x + 16y - 12$$

$$\textit{Equating } x - \textit{terms}: 3lx + 3mx = -12x$$

$$\textit{Equating coefficient of } x: \ 3l + 3m = -12 \implies l + m = -4 \ \dots(1)$$

$$\div 3$$

$$\textit{Constant term}: lm = -12$$

$$m = -\frac{12}{l} \ \textit{in} \ (1) \ l + m = -4$$

$$l - \frac{12}{l} = -4 \implies \frac{l^2 - 12}{l} = -4 \implies l^2 - 12 = -4l$$

$$l^2 + 4l - 12 = 0 \implies (l - 2)(l + 6) = 0 \implies l - 2 = 0, l + 6 = 0$$

$$l = 2 \ \textit{and} \ l = -6$$

$$\textit{sub } l = 2 \textit{ in } \ m = \frac{-12}{l} \implies m = \frac{-12}{2} \implies m = -6$$

$$l = 2, m = -6$$

$sub\ l = -6\ in\ m = \frac{-12}{-6} \Longrightarrow m = 2$

$l = -6, m = 2$

$\therefore$ *The separate equations are* $3x - 4y - 6 = 0$ *and* $3x - 4y + 2 = 0$

where $a = 3\ , b = -4\ , c_1 = -6$ *and* $c_2 = 2$

The distance between the parallel lines $= \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$

$$= \left|\frac{-6 - 2}{\sqrt{3^2 + (-4)^2}}\right| = \left|\frac{-8}{\sqrt{9 + 16}}\right|$$

$$= \left|\frac{-8}{\sqrt{25}}\right| = \frac{8}{5}\ units$$

15. $\boldsymbol{Show\ that\ 4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0\ represents\ a\ pair\ of\ parallel\ lines\ and\ find\ the\ distance\ between\ them.}$

$$4x^2 + 4xy + 1y^2 - 6x - 3y - 4 = 0$$

Comparing with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

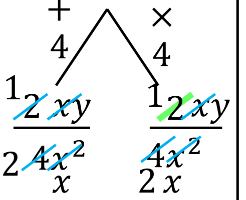
$a = 4; 2h = 4, \quad b = 1, \quad 2g = -6, 2f = -3, c = -4$

$h = 2 \qquad g = -3 \qquad f = \frac{-3}{2}$

$h^2 - ab = 2^2 - (4)(1) = 4 - 4 = 0$

$h^2 - ab = 0 \therefore$ *given equation represents a pair of parallel lines.*

Consider quadratic equation : $4x^2 + 4xy + y^2 = (2x + y)(2x + y)$



$(2x + y)(2x + y) = 4x^2 + 4xy + y^2$

$(2x + y + l\ )(2x + y + m) = 4x^2 + 4xy + 1y^2 - 6x - 3y - 4$

Equating x *– terms* : $2lx + 2mx = -6x$

Equating coefficient of x : $2l + 2m = -6 \Longrightarrow l + m = -3 \ldots (1)$
$\div 2$

Constant term : $lm = -4$

$m = -\frac{4}{l}$ in (1) $l + m = -3 \Longrightarrow l - \frac{4}{l} = -3 \Longrightarrow \frac{l^2 - 4}{l} = -3$

$l^2 - 4 = -3l \Longrightarrow l^2 + 3l - 4 = 0 \Longrightarrow (l - 1)(l + 4) = 0$

$l - 1 = 0, l + 4 = 0 \Longrightarrow l = 1$ *and* $l = -4$

$sub\ l = 1\ in\ m = \frac{-4}{l} \Longrightarrow m = \frac{-4}{1} \Longrightarrow m = -4$

$l = 1, m = -4$

$sub\ l = -4\ in m = \frac{-4}{-4} \Longrightarrow m = 1$

$l = -4, m = 1$

$\therefore$ *The separate equations are* $2x - y - 4 = 0$ *and* $2x - y + 1 = 0$

where $a = 2, b = 1, c_1 = -4$ *and* $c_2 = 1$

The distance between the parallel lines is $= \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$

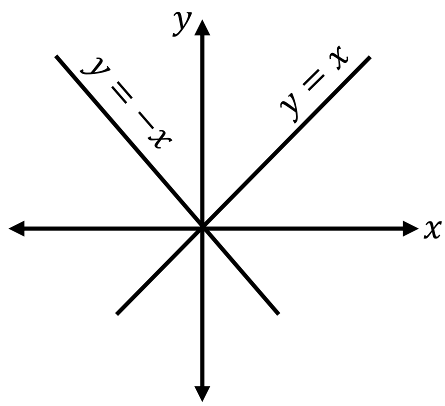
$$= \left|\frac{-4 - 1}{\sqrt{2^2 + (1)^2}}\right| = \left|\frac{-5}{\sqrt{4 + 1}}\right| = \left|\frac{-5}{\sqrt{5}}\right| = \frac{5}{\sqrt{5}} = \frac{\sqrt{5} \times \sqrt{5}}{\sqrt{5}}$$

$= \sqrt{5}$ *units*

16. Prove that one of the straight lines $ax^2 + 2hxy + by^2 = 0$ will bisect the angle between the co − ordinate axes if $(a + b)^2 = 4h^2$

Given pair of st. lines : $ax^2 + 2hxy + by^2 = 0$

If we consider $y = x$ *we get*



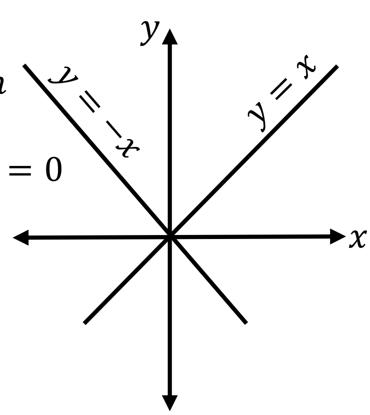
$$ax^2 + 2hx(x) + bx^2 = 0$$
$$ax^2 + 2hx^2 + bx^2 = 0$$
$$\div x^2$$
$$a + 2h + b = 0 \Longrightarrow a + b = -2h$$
$$(a + b)^2 = (-2h)^2 \Longrightarrow (a + b)^2 = 4h^2$$

If we consider $y = -x$ *we get*

$$ax^2 + 2hx(-x) + b(-x)^2 = 0$$
$$ax^2 - 2hx^2 + bx^2 = 0$$
$$\div x^2$$
$$a - 2h + b = 0 \Longrightarrow a + b = 2h$$
$$(a + b)^2 = (2h)^2 \Longrightarrow (a + b)^2 = 4h^2$$

16. Prove that one of the straight lines $ax^2 + 2hxy + by^2 = 0$ will bisect the angle between the co − ordinate axes if $(a + b)^2 = 4h^2$

Given pair of st. lines : $ax^2 + 2hxy + by^2 = 0$



$(a + b)^2 = 4h^2 \Longrightarrow a + b = \sqrt{4h^2} \Longrightarrow a + b = \pm 2h$

If $a + b = 2h$

$ax^2 + (a + b)xy + by^2 = 0 \Longrightarrow ax^2 + axy + bxy + by^2 = 0$

$ax^2 + bxy + axy + by^2 = 0$

$$(ax + by)x + (ax + by)y = 0$$
$$(ax + by)(x + y) = 0 \Longrightarrow x + y = 0$$

If $a + b = -2h$ *(or)* $2h = -(a + b)$

$ax^2 - (a + b)xy + by^2 = 0 \Longrightarrow ax^2 - axy - bxy + by^2 = 0$

$ax^2 - bxy - axy + by^2 = 0 \Longrightarrow (ax - by)x - y(ax - by) = 0$

$$(ax - by)(x - y) = 0 \Longrightarrow x - y = 0$$

one of the lines $x + y = 0$ *or* $x - y = 0$ *bisect the co − ordinate axes*

17. ***If the pair of straight lines*** $x^2 - 2kxy - y^2 = 0$ ***bisect the angle between the pair of straight lines*** $x^2 - 2lxy - y^2 = 0$ ***, Show that the later pair also bisectsthe angle between the former.***

$$x^2 - 2kxy - y^2 = 0$$

The equation of bisectors of the angle between the lines is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

$$a = 1, b = -1 \ , 2h = -2k$$
$$h = -k$$
$$\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-k} \Longrightarrow \frac{x^2 - y^2}{2} = \frac{xy}{-k}$$
$$-kx^2 + ky^2 = 2xy \ \Longrightarrow \ -kx^2 + ky^2 - 2xy = 0$$
$$kx^2 - ky^2 + 2xy = 0 \ \Longrightarrow \ kx^2 + 2xy - ky^2 = 0 \ldots (1)$$
$$x^2 - 2lxy - y^2 = 0 \ \ldots (2)$$

(1) & (2) *coefficient are proportional*

$$\frac{k}{1} = \frac{2}{-2l} = \frac{-k}{-1} \ \Longrightarrow \ k = \frac{1}{-l} = \frac{k}{1} \ \Longrightarrow k = -\frac{1}{l}$$
$$l = -\frac{1}{k}$$

Let us find the equation of bisector of $x^2 - 2lxy - y^2 = 0$

$$a = 1, b = -1 \ , 2h = -2l$$
$$h = -l$$
$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \ \Longrightarrow \frac{x^2 - y^2}{2} = \frac{xy}{-l} \ \Longrightarrow -lx^2 + ly^2 = 2xy$$
$$-lx^2 + ly^2 - 2xy = 0 \Longrightarrow \ lx^2 - ly^2 + 2xy = 0$$
$$sub \ l = -\frac{1}{k}$$
$$-\frac{1}{k}x^2 + \frac{1}{k}y^2 + 2xy = 0 \ \Longrightarrow -x^2 + y^2 + 2kxy = 0$$
$$x^2 - y^2 - 2kxy = 0 \ \Longrightarrow \ x^2 - 2kxy - y^2 = 0$$

18. ***Prove that the straight lines joining the origin to the points of intersection of*** $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ ***and*** $3x - 2y - 1 = 0$ ***are at right angles***

$$3x^2 + 5xy - 3y^2 + 2x + 3y = 0 \ldots (1)$$
$$3x - 2y - 1 = 0 \Longrightarrow 3x - 2y = 1$$

The straight lines joining the origin and the points of intersection of given equationis a second degree homogeneous equation.

$$3x^2 + 5xy - 3y^2 + (2x + 3y)(1) = 0$$
$$3x^2 + 5xy - 3y^2 + (2x + 3y)(3x - 2y) = 0$$
$$3x^2 + 5xy - 3y^2 + 6x^2 - 4xy + 9xy - 6y^2 = 0$$
$$9x^2 + 10xy - 9y^2 = 0$$

$coefficient\ of\ x^2 + coefficient\ of\ y^2 = 9 - 9 = 0$

$The\ lines\ joining\ origin\ to\ the\ point\ of\ intersection\ of$

$3x^2 + 5xy - 3y^2 + 2x + 3y = 0\ and\ 3x - 2y - 1 = 0\ are\ at\ right\ angles$

EXERCISE 4.1

Example 4.1 : Suppose one girl or one boy has to be selected for a competition from a class comprising 17 boys and 29 girls. In how many different ways can this selection be made?

Here the teacher is to perform two jobs :

(i) Selecting a boy among 17 boys = 17 ways

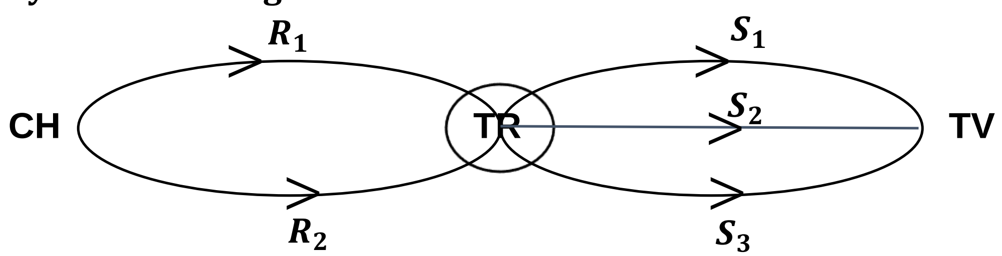
or

(ii) Selecting a girl among 29 girls = 29 ways

$\therefore$ ***The fundamental principle of addition***

Number of ways is $\mathbf{29 + 17 = 46}$.

4.2. Consider the 3 cities Chennai, Trichy and Tirunelveli. In order to reach Tirunelveli from Chennai, one has to pass through Trichy. There are 2 roads connecting Chennai with Trichy and there are 3 roads connecting Trichy with Tirunelveli. What are the total number of ways of travelling from Chennai to Tirunelveli?



Let R_1 *and* R_2 be 2 roads connecting Chennai to Trichy.

Let S_1, S_2 *and* S_3 be 3 roads connecting Trichy to Tirunelveli.

A person chooses R_1 *to travel from Chennai to Trichy and may further* choose any of the 3 roads S_1, S_2 or S_3 to travel from Trichy to Tirunelveli.

Thus the possible road choices are $(R_1, S_1)(R_1, S_2)(R_1, S_3)$.

Similarly, if the person chooses R_2 to travel from Chennai to Trichy, the choices $(R_2, S_1), (R_2, S_2), (R_2, S_3)$.

$\therefore$ *The fundamental principle of multiplication*

Number of ways is $2 \times 3 = 6$. *ways of travelling from Chennai to Tirunelveli*.

Example 4.3 : A *School* library has 75 books on Mathematics, 35 books on Physics. A student can choose only one book. In how many ways a student can choose a book on Mathematics or Physics?

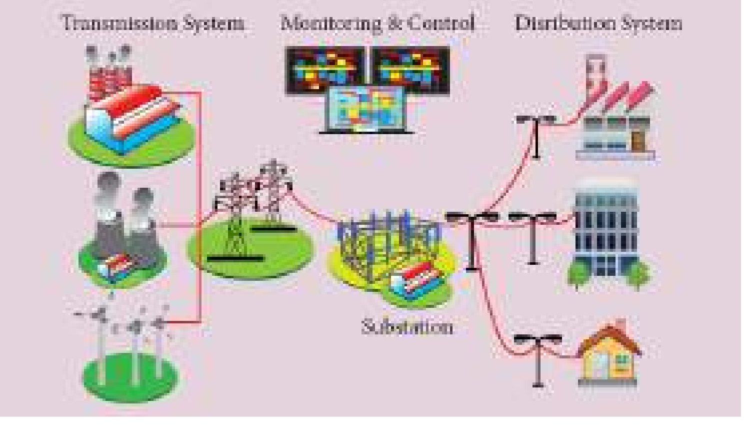
(*i*) *A* student can *choose a Mathematics book in* " 75 " different ways

(*ii*) *A student can choose a Physics book in* "35" *different ways*.

$\therefore$ ***The fundamental principle of addition***

Number of ways is $\mathbf{75 + 35 = 110}$.

Example 4.4. If an electricity consumer has the consumer number say $238:110:29$, then describe the linking and count the number of house connections upto 29^{th} *consumer connection linked to the larger capacity transformer number* 238 *subject to the condition that each smaller capacity transformer can have a maximal consumer link of say* 100.



There are 110 smaller capacity transformer attached to a larger capacity transformer. As each smaller capacity transformer can be linked with only 100 consumers, we have for the 109 transformers, there will be $109 \times 100 = 10900$ *links*.

For the 110^{th} transformer there are only 29 consumers linked. Hence, the total number of consumer linked to the 238^{th} *larger capacity transformer is* $10900 + 29 = 1029$.

Eg 4.5: A person wants to buy a car. There are two brands of car available in the market and each brand has 3 variant models and each model comes in five different colours as in figure. In how many ways she can choose a car to buy?

A car can be brought by a brand, a variant model and a colour :

(i) A brand can be choosen = 2 ways

and

(ii) A model can be choosen = 3 ways

and

(iii) A colour can be choosen = 5 ways

$\therefore$ ***The fundamental principle of multiplication***

Number of ways is $2 \times 3 \times 5 = 30$ ***ways***

4.6. A Woman wants to select one silk saree and one sungudi saree from a textile shop located at Kancheepuram. In that shop, there are 20 different varieties of silk sarees and 8 differrent varieties of sungudi sarees. In how many ways she can select her sarees?

(i) Selecting a silk saree among 20 vareties = 20 ways

and

(ii) Selecting a sungudi saree 8 vareties = 8 ways

$\therefore$ ***The fundamental principle of multiplication***

Number of ways is $\mathbf{20 \times 8 = 160}$.

4.7. In a village, out of the toal number of people, 80 percentage of the people own Coconut groves and 65 percent of the people own Paddy fields. What is the minimum percentage of people own both?

Let $n(C)$ *denote the percentage of people who own the Coconut groves and* $n(P)$ *denote the percentage of people who own Paddy fields*.

Given : $n(C) = 80$ *and* $n(P) = 65$

The number of people own coconut groves or paddy groves is 100%

$$n(C \cup P) = 100.$$

By the rule of inclusion − exclusion.

$n(C \cap P) = n(C) + n(P) - n(C \cup P)$.

$n(C \cap P) = 80 + 65 - 100$

$n(C \cap P) = 45$.

This is , the minimum percentage of the people who own both is 45.

Example 4.8: (i) Find the number of strings of length 4, which can be formed using the letters of the word BIRD without repetition of the letters.

(i) repetition is not allowed

Using : B, I, R, D

1^{st}	$2nd$	$3rd$	$4th$

1^{st} *place can be filled any of the letters* B, I, R, D = 4 *ways*

$2nd$ *place can be filled any of the letters* B, I, R,D = 3 *ways*

$3rd$ *place can be filled any of the letters* B, I, R,D = 2 *ways*

$4th$ *place can be filled any of the letters* B, I, R,D = 1 *ways*

Number of ways = $4 \times 3 \times 2 \times 1 = 24$ *ways*

(ii) How many strings of length 5 can be formed out of the letters of the word PRIME taking all the letters at a time without repetition.

(ii) repetition is not allowed

1^{st}	$2nd$	$3rd$	$4th$	$5th$

Using : P, R, I, M, E

1^{st} *place can be filled any of the letters* $P, R, I, M, E = 5$ *ways*

$2nd$ *place can be filled any of the letters* $P, R, I, M, E = 4$ *ways*

$3rd$ *place can be filled any of the letters* $P, R, I, M, E = 3$ *ways*

$4th$ *place can be filled any of the letters* $P, R, I, M, E = 2$ *ways*

$5th$ *place can be filled any of the letters* $P = 1$ *ways*

Number of ways = $5 \times 4 \times 3 \times 2 \times 1 = 120$ *ways*

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Eg. 4. 9: How many strings of length 6 can be formed using letters of the FLOWER if (i) either starts with F or ends with R?
(ii) neither starts with F nor ends with R?

(i) either starts with F or ends with R

$Using : F, L, O, W, E, R$

$String\ starts\ with\ F$

1^{st}	$2nd$	$3rd$	$4th$	$5th$	$6th$

$1^{st}\ place\ can\ be\ filled\ by\ letter\ F = 1\ ways$

$2nd\ place\ can\ be\ filled\ any\ of\ the\ letters\ L, O, W, E, R = 5\ ways$

$3rd\ place\ can\ be\ filled\ any\ of\ the\ letters\ L, O, W, E, R = 4\ ways$

$4th\ place\ can\ be\ filled\ any\ of\ the\ letters\ L, O, W, E, R = 3\ ways$

$5th\ place\ can\ be\ filled\ any\ of\ the\ letters\ L, O, W, E, R = 2\ ways$

$6th\ place\ can\ be\ filled\ any\ of\ the\ letters\ L, O, W, E, R = 1\ ways$

$Number\ of\ ways = 5 \times 4 \times 3 \times 2 \times 1 = 120\ ways$

$Using : F, L, O, W, E, R$

$String\ ends\ with\ R$

1^{st}	$2nd$	$3rd$	$4th$	$5th$	$6th$

$1^{st}\ place\ can\ be\ filledany\ of\ the\ letters\ F, L, O, W, E = 5\ ways$

$2nd\ place\ can\ be\ filled\ any\ of\ the\ letters\ F, L, O, W, E = 4\ ways$

$3rd\ place\ can\ be\ filled\ any\ of\ the\ letters\ F, L, O, W, E = 3\ ways$

$4th\ place\ can\ be\ filled\ any\ of\ the\ letters\ F, L, O, W, E = 2\ ways$

$5th\ place\ can\ be\ filled\ any\ of\ the\ letters\ F, L, O, W, E = 1\ ways$

$6th\ place\ can\ be\ filled\ by\ letter\ R = 1\ ways$

$Number\ of\ ways = 5 \times 4 \times 3 \times 2 \times 1 = 120\ ways$

$Using : F, L, O, W, E, R$

$String\ starts\ with\ F\ and\ also\ end\ with\ R$

1^{st}	$2nd$	$3rd$	$4th$	$5th$	$6th$

$1^{st}\ place\ can\ be\ filled\ by\ letter\ F = 1\ ways$

$2nd\ place\ can\ be\ filled\ any\ of\ the\ letters\ L, O, W, E = 4\ ways$

$3rd\ place\ can\ be\ filled\ any\ of\ the\ letters\ L, O, W, E, = 3\ ways$

$4th\ place\ can\ be\ filled\ any\ of\ the\ letters\ L, O, W, E, = 2\ ways$

$5th\ place\ can\ be\ filled\ any\ of\ the\ letters\ L, O, W, E, = 1\ ways$

$6th\ place\ can\ be\ filled\ by\ letter\ R = 1\ ways$

$Number\ of\ ways = 4 \times 3 \times 2 \times 1 = 24\ ways$

$By\ The\ principle\ of\ inclusion - exclusion$

The number of strings of length either starts with F or ends with R

$= 120 + 120 - 24 = 240 - 24 = 216$

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(ii) neither starts with F or ends with R

$Using : F, L, O, W, E, R$

1^{st}	$2nd$	$3rd$	$4th$	$5th$	$6th$

1^{st} $place\ can\ be\ filled\ any\ of\ the\ letters\ F, L, O, W, E, R = 6\ ways$

$2nd\ place\ can\ be\ filled\ any\ of\ the\ letters\ F, L, O, W, E, R = 5\ ways$

$3rd\ place\ can\ be\ filled\ any\ of\ the\ letters\ F, L, O, W, E, R = 4\ ways$

$4th\ place\ can\ be\ filled\ any\ of\ the\ letters\ F, L, O, W, E, R = 3\ ways$

$5th\ place\ can\ be\ filled\ any\ of\ the\ letters\ F, L, O, W, E, R = 2\ ways$

$6th\ place\ can\ be\ filledany\ of\ the\ letters\ F, L, O, W, E, R = 1\ ways$

$Number\ of\ ways = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720\ ways$

The number of strings of length neither starts with F or ends with R

$$= 720 - 216 = 504$$

Eg. 4. 10: How many licence plates may be made using either two distinct letters followed by four digits or two digits followed by 4 distinct letters where all digits and letters are distinct?

$Case\ 1: The\ number\ of\ license\ plates\ having\ two\ letters\ followed\ by\ four\ digits\ is$

$1000's$	$100's$	$10's$	$unit's$

$1st\ place\ can\ be\ filled\ any\ of\ the\ letters\ from\ A - Z\ in\ 26\ ways$

$2nd\ place\ can\ be\ filled\ any\ of\ the\ letters\ from\ A - Z\ in\ 25\ ways$

$1000's\ place\ can\ be\ filled\ in\ 10\ ways$

$100's\ place\ can\ be\ filled\ in\ 9\ ways$

$10's\ place\ can\ be\ filled\ in\ 8\ ways$

$unit's\ place\ can\ be\ filled\ in\ 7\ ways$

$Number\ of\ ways = 26 \times 25 \times 10 \times 9 \times 8 \times 7 = 32{,}76{,}000$

$Case\ 2: The\ number\ of\ license\ plates\ having\ two\ digits\ followed\ by\ four\ letters\ is$

$10's$	$unit's$

$10's\ place\ can\ be\ filled\ in\ 10\ ways$

$unit's\ place\ can\ be\ filled\ in\ 9\ ways$

$3rd\ place\ can\ be\ filled\ any\ of\ the\ letters\ from\ A - Z\ in\ 26\ ways$

$4th\ place\ can\ be\ filled\ any\ of\ the\ letters\ from\ A - Z\ in\ 25\ ways$

$5th\ place\ can\ be\ filled\ any\ of\ the\ letters\ from\ A - Z\ in\ 24\ ways$

$6th\ place\ can\ be\ filled\ any\ of\ the\ letters\ from\ A - Z\ in\ 23\ ways$

$Number\ of\ ways = 10 \times 9 \times 26 \times 25 \times 24 \times 23 = 3{,}22{,}92{,}000$

$The\ total\ number\ of\ license\ plates = 32{,}76{,}000 + 3{,}22{,}92{,}000$

$= 3{,}55{,}68{,}000$

Example 4.11: Count the number of positive integers greater than 7000 and less than 8000 which are divisible by 5, provided that no digits are repeated.

$Using\ the\ digits: 0, 1, 2, 3, 4, 5, 6, 8, 9$

$1000's$	$100's$	$10's$	$unit's$

$1000's\ place\ can\ be\ filled\ in = 1\ way$

$unit's\ place\ can\ be\ filled\ in\ = 2\ ways$

$100's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 0, 1, 2, 3, 4, 5, 6, 8, 9 = 8\ ways$

$10's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 0, 1, 2, 3, 4, 5, 6, 8, 9 = 7 ways$

$$Number\ of\ ways = 1 \times 2 \times 7 \times 8 = 112\ ways$$

Eg. 4.12: How many 4 − digit even number can be formed using the digits 0, 1, 2, 3 and 4, if repetition of digits are not permitted?

$Using\ the\ digits: 0, 1, 2, 3, 4$

$Case\ 1:\ unit\ place\ filled\ by\ 0$

$1000's$	$100's$	$10's$	$unit's$

$1000's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 1,2,3,4 = 4 ways$

$100's place\ can\ be\ filled\ any\ of\ the\ digits\ 1,2,3,4 = 3 ways$

$10's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 1,2,3,4 = 2 ways$

$unit\ place\ can\ be\ filled\ by\ digit\ 0 = 1 ways$

$$Number\ of\ ways = \ 4 \times 3 \times 2 \times 1 \ = 24$$

$Using\ the\ digits: 0, 1, 2, 3, 4$

$Case\ 2:\ unit\ place\ filled\ by\ 2\ or\ 4$

$1000's$	$100's$	$10's$	$unit's$

$unit\ place\ can\ be\ filled\ by\ digit\ 2\ or\ 4 = 2\ ways$

$1000's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 1,2,3 = 3\ ways$

$100's place\ can\ be\ filled\ any\ of\ the\ digits\ 0,1,2 = 3 ways$

$10's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 0,1 = 2 ways$

$$Number\ of\ ways = \ 3 \times 3 \times 2 \times 2 \ = 36$$

$$Number\ of\ 4 - digits\ even\ numbers\ = 24 + 36\ = 60$$

Eg : 4.13. Find the total number of outcome when 5 coins are tossed once.

When a coin is tossed, the outcomes are in two ways which are {Head, Tail}.

$\therefore$ ***The fundamental principle of multiplication***

The number of outcomes when 5 coins are tossed is

$$2^5 = 2 \times 2 \times 2 \times 2 \times 2 \ = 32.$$

4.14. In how many way (i) 5 different balls be distributed among 3 boxes? (ii) 3 different balls he distributed among 5 boxes?

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(i) 5 *different balls be distributed among* 3 *boxes.*

$5ways$	$5ways$	$5ways$

1^{st} *box can be filled any of the* 5 *different balls* $= 5ways$
$2nd$ *box can be filled any of the* 5 *different balls* $= 5ways$
$3rd$ *box can be filled any of the* 5 *different balls* $= 5ways$

$\therefore$ *The fundamental principle of multiplication*

The number of ways of distributing 5 different balls among three boxes

$$Number\ of\ ways = 5 \times 5 \times 5 = 125\ ways$$

(ii) 3 *different balls he distributed among* 5 *boxes*

$3ways$	$3ways$	$3ways$	$3ways$	$3ways$

1^{st} *box can be filled any of the* 5 *different balls* $= 3ways$
$2nd$ *box can be filled any of the* 5 *different balls* $= 3ways$
$3rd$ *box can be filled any of the* 5 *different balls* $= 3ways$
$4th$ *box can be filled any of the* 5 *different balls* $= 3ways$
$5th$ *box can be filled any of the* 5 *different balls* $= 3ways$

$\therefore$ *The fundamental principle of multiplication*

The number of ways of distributing 3different balls among five boxes

$$Number\ of\ ways = 3 \times 3 \times 3 \times 3 \times 3 = 3^5\ ways$$

4. 15. There are 10 bulbs in a room Each one of them can be operated independently. Find the number of ways in which the room can be illuminated.


ON or OFF 2 *ways*


ON or OFF 2 *ways*


ON or OFF 2 *ways*


ON or OFF 2 *ways*

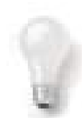

ON or OFF 2 *ways*


ON or OFF 2 *ways*


ON or OFF 2 *ways*


ON or OFF 2 *ways*


ON or OFF 2 *ways*


ON or OFF 2 *ways*

Each of the 10 bulbs are operated independently means that each buld can be operated in two ways. That is in off mode or on mode.

The total number of doing this $= 2^{10}$

which includes the case in which 10 bulbs are off.

Keeping all 10 bulbs in 'off' mode, the room cannot be illuminated

Hence, the total number of ways $= 2^{10} - 1 = 1024 - 1 = 1023.$

1. (i) A person went to a restaurant for dinner. In the menu card, the person saw 10 Indian and 7 Chinese food items. In how many ways the person can select either an Indian or a Chinese food?

Here the person is to perform two jobs :

(i) Selecting a indian food among 10 items = 10 ways

or

(ii) Selecting a chinese food among 7 items = 7 ways

$\therefore$ ***The fundamental principle of addition***

Number of ways is $10 + 7 = 17$

(ii) There are 3 types of toy car and 2 types of toy train available in a shop. Find the number of ways a baby can buy a toy car and a toy train?

(i) Selecting a toy car among 3 types = 3 ways

and

(ii) Selecting a toy train among 2 types = 2 ways

$\therefore$ *The fundamental principle of multiplication*

Number of ways is $3 \times 2 = 6$

(iii) How many two − digit numbers can be formed using 1, 2, 3, 4, 5 without repetition of digits?

Two digits number using 1, 2, 3, 4, 5 *without repetition of digits*

unit's place can be filled any of the digits 1,2 3,4,5 = 5 *ways*

10's place can be filled up by remaining 4 *digits* = 4 *ways*

Number of ways $= 5 \times 4 = 20$ *ways*

10's	unit's
4 *ways*	5 *ways*

(iv) Three persons enter in to a conference hall in which there are 10 seats. In how many ways they can take their seats?

This first person can take any one of the 10 *seats in* 10 *ways*

The second person can take any one of the remaining 9 *seats in* 9 *ways.*

The third person can take any one of the remaining 8 *seats in* 8 *ways.*

$\therefore$ *Number of ways* $= 10 \times 9 \times 8 = 720$.

(v) *In how many ways* 5 *persons can be seated in a row*?

1^{st}	2nd	3rd	4^{th}	5^{th}

1^{st} *place can be filled any of the* 5 *persons* = 5*ways*

2nd place can be filled remaining 4 *persons* = 4*ways*

3rd place can be filled remaining 3 *persons* = 3 *ways*

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Example 4.16: Find the value of (i) 5! (ii) 6! − 5! (iii) $\frac{8!}{5! \times 2!}$

(i) $\mathbf{5!} = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$$(ii)\ \mathbf{6! - 5!} = 6 \times 5! - 5!$$
$$= 5!\ (6-1) = (5 \times 4 \times 3 \times 2 \times 1)$$
$$= 120 = 600$$

$$(iii)\ \frac{\mathbf{8!}}{\mathbf{5! \times 2!}} = \frac{8 \times 7 \times 6 \times \cancel{5!}}{\cancel{5!} \times 2!} = \frac{\overset{4}{\cancel{8}} \times 7 \times 6}{\cancel{2} \times 1} = 168$$

Example 4.17: Simplify $\frac{7!}{2!}$

$$\frac{\mathbf{7!}}{\mathbf{2!}} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}} = 7 \times 6 \times 5 \times 4 \times 3 = 2520$$

Example 4.18: Evaluate $\frac{n!}{r!\,(n-r)!}$ when (i) $n = 7, r = 5$ (ii) $n = 50, r = 47$, (iii) For any with $r = 3$.

(i) $n = 7, r = 5$

$$\frac{n!}{r!\,(n-r)!} = \frac{7!}{5!\,(7-5)!} = \frac{7!}{5! \times 2!} = \frac{7 \times 6 \times \cancel{5!}}{\cancel{5!} \times 2!} = \frac{7 \times \overset{3}{\cancel{6}}}{\cancel{2} \times 1} = 21$$

(ii) $n = 50, r = 47$

$$\frac{n!}{r!\,(n-r)!} = \frac{50!}{47!\,(50-47)!}$$

$$= \frac{50!}{47! \times 3!} = \frac{50 \times 49 \times 48 \times \cancel{47!}}{\cancel{47!} \times 3!} = \frac{50 \times 49 \times \overset{\overset{8}{\cancel{24}}}{\cancel{48}}}{\cancel{3} \times \cancel{2} \times 1} = 19600$$

(iii) For any n with $r = 3$

$$\frac{n!}{r!\,(n-r)!} = \frac{n!}{3!\,(n-3)!} = \frac{n(n-1)(n-2)\cancel{(n-3)!}}{3 \times 2 \times 1\cancel{(n-3)!}}$$

$$= \frac{n(n-1)(n-2)}{6}$$

Example 4.19: Let N denote the number of days. If the value of N! is equal to the toatal number of hours in N days then find the value of N

We need to solve the equation $N! = 24 \times N$.

$$\cancel{N}(N-1)! = 24 \times \cancel{N} \Longrightarrow (N-1)! = 1 \times 2 \times 3 \times 4$$
$$(N-1)! = 4! \Longrightarrow N - 1 = 4$$
$$N = 4 + 1 \Longrightarrow N = 5$$

Example 4.20: If $\frac{6!}{n!} = 6$, then find the value of n.

$$\frac{6!}{n!} = 6 \Longrightarrow \frac{6!}{6} = n!$$

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$$\frac{\cancel{6} \times 5!}{\cancel{6}} = n!$$

$$n! = 5! \implies n = 5$$

Example 4.21: If $n! + (n-1)! = 30$, then find the value of n.

$$n! + (n-1)! = 30$$

$$n(n-1)! + (n-1)! = 30$$

$$(n-1)!(n+1) = 30 \implies (n-1)!(n+1) = 5 \times 3!$$

$$(n-1)! = 3! \quad or \; n+1 = 5$$

$$n - 1 = 3, \; n = 4$$

Example 4.22: What is the unit digit of the sum $2! + 3! + 4! + \cdots. +22!$?

$$2! + 3! + 4! + \cdots + 22!$$

From $5!$ *onwards for all* $n!$ *the unit digit is zero*

$$2! + 3! + 4! = 2 + 6 + 24 = 32$$

Therefore the required unit digit is 2.

Example 4.23: If $\frac{1}{7!} + \frac{1}{8!} = \frac{A}{9!}$ *then find the value of A.*

$$\frac{1}{7!} + \frac{1}{8!} = \frac{A}{9!} \implies \frac{1}{7!} + \frac{1}{8 \times 7!} = \frac{A}{9 \times 8 \times 7!}$$

$$\frac{1}{\cancel{7!}}\left[1 + \frac{1}{8}\right] = \frac{1}{\cancel{7!}} \times \frac{A}{9 \times 8} \implies \frac{8+1}{8} = \frac{A}{72}$$

$$\frac{9}{8} = \frac{A}{72} \implies \frac{9}{\cancel{8}} \times \cancel{72}^{9} = A \implies A = 81$$

Example 4.24: *Prove that* $\frac{(2n)!}{n!} = 2^n\left(1.3.5 \ldots (2n-1)\right)$

$$\frac{2n!}{n!} = \frac{2n\,(2n-1)\,(2n-2) \ldots \ldots \ldots 4 \times 3 \times 2 \times 1}{n!}$$

Grouping odd and even number separately

$$= \frac{[1 \times 3 \times 5 \ldots \ldots (2n-1)][2 \times 4 \ldots \ldots 2n]}{n!}$$

$$= \frac{[1 \times 3 \times 5 \ldots \ldots (2n-1)]\, 2^n\, [1 \times 2 \times 3 \ldots \ldots n]}{n!}$$

$$= \frac{[1 \times 3 \times 5 \ldots \ldots (2n-1)]\; 2^n\; \cancel{n!}}{\cancel{n!}} = 2^n\, [1 \times 3 \times 5 \ldots \ldots (2n-1)]$$

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4^{th} *place can be filled remaining* 2 *persons* = 2 *ways*
5^{th} *place can be filled remaining* 1 *persons* = 1*ways*
Number of ways = $5 \times 4 \times 3 \times 2 \times 1 = 120$ *ways*

2. (i) A mobile phone has a passcode of 6 distinct digits. What is the maximum number of attempts one makes to retrieve the passcode?

Using the digits: 1, 2, 3, 4,5, 6, 7,8,9, 0

1^{st}	2nd	3rd	4^{th}	5^{th}	6^{th}

1^{st} *place can be filled up in* = 10 *ways*
2nd *place can be filled up in* = 9 *ways*
3rd *place can be filled up in* = 8 *ways*
4^{th} *place can be filled up in* = 7 *ways*
5^{th} *place can be filled up in* = 6 *ways*
6^{th} *place can be filled up in* = 5 *ways*
Number of ways = $10 \times 9 \times 8 \times 7 \times 6 \times 5 = 1,51,200$

(ii) Given four flags of different colours, how many different signals can be generated if each signal requires the use of three flags, one below the other?

(i) Selecting 1 flag among 4 flags = 4 ways

and

(ii) Selecting *anthor* flag among 3 flags = 3 ways

and

(ii) Selecting *anthor* flag among 2 flags = 2 ways

∴ *The fundamental principle of multiplication*
Number of different signals *is* $4 \times 3 \times 2 = 24$

3. Four children are running a race.

(i) In how many ways can the first two places be filled?

(ii) In how many different ways could they finish the race?

(*i*) *First place can be given to any one of the* 4 *children in* 4 *ways*.

The second place can be given to any one of the remaining 3 children in 3 ways.

∴ *Number of ways* $= 4 \times 3 = 12$.

(*ii*) *The* winner *may be any one of* 4 *children*
The runner cup may be any one of remaining 3 *children*

4. Count the number of three − digit numbers which can be formed from the digits 2, 4, 6, 8 if (i) repetitions of digits is allowed. (ii) repetitions of digits is not allowed.

(i) repetition is allowed

1^{st}	$2nd$	$3rd$

$Using : 2, 4, 6, 8$

1^{st} $place\ can\ be\ filled\ any\ of\ the\ digits\ 2, 4, 6, 8 = 4\ ways$

$2nd\ place\ can\ be\ filled\ any\ of\ the\ digits\ 2, 4, 6, 8 = 4\ ways$

$3rd\ place\ can\ be\ filled\ any\ of\ the\ digits\ 2, 4, 6, 8 = 4\ ways$

$Number\ of\ ways = 4 \times 4 \times 4 = 64\ ways$

(ii) repetition is not allowed

1^{st}	$2nd$	$3rd$

$Using : 2, 4, 6, 8$

1^{st} $place\ can\ be\ filled\ any\ of\ the\ digits\ 2, 4, 6, 8 = 4\ ways$

$2nd\ place\ can\ be\ filled\ any\ of\ the\ digits\ 2, 4, 6, 8 = 3\ ways$

$3rd\ place\ can\ be\ filled\ any\ of\ the\ digits\ 2, 4, 6, 8 = 2\ ways$

$Number\ of\ ways = 4 \times 3 \times 2 = 24\ ways$

5. ***How many three − digit numbers are there with 3 in the unit place? (i) with repetition (ii) without repetition.***

(i) repetition is allowed

$100's$	$10's$	$unit's$

$Using : 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$

$unit's\ place\ can\ be\ filled\ in = 1\ way$

$10's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 = 10\ ways$

$100's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 1, 2, 3, 4, 5, 6, 7, 8, 9 = 9\ ways$

$Number\ of\ ways = 10 \times 9 \times 1 = 90\ ways$

(ii) without repetition

$100's$	$10's$	$unit's$

$Using : 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$

$unit's\ place\ can\ be\ filled\ in = 1\ way$

$10's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 0, 1, 2, 4, 5, 6, 7, 8, 9 = 8\ ways$

$100's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 1, 2, 3, 4, 5, 6, 7, 8, 9 = 8\ ways$

$Number\ of\ ways = 8 \times 8 \times 1 = 64\ ways$

6. How many numbers are there between 100 and 500 wih the digits 0, 1, 2, 3, 4, 5? if (i) repetition of digits allowed (ii) the repetition of digits is not allowed.

(i) repetition is allowed

$100's$	$10's$	$unit's$

$Using : 0, 1, 2, 3, 4, 5$

$unit's\ place\ can\ be\ filled\ in = 6\ way$

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$10's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 0,1,2,3,4,5 = 6\ ways$

$100's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 1,2,3,4 = 4\ ways$

$Number\ of\ ways = 6 \times 6 \times 4 = 144\ ways$

$100's$	$10's$	$unit's$

(ii) without repetition

$Using : 0,1,2,3,4,5$

$100's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 1,2,3,4 = 4\ ways$

$10's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 0,1,2,3,4,5 = 5\ ways$

$unit's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 0,1,2,3,4,5 = 4\ way$

$Number\ of\ ways = 4 \times 5 \times 4 = 80\ ways$

7. How many three − digit odd numbers can be formed by using the digits 0, 1, 2, 3, 4, 5? If (i) the repetition of digits is not allowed (ii) the repetition of digits is allowed.

$100's$	$10's$	$unit's$

(i) without repetition

$Using : 0,1,2,3,4,5$

$unit's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 1,3,5 = 3\ ways$

$100's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 1,2,3,4,5 = 4\ ways$

$10's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 0,1,2,3,4,5 = 4\ ways$

$Number\ of\ ways = 3 \times 4 \times 4 = 48\ ways$

(ii) repetition is allowed

$Using : 0,1,2,3,4,5$

$unit's\ place\ can\ be\ filled\ in = 3\ ways$

$10's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 0,1,2,3,4,5 = 6\ ways$

$100's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 1,2,3,4 = 5\ ways$

$Number\ of\ ways = 3 \times 6 \times 5 = 90\ ways$

8. Count the numbers between 999 and 10000 subject to the condition that there are (i) no restriction, (ii) no digit is repeated, (iii) at least one of the digits is repeated.

$1000's$	$100's$	$10's$	$unit's$

$(i)\ no\ restriction$

$Using : 0,1,2,3,4,5,6,7,8,9$

$unit's\ place\ can\ be\ filled\ any\ of\ the\ digits = 10\ ways$

$10's\ place\ can\ be\ filled\ any\ of\ the\ digits = 10\ ways$

$100's\ place\ can\ be\ filled\ any\ of\ the\ digits = 10\ ways$

$1000's\ place\ can\ be\ filled\ any\ of\ the\ digits = 9\ ways$

$Number\ of\ ways = 10 \times 10 \times 10 \times 9 = 9000\ ways$

(ii) ***no digit is repeated***

$Using : 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$

$1000's\ place\ can\ be\ filled\ any\ of\ the\ digits = 9\ ways$

$100's\ place\ can\ be\ filled\ any\ of\ the\ digits = 9\ ways$

$10's\ place\ can\ be\ filled\ any\ of\ the\ digits = 8\ ways$

$unit's\ place\ can\ be\ filled\ any\ of\ the\ digits = 7\ ways$

$Number\ of\ ways = 9 \times 9 \times 8 \times 7 = 4536\ ways$

(iii) ***atleast one of the digit is repeated***

$Number\ of\ ways = 9000 - 4536$

$= 4464$

9. How many three − digit numbers, which are divisible by 5, can be formed using the digits 0, 1, 2, 3, 4, 5 if (i) repetition of digits are not allowed? (ii) repetition of digits are allowed?

$100's$	$10's$	$unit's$

(i) without repetition $Using : 0, 1, 2, 3, 4, 5$

$unit's\ place\ can\ be\ filled\ by\ the\ digits\ 0 = 1\ ways$

$100's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 1, 2, 3, 4, 5 = 5\ ways$

$10's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 1, 2, 3, 4, 5 = 4\ ways$

$Number\ of\ ways = 1 \times 4 \times 5 = 20\ ways$

or

$unit's\ place\ can\ be\ filled\ by\ the\ digits\ 5 = 1\ ways$

$100's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 1, 2, 3, 4 = 4\ ways$

$10's\ place\ can\ be\ filled\ any\ of\ the\ digits\ 0, 1, 2, 3, 4 = 4\ ways$

$Number\ of\ ways = 1 \times 4 \times 4 = 16\ ways$

$Total\ number\ of\ ways = 20 + 16 = 36\ ways$

10. To travel from a place A to place B, , there are two different bus routes B_1, B_2, two different train routes T_1, T_2, and one air route A_1. From place B to place C there is one bus route say B'_1, two different train routes say $T'_1. T'_2$ and one air route A'_1. Find the number of routes of commuting from place A to place C via place B without using similar more of transportaton.

From A to B

$no. of\ bus\ routes = 2 \quad no. of\ train\ routes = 2 \quad no. of\ air\ route = 1$

$\therefore Number\ of\ routes\ of\ communting\ from\ A\ to\ B = 2 + 2 + 1 = 5$

From B to C

$no. of\ bus\ routes = 1 \quad no. of\ train\ routes = 2 \quad no. of\ air\ route = 1$

$\therefore Number\ of\ routes\ of\ communting\ from\ B\ to\ C = 1 + 2 + 1 = 4$

$\therefore Number\ of\ routes\ of\ communting\ from\ A\ to\ C = 5 \times 4 = 20$

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From A to C through B

$Using\ bus = \ 2 \times 1 = 2$

$Using\ train = 2 \times 2 = 4$

$Using\ air = \ 1 \times 1 = 1$

$Total = 2 + 4 + 1 = 7$

$\therefore No.of\ routes = \ 20 - 7 = 13$

11. How many numbers are there between 1 and 1000 (both inclusive) which are divisible neither by 2 nor by 5?

no. of numbers divisible by 2 $n(A) = 500$

no. of numbers divisible by 5 $n(B) = 200$

no. of numbers divisible by 10 $n(A \cap B) = \ 100$

$^{5}\dfrac{\cancel{1000}}{\cancel{2}} = 500$

$^{2}\dfrac{\cancel{1000}}{\cancel{5}} = 200$

$^{1}\dfrac{\cancel{1000}}{\cancel{10}} = 100$

no. of numbers divisible either by 2 or by 5

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$= 500 + 200 - 100$$
$$= 700 - 100 = 600$$

$\therefore$ **No. of numbers divisible neither by 2 nor by 5**

$$= 1000 - 600 = 400$$

12. How many strings can be formed using the letters of the word LOTUS if the word (i) either starts with L or ends with S? (ii) neither starts with L nor ends with S?

(i) either starts with L or ends with S

$Using : L, O, T, U, S$

$Let\ A = \ string\ starts\ with\ L$

1^{st}	$2nd$	$3rd$	$4th$	$5th$

$1^{st}\ place\ can\ be\ filled\ by\ letter\ L = 1\ ways$

$2nd\ place\ can\ be\ filled\ any\ of\ the\ letters\ O, T, U, S = 4\ ways$

$3rd\ place\ can\ be\ filled\ any\ of\ the\ letters\ O, T, U, S = 3\ ways$

$4th\ place\ can\ be\ filled\ any\ of\ the\ letters\ O, T, U, S = 2\ ways$

$5th\ place\ can\ be\ filled\ any\ of\ the\ letters\ O, T, U, S = 1\ ways$

$$n(A) = 1 \times 4 \times 3 \times 2 \times 1 = 24$$

$Using : L, O, T, U, S$

$Let\ B:\ String\ ends\ with\ S$

$1st\ place\ can\ be\ filled\ any\ of\ the\ letters\ L, O, T, U \ = 4\ ways$

$2nd\ place\ can\ be\ filled\ any\ of\ the\ letters\ L, O, T, U \ = 3\ ways$

$3rd\ place\ can\ be\ filled\ any\ of\ the\ letters\ L, O, T, U \ = 2\ ways$

$4th\ place\ can\ be\ filled\ any\ of\ the\ letters\ L, O, T, U \ = 1\ ways$

$5th\ place\ can\ be\ filled\ by\ letter\ S = 1\ ways$

$$n(B) = \ 4 \times 3 \times 2 \times 1 = 24$$

$$Using : L, O, T, U, S$$

$$Let\ A \cap B = String\ starts\ with\ L$$

$$1^{st}\ place\ can\ be\ filled\ by\ letter\ L = 1\ ways$$

$$2nd\ place\ can\ be\ filled\ any\ of\ the\ letters\ O, T, U = 3\ ways$$

$$3rd\ place\ can\ be\ filled\ any\ of\ the\ letters\ O, T, U = 2\ ways$$

$$4th\ place\ can\ be\ filled\ any\ of\ the\ letters\ O, T, U = 1\ ways$$

$$5th\ place\ can\ be\ filled\ by\ letter\ S = 1\ ways$$

$$n(A \cap B) = 3 \times 2 \times 1 = 6$$

either starts with L or ends with S

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$= 24 + 24 - 6 = 48 - 6$$

$$= 42$$

(i) neither starts with L nor ends with S

$$= 5! - 42 = 120 - 42 = 78$$

13. (i) Count the total number of ways of answering 6 objective type questions, each question having 4 choices. (ii) In how many ways 10 pigeons can be placed in 3 different pigeon holes? (iii) Find the number of ways of distributing 12 distinct prizes to 10 students?

The no. of ways of answering 6 objective type questions, each question having 4 choices $= 4^6$

The no. of ways 10 pigeons can be placed in 3 different pigeon holes $= 3^{10}$

The no. of ways of distributing 12 distinct prizes to 10 students $= 10^{12}$

14. Find the value of (i) 6!, (ii) 4! + 5!, (iii) 3! − 2!, (iv) 3! × 4!, (v) $\frac{12!}{9! \times 3!}$, (vi) $\frac{(n+3)!}{(n+1)!}$.

(i) **6!** $= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

(ii) **4! + 5!** $= (4 \times 3 \times 2 \times 1) + (5 \times 4 \times 3 \times 2 \times 1)$

$$= 24 + 120 = 144$$

(iii) **3! − 2!** $= (3 \times 2 \times 1) - (2 \times 1) = 6 - 2 = 4$

(iv) **3! × 4!** $= (3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1) = 6 \times 24$

(v) $\frac{\mathbf{12!}}{\mathbf{9! \times 3!}}$

$$= \frac{\overset{4}{\cancel{12}} \times 11 \times \overset{5}{\cancel{10}} \times \cancel{9!}}{\cancel{9!} \times \cancel{3} \times \cancel{2} \times 1} = 4 \times 11 \times 5 = 220$$

(vi) $\frac{(\boldsymbol{n}+\mathbf{3})!}{(\boldsymbol{n}+\mathbf{1})!} = \frac{(n+3)(n+2)\cancel{(n+1)!}}{\cancel{(n+1)!}} = (n+3)(n+2)$

$$\mathbf{15.}\ \textbf{\textit{Evaluate}}\ \frac{n!}{r!\,(n-r)!}\ \textbf{\textit{when}}\ (i)\ n=6, r=2, (ii)\ n=10, r=3,$$

$$(iii)\ \textbf{\textit{For any n with r}} = 2.$$

$(i)\ n = 6, r = 2$

$$\frac{n!}{r!\,(n-r)!} = \frac{6!}{2!\,(6-2)!} = \frac{6!}{2!\times 4!} = \frac{\overset{3}{\cancel{6}}\times 5\times \cancel{4!}}{\cancel{2}\times 1\times \cancel{4!}} = 15$$

$(ii)\ n = 10, r = 3$

$$\frac{n!}{r!\,(n-r)!} = \frac{10!}{3!\,(10-3)!} = \frac{10!}{3!\times 7!} = \frac{10\times \overset{3}{\cancel{9}}\times \overset{4}{\cancel{8}}\times \cancel{7!}}{\cancel{3}\times \cancel{2}\times 1\times \cancel{7!}} = 120$$

(iii) ***For any n with r*** $= 2$

$$\frac{n!}{r!\,(n-r)!} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)\cancel{(n-2)!}}{2\cancel{(n-2)!}} = \frac{n(n-1)}{2}$$

$$\mathbf{16.}\ \textbf{\textit{Find the value of n if}}\ (i)\ (n+1)! = 20(n-1)!\ (ii)\ \frac{1}{8!}+\frac{1}{9!}=\frac{n}{10!}.$$

$(i)\ (n+1)! = 20\,(n-1)!$

$(n+1)\,n\cancel{(n-1)!} = 20\cancel{(n-1)!}$

$n(n+1) = 20 \Rightarrow n^2 + n - 20 = 0$

-20

$5 \quad -4$

$(n+5)(n-4) = 0 \Rightarrow n+5 = 0,\ n-4 = 0$

$n = -5(\text{not valid})$

$\boxed{n = 4}$

$$(ii)\ \frac{1}{8!}+\frac{1}{9!}=\frac{n}{10!}$$

$$\frac{1}{8!}+\frac{1}{9\times 8!} = \frac{n}{10\times 9\times 8!} \Rightarrow \frac{1}{\cancel{8!}}\left[1+\frac{1}{9}\right] = \frac{n}{10\times 9\times \cancel{8!}}$$

$$\frac{10}{9} = \frac{n}{90} \Rightarrow \frac{10}{\cancel{9}}\times \overset{10}{\cancel{90}} = n \Rightarrow \boxed{n = 100}$$

$EXERCISE\ 4.2$

Example 4.8: (i) Find the number of strings of length 4, which can be formed using the letters of the word BIRD without repetition of the letters.

(i) repetition is not allowed

$Using : B, I, R, D$

1^{st}	$2nd$	$3rd$	$4th$

1^{st} place can be filled any of the letters B, I, R, D $= 4$ ways

$2nd$ place can be filled any of the letters B, I, R, D $= 3$ ways

$3rd$ place can be filled any of the letters B, I, R, D $= 2$ ways

$4th$ place can be filled any of the letters B, I, R, D $= 1$ ways

$$Number\ of\ ways = 4 \times 3 \times 2 \times 1 = 24\ ways$$

Example 4.25: Evaluate: (i)$4P_4$. (ii)$5P_3$, (iii)$8P_4$, (iv)$6P_5$,

$(i)\ \mathbf{4P_4} = 4 \times 3 \times 2 \times 1 = 4! = 24$

$(ii)\ \mathbf{5P_3} = 5 \times 4 \times 3 = 60$

$(iii)\ \mathbf{8P_4} = 8 \times 7 \times 6 \times 5 = 1680$

$(iv)\ \mathbf{6P_5} = 6 \times 5 \times 4 \times 3 \times 2 = 6! = 720$

Example 4.26: If $(n+2)P_4 = 42 \times nP_2$ find n.

$$(n+2)P_4 = 42 \times P_2^n$$

$$\frac{(n+2)P_4}{nP_2} = 42 \implies \frac{(n+2)(n+1)\cancel{(n)(n-1)}}{\cancel{n(n-1)}} = 42$$

$$(n+2)(n+1) = 42 \implies (n+2)(n+1) = 7 \times 6$$

$$n+2 = 7 \implies n = 7-2$$

$$\boxed{n = 5}$$

Example 4.27: If $10P_r = 7P_{r+2}$ find r.

$$10P_r = 7P_{r+2}$$

$$\frac{10!}{(10-r)!} = \frac{7!}{(5-r)!}$$

$$\frac{10 \times 9 \times 8 \times \cancel{7!}}{(10-r) \times (9-r) \times (8-r) \times (7-r) \times (6-r) \times \cancel{(5-r)!}} = \frac{\cancel{7!}}{\cancel{(5-r)!}}$$

$$\frac{10 \times 9 \times 8}{(10-r) \times (9-r) \times (8-r) \times (7-r) \times (6-r)} = 1$$

$$(10-r) \times (9-r) \times (8-r) \times (7-r) \times (6-r) = 10 \times 9 \times 8$$

$$(10-r)\times(9-r)\times(8-r)\times(7-r)\times(6-r)=5\times2\times3\times3\times4\times2$$

$$10-r=6 \Rightarrow 10-6=r \Rightarrow \boxed{r=4}$$

Example 4.28: How many 'letter strings' together can be formed with the letters of the word VOWELS" so that (i) the strings begin with E, (ii) the strings begin with E and end with W.

The given strings contains 6 letters (V, O, W, E, L, S).

(i) Since all strings must begin with E,

E	5	4	3	2	1
1					

$5P_5$ *ways*

we have the remaining 5 letters which can be arranged in

$$5P_5 = 5!\text{ ways}$$

Therefore the total number of strings with E as the starting letter is

$$5! = 5\times4\times3\times2\times1 = 120$$

(ii) Since all strings must begin with E, and end with W, we need to fix E and W.

The remaining 4 letters can be arranges in $4P_4 = 4!$ Ways

E	4	3	2	1	W

$4P_4$*ways*

∴ *the total no. of strings with E as the starting letter and W as the final letter is*

$$4! = 4\times3\times2\times1 = 24$$

Example 4.29: A number of four different digits is formed with the use of the digits 1, 2, 3, 4 and 5 in all possible ways. Find the following (i) How many such numbers can be formed? (ii) How many of these are even? (iii) How many of these are exactly divisible by 4 ?

(i) The solution for this is the same as the number of permutations taking four − digits out of 5 digits is

$5P_4$*ways*

5 *ways*	4 *ways*	3 *ways*	2 *ways*

$5P_4$*ways*

$$5P_4 = 5\times4\times3\times2 = 120$$

$(\boldsymbol{ii})$

$4\ ways$	$3\ ways$	$2\ ways$	$2/4$

$\underbrace{4\ ways \quad 3\ ways \quad 2\ ways}_{4P_3 ways}$ $\quad 2P_1 ways$

For even number last digits must be 2 *or* 4 *which is filled in* $2P_1 ways$ *and remaining* 3 *placess filled from remaining* 4 *digits in* $4P_3 ways$

$$2P_1 \times 4P_3 = 2 \times 4 \times 3 \times 2 = 48$$

$\therefore$ *The required number of ways is* 48

(iii) *Since last two digit must be divisible by* 4.

use of the digits 1,2,3,4 and 5

The Last two digits become $12, 24, 32, 52$ (4 ways).

The remaining first two places filled from remaining 3 digits in $3P_2$ways.

$3\ ways$	$2\ ways$	$12/24/32/52$

$\underbrace{3\ ways \quad 2\ ways}_{3P_2 ways}$ $\quad 4P_1 ways$

The required number of numbers which are divisible by 4 is

$$4P_1 \times 3P_2 = 4 \times 3 \times 2 = 24$$

Example 4.30: How many different strings can be formed together using the letters of the word "EQUATION" so that (i) the vowels always come together? (ii) the vowels never come together?

(i) *There are* 8 *letters in the word "EQUATION"*

which includes 5 vowels (E, U, A, I, O) *and* 3 *consonants*(Q, T, N).

(i) *The vowels always come together*?

$\boldsymbol{Q, T, N}$ Considering 5 vowels $(\mathbf{E, U, A, I}, \boldsymbol{O})$ as one letter,

we have 4 *letters which can be arranged in* $4P_4 = 4!$ ways

Now the vowels E, U, A, I, O can be arrangements by itself

$$5P_5 = 5! \text{ ways}$$

Hence, by the rule of product required number of words is

$$4! \times 5! = (4 \times 3 \times 2 \times 1) \times (5 \times 4 \times 3 \times 2 \times 1)$$
$$= 24 \times 120 = 2880$$

$(\boldsymbol{ii})$ ***the vowels never come together***?

The total number of strings formed by using all the eight letters of the word **"EQUATION"**

$$8P_8 = 8!$$
$$= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$$

So, the total number of strings in which vowels are never together is

$= \text{the total no. of strings} - \text{the no. of strings in which vowels are together}$

$= 40320 - 2880 = 37440$

Example 4.31: There are 15 candidates for an examination. 7 candidates are appearing for mathematics examination while the remaining 8 are appearing for different subjects. In how many ways can they be seated in a row so that no two mathematics candidates are together?

$Total\ number\ of\ canditate = 15$

$Number\ of$ **mathematics** $canditate = 7$

$Number\ of$ **non − mathematics** $canditate = 8$

Let us arrange the 8 non − mathematics candidates in $8P_8 = 8!$ ways

No two mathematics candidates are together

$\_\_ \; O_1 \; \_\_ \; O_2 \; \_\_ \; O_3 \; \_\_ \; O_4 \; \_\_ \; O_5 \; \_\_ \; O_6 \; \_\_ \; O_7 \; \_\_ \; O_8 \; \_\_$

Each of these arrangements create 9 gaps.

Therefore, the 7 mathematics candidates can be placed in these 9 gaps in $9P_7$ ways

By the rule of product, the required number of arrangements is

$$8! \times 9P_7 = 8! \times \frac{9!}{(9-7)!}$$

$$= 8! \times \frac{9!}{2!} = \frac{8! \times 9!}{2!}$$

Example 4.32: In how many ways 5 boys and 4 girls can be seated in a row so that no two girls are together.

The 5 *boys can be seated in the row in* $5P_5 = 5!$ *ways*

No two GIRLS are seated together

$\_\_ \; B_1 \; \_\_ \; B_2 \; \_\_ \; B_3 \; \_\_ \; B_4 \; \_\_ \; B_5 \; \_\_$

In each of these arrangements 6 gaps are created.

Since no two girls are to sit together, we any arrange 4 girls in this 6 gaps. This can be done in $6P_4$ ways.

Hence, the total number of seating arrangements is

$$5! \times 6P_4 = (5 \times 4 \times 3 \times 2 \times 1) \times (6 \times 5 \times 4 \times 3)$$

$$= 120 \times 360 = 43200$$

Example 4.33: 4 boys and 4 girls form a line with the boys and girls alternating. Find the of ways of making this line.

4 boys can be arranged in a line in $4P_4 = 4!$ ways

By keeping boys as first in each of these arrangements,

$B_1 \; \_\_ \; B_2 \; \_\_ \; B_3 \; \_\_ \; B_4 \; \_\_$ 4 gaps are created.

In these 4 gaps, 4 girls can be arranged in $4P_4 = 4!$ ways.

$The\ total\ number\ of\ arrangement\ are = 4! \times 4!$

OR

Similarly, keeping girls as first,

G_1 — G_2 — G_3 — G_4 — 4 gaps are created.

$The\ total\ number\ of\ arrangement\ are = 4! \times 4!$

$Hence, by\ the\ rule\ of\ sum\ keeping\ either\ a\ boy\ or\ a\ girl\ first,$

$The\ total\ numberr\ of\ arrangements\ are$

$$(4! \times 4!) + (4! \times 4!) = 2\,(4!)^2 = 2\,(4 \times 3 \times 2 \times 1)^2$$

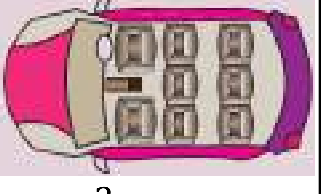
$$= 2\,(24)^2 = 2 \times 576 = 1152$$

Example 4.34: A van has 8 seats. It has two seats in the front with two rows of three seats behind. The van belongs to a family, consisting of seven members, F, M, S_1, S_2, S_3, D_1, D_2. How many ways can the family sit in the van if (i) There are no restriction, (ii) Either F or M drives the van, (iii) D_1, D_2 sits next to a window and F is driving?

(i)There are no restrictions any one can drive the van.

Hence the number of ways of occupying the driver seat is $7P_1 = 7$ ways.

The number of ways of occuping the remaining 7 seats by the remaining 6 people is $7P_6 = 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 5040$



Hence the total number of ways the family can be seated in the car is $7 \times 5040 = 35280$

(ii) Either F or M drives the van,

The driver seat can be occupied by only F or M, Hence there are 2 ways

The number of ways of occuping the remaining 7 seats by the remaining 6 people is $7P_6 = 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 5040$

Hence the total number of ways the family can be seated in the car is

$$2 \times 5040 = 10080$$

(iii)D_1, D_2 sits next to a window and F is driving

5 window seats avialable for D_1 & D_2

$$5P_2 = 5 \times 4 = 20$$

As the driver seat is occupied by F, $-$ 1 way

The remaining 4 people can be seated in the available 5 seats in

$$5P_4 = 5 \times 4 \times 3 \times 2 = 120$$

Hence the total number of ways the family can be seated in the car is

$$20 \times 1 \times 120 = 2400$$

Example 4.35: If the letters of the word TABLE are permuted in all possible ways and the words thus formed are arranged in the dictionary order (alphabetical order), find the ranks of the words (i) TABLE, (ii) BLEAT.

The dictionary order of the letters of given word is, A, B, E, L, T.

In the dictionary order of the words which begin with A come first.

If we fill the first place with A,

remaining 4 letters (B, E, L, T) can be arranged in $4P_4 = 4!$ ways.

(i) The rank of the word TABLE

A $- - - - = 4! = 24$ ways

B $- - - - = 4! = 24$ ways

E $- - - - = 4! = 24$ ways

L $- - - - =$ 1 way $= 1$ ways

TABEL $= 4!$

TABLE $= 1$ way

The rank of the word TABLE is $4 \times 4! + 1 + 1$

$= 4 \times 24 + 2 = 96 + 2 = 98$

(ii) The rank of the word BLEAT

A $- - - - = 4! = 24$ ways

BA $- - - = 3! = 6$ ways

BE $- - - = 3! = 6$ ways

BLA $- - = 2! = 2$ ways

BLEAT $= 1$ way

The rank of the word BLEAT is $24 + 6 + 6 + 2 + 1 = 39$

Eg 4.36: Find the number of ways of arranging the letters of the word BANANA.

This word has 6 letters in which there are 3 A'S, 2N's and one B.

The number of ways of arrangements is

$$\frac{6!}{3! \times 2!} = \frac{6 \times 5 \times \overset{2}{\cancel{4}} \times \cancel{3!}}{\cancel{3!} \times \cancel{2} \times 1} = 60$$

Example 4.37 : Find the number of ways of arranging the letters of the word RAMANUJAN so that the relative positions of vowels and consonants are not changed.

In the word RAMANUJAN there are 4 vowels (A, A, U, A)

in that 3 A's, 1U and 5 consonants (R, M, N, J, N)

in that two N's and rest are distinct

The 4 vowels (A, A, A, U) can be arranged themselves in $\frac{4!}{3!} = \frac{4 \times \cancel{3!}}{\cancel{3!}} = 4\ ways$

The consonants (R, M, N, J, N) can be arranged themselves in $\frac{5!}{2!} = \frac{5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}}$

$= 60\ ways$

Therefore the number of required arrangements are $= 4 \times 60 = 240$

Example 4.38: Three twins pose for a photoggraph standing in a line. How many arrangements are there (i) when there are no restrictions. (ii)when each person is standing next to his or her to win?

(i) The six persons without any restriction may be arranged in

$$6P_6 = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \text{ ways}$$

(ii) Let us consider three twins as T_1, T_2, T_3.

Each twin is considered as a single unit and these three can be permuted in 3! ways.

Again each twin can be permuted between themselves in 2! ways.

Hence, the total number of arrangements is

$$3! \times 2! \times 2! \times 2! = (3 \times 2 \times 1)(2 \times 1)(2 \times 1)(2 \times 1)$$

$$= 6 \times 2 \times 2 \times 2 = 48 \text{ ways}$$

Eg 4.41: If the different permutations of all letters of the word BHASKARA are listed as in a dictionary, how many strings are there in this list before the first word starting with B?

The required number of strings is

the total number of strings starting with A and using the letters A, A, B, H, K, R, S is

$$\frac{7!}{2!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}} = 2520$$

Eg 4.42: If the letters of the word IITJEE are permuted in all possible ways and the strings thus formed are arranged in the lexicographic order, find the rank of the word IITJEE.

The lexicographic order of the letters of given word is E, E, I, I, J, T.

In the lexicographic order, the strings which begin with E come first.

If we fill the first place with E,

remaining 5 letters(E, I, I, J, T) can be arranged in $\frac{5!}{2!}$ ways.

On proceeding like this we get,

$$E - - - - = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}} = 60\ ways$$

$$IIE - - - = 3! = 3 \times 2 \times 1 = 6\ ways$$

$$IIJ - - - = \frac{3!}{2!} = \frac{3 \times \cancel{2!}}{\cancel{2!}} = 3\ ways$$

$IITE - - = 2! = 2 \times 1 = 2\ ways$

IITJEE $= 1\ way$

$The\ rank\ of\ the\ word\ IITJEE\ is\ 60 + 6 + 3 + 2 + 1 = 72$

***Example 4.43**: Find the sum of all 4 − digit numbers that can be formed using the digits 1, 2, 4, 6, 8.*

The number of 4 − digitits numbers that can be formed using the given 5 digits is

$$5P_4 = 5 \times 4 \times 3 \times 2 = 120$$

We first find the sum of the digits in the unit place of all these 120 numbers.

By filling the 1 in unit place,

the remaining three places can be filled with remaining 4 digits in

$$4P_3 = 4 \times 3 \times 2 = 24\ ways$$

This means, the number of 4 − digit numbers having 1 in units place is

$$4P_3 = 4 \times 3 \times 2 = 24\ ways$$

Similarly, each of the digits 2,4,6,8 appear 24 times in units place.

An addition of all these digits givens the sum of all the unit digits of all 120 numbers.

$$(4P_3 \times 1) + (4P_3 \times 2) + (4P_3 \times 4) + (4P_3 \times 6) + (4P_3 \times 8)$$

$$= 4P_3 \times (1 + 2 + 4 + 6 + 8) = 4P_3 \times 21$$

Similarly, we get the sum of the digits in 10^{th} *place as* $4P_3 \times 21$.

Since it is in 10^{th} *place,* its value is $4P_3 \times 21 \times 10$

Similarly, the values of the sum of the digits in 100^{th} place and 1000^{th} place

$4P_3 \times 21 \times 100$ and $4P_3 \times 21 \times 1000$ respectively

Hence the sum of all the 4 digit numbers formed by using the digits 1, 2, 4, 6,8 is

$$(4P_3 \times 21) + (4P_3 \times 21 \times 10) + (4P_3 \times 21 \times 100) + (4P_3 \times 21 \times 1000)$$

$$= 4P_3(21 \times 1111)$$

$$= 24 \times 21 \times 1111 = 5559944$$

1. If $(n-1)P_3 : nP_4 = 1 : 10$, find n

$$(n-1)P_3 : nP_4 = 1 : 10$$

$$\frac{(n-1)P_3}{nP_4} = \frac{1}{10}$$

$$\frac{\cancel{(n-1)(n-2)(n-3)}}{n\cancel{(n-1)(n-2)(n-3)}} = \frac{1}{10}$$

$$\frac{1}{n} = \frac{1}{10} \Longrightarrow n = 10$$

$$\boxed{nP_r = \frac{n!}{(n-r)!}}$$

$$\boxed{5P_3 = 5 \times 4 \times 3}$$

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2. If $10P_{r-1} = 2 \times, 6P_r$, find r.

$$\boxed{nP_r = \frac{n!}{(n-r)!}}$$

$$10\,P_{r-1} = 2 \times 6\,P_r$$

$$\frac{10!}{[10-(r-1)!]} = 2 \times \frac{6!}{(6-r)!}$$

$$\frac{10!}{(10-r+1)!} = 2 \times \frac{6!}{(6-r)!}$$

$$\frac{10!}{(11-r)!} = 2 \times \frac{6!}{(6-r)!}$$

$$\frac{10 \times 9 \times 8 \times 7 \times \cancel{6!}}{(11-r)(10-r)\,(9-r)\,(8-r)\,(7-r)\,\cancel{(6-r)!}} = \frac{2 \times \cancel{6!}}{\cancel{(6-r)!}}$$

$$\frac{10 \times 9 \times \overset{4}{\cancel{8}} \times 7}{(11-r)(10-r)(9-r)(8-r)(7-r)} = \cancel{2}\ 1$$

$$10 \times 9 \times 4 \times 7 = (11-r)(10-r)(9-r)(8-r)(7-r)$$

$$(11-r)(10-r)(9-r)(8-r)(7-r) = 5 \times 2 \times 3 \times 3 \times 4 \times 7$$

$$(11-r)(10-r)(9-r)(8-r)(7-r) = 7 \times 6 \times 5 \times 4 \times 3$$

$$11 - r = 7 \Longrightarrow 11 - 7 = r$$

$$\boxed{r = 4}$$

3. (i) Suppose 8 people enter an event in a swimming meet. In how many ways could the gold, silver, and bronze prizes be awarded?
(ii) Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them?

(i) 8 people, 3 prizes (G, S, B) Number of ways of awarding

$$= 8P_3 = 8 \times 7 \times 6 = 336$$

(ii) Number of ways they can wear is

$$= 4P_3 \times 5P_3 \times 6P_3$$
$$= (4 \times 3 \times 2) \times (5 \times 4 \times 3) \times (6 \times 5 \times 4)$$
$$= 24 \times 60 \times 120$$
$$= 172800$$

4. Determine the number of permutations of the letters of the word SIMPLE if all are taken at a time?

SIMPLE

$$\textit{Number of letters} = 6$$

$$\textit{Required number of permutations} = 6P_6$$
$$= 6!$$
$$= 6 \times 5 \times 4 \times 3 \times 2 \times 1$$
$$= 720$$

5. A test consists of 10 multiple choice questions. In how many ways can the test be answered if (i) Each question has four choices? (ii) The first four questions have three choices and the remaining have five choices? (iii) Question $number\ n\ has\ n+1\ choices$?

(i) $Number\ of\ ways\ the\ test\ can\ be$ answered $is = 4^{10}$

(ii) The first four question have three choices and the remainder have five choices, then the number of ways the test can be answered is $= 3^4 \times 5^6$.

(iii) $Question\ number\ n\ has\ n+1\ choices$

$First\ question\ has\ 2\ choices$

$Second\ questions\ has\ 3\ choices$

$Third\ question\ has\ 4\ choices\ etc.$

$Number\ of\ ways\ the\ test\ can\ be\ answered\ is$

$$2 \times 3 \times 4 \times \cdots 11 = 11!$$

6. A student appears in an objective test which contain 5 multiple choice questions. Each question has four choices out of which one correct answer(i) What is the maximum number of different answers can the students give? (ii) How *will the answer change if each question may have more than onecorrect answers*?

$5\ multiple\ choice\ questions\ each\ having\ 4\ choices.$

(i) $The\ maximum\ number\ of\ answers = 4^5,$

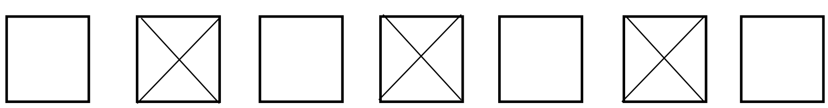
(ii) $The\ question\ may\ have\ 1\ correct\ answer, or\ 2\ correct\ answers\ or\ 3\ or\ 4\ or\ 5\ correct\ answers.$

$Number\ of\ correct\ answers = 1 + 2 + 3 + 4 + 5 = 15$

$\therefore Maximum\ number\ of\ answers = 15^5$

7. How many strings can be formed from the letters of the word ARTICLE, so that vowel occupy the even places?

$ARTICLE = Number\ of\ letters = 7$



Vowels = A E I

consonants = R T C L

Let the 3 vowels occupy 3 even places in $3!$ ways.

The remaining 4 letters will occupy the remaining 4 places in $4!$ ways.

$\therefore$ Number of strings $= 3! \times 4!$

$= (3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1)$

$= 6 \times 24 = 144$

8. 8 women and 6 men are standing in a line.

(i) How many arrangements are possible if any individual can stand in any position?

(ii) In how many arrangements will all 6 men be standing next to one another?

(iii) In how many arrangements will no two men be standing next to one another?

8 women and 6 men. Total 14 persons.

(i) 14 persons can stand in a line in 14! persons.

(ii) $n = 14, m = 6$

$$\text{Number of arrangements} = m! \times (n - m + 1)!$$
$$= 6! \times (14 - 6 + 1)!$$
$$= 6! \times 9!$$

(iii) $n = 14, k = 6, m = n - k$

$$m = 14 - 6$$
$$m = 8$$

$$\text{Number of arrangemets} = m!\,(m + 1)P_k$$
$$= 8!\,(8 + 1)P_6$$
$$= 8!\,9P_6$$

9. Find the distinct permutations of the letters of the word MISSISSIPPI?

MISSISSIPPI = Number of letters = 11!

Number of S′s = 4!

Number of I′s = 4!

Number of P′s = 2!

$$\text{Required number of arrangements} = \frac{11!}{4!\,4!\,2!}$$
$$= \frac{11 \times 10 \times 9 \times \cancel{8}^{4} \times 7 \times \cancel{6}^{3} \times 5 \times \cancel{4!}}{\cancel{4!} \times (\cancel{4} \times \cancel{3} \times \cancel{2} \times 1) \times (\cancel{2} \times 1)}$$
$$= 34650$$

10. How many ways can the product $a^2\,b^3\,c^4$ be expressed without exponents?

$$\textit{Total number of letters} = 9$$
$$a = 2, b = 3, c = 4$$

$$\textit{Total number of ways} = \frac{9!}{2!\,3!\,4!} = \frac{9 \times \cancel{8}^{4} \times 7 \times \cancel{6}^{\cancel{3}} \times 5 \times \cancel{4!}}{(\cancel{2} \times 1) \times (\cancel{3} \times \cancel{2} \times 1)\cancel{4!}}$$
$$= 1260$$

11. In how many ways 4 mathematics books, 3 physics books, 2 chemistry books and 1 biology book can be arranged on a shelf so that all books of the same subjects are together.

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$Take\ 4\ Maths\ books\ as\ 1\ unit$

$3\ Physics\ books\ as\ 1\ unit$

$2\ Chemistry\ book\ as\ 1\ unit$

$1\ Biology\ book\ as\ 1\ unit$

$There\ are\ 4\ units\ which\ can\ be\ arranged\ in\ 4!\ ways.$

$4\ Maths\ books\ can\ be\ arranged\ in = 4!$

$3\ Physics\ books\ can\ be\ arranged\ in = 3!$

$2\ Chemistry\ book\ can\ be\ arranged\ in = 2!$

$1\ Biology\ bookcan\ be\ arranged\ in = 1!$

$$\text{Number of arrangements} = 4! \times 4! \times 3! \times 2!$$
$$= 4 \times 3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1 \times 2$$
$$= 6912$$

12. In how many ways can the letters of the word SUCCESS be arranged so that all Ss are together?

SUCCESS, Let all the 3 S′s be considered as 1 unit.

There will be 5 units with 2 C′s.

$$\therefore \text{Number of arragemens} = \frac{5!}{2!}$$
$$= \frac{5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}} = 60$$

13. A coin is tossed 8 times,

(i) How many different sequences of heads and tails are possible?

(ii) How many different sequences containing sixheads and two tails are possible?

(i) Total Possible ways $= 2^8$

$$\text{(ii) Number of ways} = \frac{8!}{6!\ 2!}$$
$$= \frac{\overset{4}{\cancel{8}} \times 7 \times \cancel{6!}}{\cancel{6!} \times \cancel{2} \times 1} = 28$$

14. How many strings are there using the letterrs of the word INTERMEDIATE, if (i) The vowels and consonants are alternative (ii) All the vowels are together (iii) Vowels are never together (iv) No two vowels are together.

INTERMEDIATE

Number of letters $= 12$

Number of I′s $= 2$

Number of T′s $= 2$

Number of E′s $= 3$

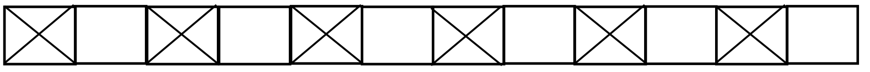
The consonants are
$N = 1$
$T = 2$
$R = 1$
$M = 1$
$D = 1$

The vowels are

$$A = 1$$
$$I = 2$$
$$E = 3$$

$\text{No. of vowels} = 1 + 2 + 3 = 6$

(i) The vowels and consonants are alternative



Let the first place be given to a vowel.

There are six places available for 6 vowels

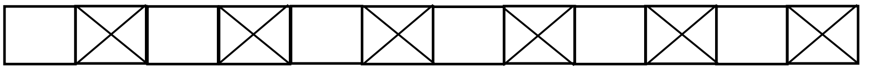
$$\therefore \textit{Number of permutation} = \frac{6!}{2!\,3!}$$

And the remaining 6 places can be given to consonants.

$$\textit{Number of permutation} = \frac{6!}{2!}$$

$$\textit{Number of ways} = \frac{6!}{2!\,3!} \times \frac{6!}{2!}$$

Let the second place be given to a vowel.



First place consonants and second place vowels are $\frac{6!}{2!} \times \frac{6!}{2!\,3!}$

$$\therefore \text{Required number of ways} = \frac{6!}{2!\,3!} \times \frac{6!}{2!} + \frac{6!}{2!} \times \frac{6!}{2!\,3!}$$

$$= 2 \times \frac{6!}{2!\,3!} \times \frac{6!}{2!}$$

$$= 2 \times \frac{6 \times 5 \times \cancelto{2}{4} \times \cancel{3!}}{\cancel{2} \times \cancel{3!}} \times \frac{6 \times 5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}}$$

$$= 2 \times 60 \times 360 = 43200$$

(ii) All the vowels are together

Take all the vowels as 1 unit.

Therefore there will be 7 units available for arrangemet.

$$\text{Number of arrangements} = \frac{7!}{2!} \text{ (There are 2 T's)}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}} = 2520$$

Among the vowels the number of arrangements $= \dfrac{6!}{2!\,3!}$

$$= \frac{6 \times 5 \times \cancel{4}^{2} \times \cancel{3!}}{\cancel{2} \times \cancel{3!}} = 60$$

$\therefore$ Required number of permutation $= 2520 \times 60$

$= 151200$

(iii) Vowels are never together

Total number of arrangements $= \dfrac{12!}{2!\ 2!\,3!}$

$$= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times \cancel{4}^{\cancel{2}} \times \cancel{3!}}{\cancel{2} \times \cancel{2} \times \cancel{3!}} = 19958400$$

Number of arrangements where all vowels are together is $= 151200$

Number of arrangements where all vowels are never together

$= 19958400 - 151200 = 19807200$

(iv) No two vowels are together

$n = 12, k = 6, m = n - k$

$m = 12 - 6$

$m = 6$

No two vowels are together $= m!\,(m+1)P_k$

$= 6!\,(6+1)P_6$

$= 6! \times 7P_6$

Number of arrangemets $= \dfrac{6! \times 7P_6}{3! \times 2! \times 2!}$

$$= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{6 \times 2 \times 2}$$

$= 151200$

15. Each of the digits 1, 1, 2, 3, 3 and 4 is written on a separate card. The six cards then laid out in a row to form a 6 − digit number. (i) How many distinct 6 − digit numbers? (ii) How many of these 6 − digit numbers are even? (iii) How many of these 6 − digit numbers are divisible by 4?

The digits are 1, 1, 2, 3, 3 and 4

(i) Number of distinct 6 digit numbers $= \dfrac{6!}{2!\,2!} = \dfrac{6 \times 5 \times \cancel{4}^{2} \times 3 \times \cancel{2!}}{\cancel{2} \times \cancel{2!}} = 180$

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(ii) *If a number is even, the unit place must be either* 2 *or* 4.

$$\text{Number of cards } = \frac{5!}{2!\,2!} \text{ if 2 is in the unit place}$$

$$= \frac{5!}{2!\,2!} \text{ if 4 is in the unit place}$$

$$\text{Total number of cards} = \frac{5!}{2!\,2!} + \frac{5!}{2!\,2!} = \frac{5!+5!}{2!\,2!}$$

$$= \frac{\cancel{2}(5!)}{\cancel{2}\times 2} = \frac{5\times4\times3\times\cancel{2}}{\cancel{2}} = 60$$

(iii) If a number is divisible by 4,

The digits are 1, 1, 2, 3, 3 and 4

then the last two digits (as a number) mus be divisible by 4.

Therefore the last two digits should be 24 *or* 32 *or* 12.

$$\text{Number of cards with 24 in last two digits } = \frac{4!}{2!\,2!}$$

$$\text{Number of cards with 32 in last two digits } = \frac{4!}{2!}$$

$$\text{Number of cards with 12 in last two digits } = \frac{4!}{2!}$$

$$\text{Number of cards } = \frac{4!}{2!\,2!} + \frac{4!}{2!} + \frac{4!}{2!}$$

$$= \frac{\overset{2}{\cancel{4}}\times 3\times\cancel{2!}}{\cancel{2}\times\cancel{2!}} + \frac{4\times3\times\cancel{2!}}{\cancel{2!}} + \frac{4\times3\times\cancel{2!}}{\cancel{2!}}$$

$$= 6 + 12 + 12 = 30$$

16. If the letters of the word GARDEN are permuted in all possible ways and the strings thus formed are arranged in the dictionary order, then find the ranks of he words (i) GARDEN (ii) DANGER.

(i) $GARDEN \longrightarrow A, D, E, G, N, R$

$A - - - - - = 5! = 120$

$D - - - - - = 5! = 120$

$E - - - - - = 5! = 120$

G starts

GA starts

GAD $- - - = 3! = 6$

GAE $- - - = 3! = 6$

GAN $- - - = 3! = 6$

GAR starts

GARD starts

GARDE starts

GARDEN $= 1$

Total $= \underline{379}$

$DANGER \rightarrow A, D, E, G, N, R$

$A - - - - -$ $= 120$

D starts

DA starts

DAE $- - -$ $= 6$

DAG $- - -$ $= 6$

DAN starts

DANE $- -$ $= 2$

DANG starts

DANGE starts

DANGER $= 1$

Total $= 135$

17. Find the number of strings that can be made using all letters of the word THING. If these words are written as in a dictionary, what will be the 85$^{\text{th}}$ string?

$THING \Rightarrow GHINT$

$G - - - - \Rightarrow 4! = 24$

$H - - - - \Rightarrow 4! = 24$

$I - - - \Rightarrow 4! = 24$

N starts

NG $- - - \Rightarrow 3! = 6$

NH $- - - \Rightarrow 3! = 6$

NI starts

NIG starts

NIGH starts

NIGHT $= 1$

Total $= 85$ $\therefore 85^{th}$ *word is NIGHT.*

18. If the letters of the word FUNNY are permuted in all possible ways and the strings thus formed are arranged in the dictionary order, find the rank the word FUNNY.

$FUNNY = FNNUY$

F starts

$FN - - - \Rightarrow 3! = 6$

FU starts

FUN starts

$FUNN\ starts$

$FUNNY \qquad = 1$

$Rank \qquad = \underline{7}$

19. Find the sum of all 4 − digit numbers that can be formed using digits 1, 2, 3, 4, and 5 repetitions not allowed?

$The\ sum\ of\ all\ r-digit\ number\ can\ be\ formed\ using\ the\ given\ n$
$non-zero\ digits$

$(n-1)P_{(r-1)} \times (sum\ of\ the\ digits) \times 111 \dots\dots\dots 1(\text{r times})$

$Sum\ of\ the\ digits = 1+2+3+4+5$

$Sum\ of\ the\ digits = 15$

$n = 5, \quad r = 4$

$Sum\ = 4P_3 \times 15 \times 1111$

$= 4 \times 3 \times 2 \times 15 \times 1111 = 399960$

20. Find the sum of all 4 − digit numbers that can be formed using digits 0, 2, 5, 7, 8 without repetition?

$Sum\ of\ the\ digits = 0+2+5+7+8$

$Sum\ of\ the\ digits = 22$

$n = 5, \quad r = 4$

$Sum\ = (n-1)P_{(r-1)} \times (sum\ of\ the\ digits) \times 111 \dots\dots\dots (r\ times)$

$-(n-2)P_{(r-2)} \times (sum\ of\ the\ digits) \times 111 \dots\dots\dots (r-1)\ times$

$= 4P_3 \times 22 \times 1111 - 3P_2 \times 22 \times 111$

$= 4 \times 3 \times 2 \times 22 \times 1111 - 3 \times 2 \times 22 \times 111$

$= 586608 - 14652$

$= 571956$

EXERCISE 4.3

44. Evaluate the following: (i) $10C_3$, (ii) $15C_{13}$, (iii) $100C_{99}$, (iv) $50\,C_{50}$.

$$\text{(i) } \boldsymbol{10c_3} = \frac{10!}{7!\times 3!} = \frac{10\times \overset{3}{\cancel{9}}\times \overset{4}{\cancel{8}}\times \cancel{7}!}{\cancel{7}!\times \cancel{3}\times \cancel{2}\times 1} = 120$$

$$(ii)\ \boldsymbol{45C_{13}} = \frac{15!}{2!\times 3!} = \frac{15\times \overset{7}{\cancel{14}}\times \cancel{13!}}{\cancel{2}\times 1\times \cancel{13!}} = 105$$

$$(iii)\ \boldsymbol{100C_{99}} = \frac{100\times \cancel{99!}}{\cancel{99!}} = 100$$

$$(iv)\ \boldsymbol{50C_{50}} = \frac{\cancel{50!}}{\cancel{50!}} = 1$$

45. Find the value of $5C_2$ and $7C_3$ using the property 5.

$$nC_r = \frac{n}{r}\times n-1C_{r-1}$$

Substituting $n = 5$ *and* $r = 2$, *we get*

$$5C_2 = \frac{5}{2}\times(5-1)C_{2-1} = \frac{5}{2}\times 4C_1 = \frac{5}{\cancel{2}}\times\frac{\overset{2}{\cancel{4}}}{1} = 10$$

Substituting $n = 7$ *and* $r = 3$, *we get*

$$7C_3 = \frac{7}{3}\times(7-1)C_{3-1} = \frac{7}{3}\times 6C_2 = \frac{7}{\cancel{3}}\times\frac{\overset{\cancel{2}}{\cancel{6}}\times 5}{\cancel{2}} = 35$$

46. If $nC_4 = 495$. What is n?

$$nC_4 = 495$$

$$\frac{n\times(n-1)\times(n-2)\times(n-3)}{4\times 3\times 2\times 1} \longrightarrow 495$$

$$n\times(n-1)\times(n-2)\times(n-3) = 495\times 4\times 3\times 2\times 1$$

Factoring $495 = 3\times 3\times 5\times 11$,

and writing this product as a product of 4 *consecutive numbers in the descending order*

we get, $n(n-1)(n-2)(n-3) = 12\times 11\times 10\times 9$

Equating n with the maximum number, we obtain

$$n = 12$$

47. If $nP_r = 11880$ and $nC_r = 495$, Find n and r.

$$\frac{nP_r}{nC_r} = r!$$

$$r! = \frac{\overset{24}{\cancel{792}}\;\cancel{2376}\;\cancel{11880}}{\cancel{495}\;\cancel{99}\;\underset{1}{\cancel{33}}} = 24 = 4!$$

$$r! = 4! \implies r = 4$$

Using this $r = 4$, in $nC_4 = 495$,

$$\frac{n \times (n-1) \times (n-2) \times (n-3)}{4 \times 3 \times 2 \times 1} = 495$$

$$n \times (n-1) \times (n-2) \times (n-3) = 495 \times 4 \times 3 \times 2 \times 1$$

$$Factoring\ 495 = 3 \times 3 \times 5 \times 11,$$

$$n(n-1)(n-2)(n-3) = 12 \times 11 \times 10 \times 9$$

$$n = 12.$$

48. Prove that $\mathbf{24C_4 + \sum_{r=0}^{4}(28-r)C_3 = 29C_4}$.

$$L.H.S = 24C_4 + \sum_{r=0}^{4}(28-r)\ C_3$$

$$= 24C_4 + 28C_3 + 27C_3 + 26C_3 + 25C_3 + 24C_3$$

$$= 24C_4 + 24C_3 + 25C_3 + 26C_3 + 27C_3 + 28C_3$$

$$= 25C_4 + 25C_3 + 26C_3 + 27C_3 + 28C_3$$

$$= 26C_4 + 26C_3 + 27C_3 + 28C_3 = 27C_4 + 27C_3 + 28C_3$$

$$= 28C_4 + 28C_3 = 29C_4 = R.H.S$$

Example 4.49: Prove that $\mathbf{10C_2 + 2 \times 10C_3 + 10C_4 = 12C_4}$.

$$L.H.S = 10C_2 + 2 \times 10C_3 + 10C_4$$

$$= 10C_2 + (10C_3 + 10C_3) + 10C_4$$

$$= (10C_2 + 10C_3) + (10C_3 + 10C_4) = 11C_3 + 11C_4$$

Example 4.50: If $\mathbf{(n+2)C_7 : (n-1)P_4 = 13:24}$ find n.

$$(n+2)C_7 : (n-1)P_4 = 13:24$$

$$\frac{(n+2)C_7}{(n-1)P_4} = \frac{13}{24}$$

$$\frac{(n+2)!}{\cancel{(n-5)!}\,7!} \times \frac{\cancel{(n-5)!}}{(n-1)!} = \frac{13}{24} \Rightarrow \frac{(n+2)(n+1)n\cancel{(n-1)!}}{\cancel{(n-1)!}\cdot 7!} = \frac{13}{24}$$

$$(n+2)(n+1)(n) = \frac{13}{24} \times 7!$$

$$(n+2)(n+1)(n) = \frac{13}{\cancel{24}_{\cancel{6}}} \times 7 \times \cancel{6} \times 5 \times \cancel{4} \times 3 \times 2 \times 1$$

$$(n+2)(n+1)(n) = 13 \times 14 \times 15 \Rightarrow n+2 = 15$$

$$n = 15 - 2 \Rightarrow \boxed{n = 13}$$

Example 4.51: A salad at a certain restaurant consists of 4 of the following fruits: apple, bannana, guava, pomegranate, grapes, papaya and pineapple. Find the total possible number of fruit salads.

There are seven fruits and we have to select four fruits for the fruit salad. Hence, the total number of possible ways of making a fruit salad is

$$7C_4 = 7C_{7-4}$$

$$= 7C_3$$

$$\boxed{nC_r = \frac{n!}{r!\,(n-r)!}}$$

where $n = 7,\ r = 3$

$$7C_3 = \frac{7!}{3!\,(7-3)!} = \frac{7!}{3!\,4!} = \frac{7 \times \overset{2}{\cancel{6}} \times 5 \times \cancel{4!}}{\cancel{3} \times \cancel{2} \times 1 \times \cancel{4!}} = 35$$

***Example* 4.52: A Mathematics club has 15 members. In that 8 are girls. 6 of the members are to be selected for a competition and half of them should be girls. How many ways of these selections are possible?**

There are 8 girls and 7 boys in the mathematics club.

The number of ways of selecting 6 members in that half of them girls (3 girls and 3 others) is

$$8C_3 \times 7C_3 = \frac{8!}{3!\,(8-3)!} \times \frac{7!}{3!\,(7-3)!}$$

$$= \frac{8!}{3!\,5!} \times \frac{7!}{3!\,4!} = \frac{8 \times 7 \times \overset{3}{\cancel{6}} \times \cancel{5!}}{\cancel{3} \times \cancel{2} \times \cancel{5!}} \times \frac{7 \times \overset{3}{\cancel{6}} \times 5 \times \cancel{4!}}{\cancel{3} \times \cancel{2} \times 1 \times \cancel{4!}}$$

$$= 56 \times 35 = 1960$$

Example 4.53: In rating 20 brands of cars, a car magazine picks a first, second, third, fourth and fifth best brand and the 7 more as acceptable. In how many ways can it be done?

The picking of 5 brands for a first, second, third, fourth and fifth best brand from 20 brands in $20P_5$ ways.

From the remaining 15 we need to select 7 acceptable in $15C_7$ ways.

By the rule of product it can be done in $20P_5 \times 15P_7$ ways.

Example 4.54: From a class of 25 students, 10 students are to be chosen for an excursion party. There was4 students who decide that either all of them will join or none of them will join. In how manyways can the excursion party be chosen?

There are two possibilities.

(i) All the 4 students will got to the excursion party then,

we need to select 6 students out of 21 students.

It can be done in $21C_6 = \dfrac{21!}{6! \times (21-6)!} = \dfrac{21!}{6! \times 15!}$ ways.

(ii) All the 4 students will not go to the excursion party then,

we need to select 10 students out of 21 students.

It can be done in $21C_{10} = \dfrac{21!}{10! \times (21-10)!} = \dfrac{21!}{10! \times 11!}$.

Hence, the total number of ways is $21C_6 + 21C_{10} = \dfrac{21!}{6! \times 15!} + \dfrac{21!}{10! \times 11!}$

Example 4.55: A box of one dozen apple contains a rotten apple. If we are choosing 3 apples simultaneously, in how many ways, one can get only good apples.

$The\ total\ number\ of\ ways\ of\ selecting\ 3\ apples\ from\ 12\ apples\ is\ 12C_3$

$$= \frac{12!}{3! \times (12-3)!} = \frac{12!}{3! \times 9!} = \frac{\overset{4}{\cancel{12}} \times 11 \times \overset{5}{\cancel{10}} \times \cancel{9!}}{\cancel{3} \times \cancel{2} \times \cancel{9!}} = 220.$$

$The\ total\ number\ of\ ways\ of\ getting\ a\ rotten\ apple\ when\ selecting$ $3\ apples\ from\ 12\ apples\ is\ equal\ to$

$selecting\ 1\ rotten\ apple\ and\ remaining\ 2\ apple\ can\ be\ selected\ from$ $11\ apples\ is$

$$11C_2 = \frac{11!}{2! \times (11-2)!} = \frac{11!}{2! \times 9!} = \frac{11 \times \overset{5}{\cancel{10}} \times \cancel{9!}}{\cancel{2} \times \cancel{9!}} = 55.$$

$Therefore, the\ total\ number\ of\ ways\ of\ getting\ only\ good\ apple\ is$

$$12C_3 - 11C_2 = 220 - 55 = 165$$

Example 4.56: $\boldsymbol{An\ exam\ paper\ contains\ 8\ questions, 4\ in\ Part\ A\ and\ 4\ in\ Part\ B. Examiners\ are\ required\ to\ answer\ 5\ questions.}$ $\boldsymbol{In\ how\ many\ ways\ can\ this\ be\ done\ if}$

(i) There are no restrictions of choosing a number of questions in either parts.

(ii) At least two questions from Part A must be answered.

(i) There are no restrictions.

Totally there are 8 questions in both Part A and Part B.

The total number of ways of attempting 5 questions is

$$8C_5 = 8C_{8-5} = 8C_3$$

$$= \frac{8!}{3! \times (8-3)!} = \frac{8!}{3! \times 5!} = \frac{8 \times 7 \times \cancel{6}^{\,3} \times \cancel{5!}}{\cancel{3} \times \cancel{2} \times \cancel{5!}} = 56$$

(ii) Atleast two questions from Part A needs to be answered.

Accordingly, various choices are tabulated as follows.

Part A	*Part B*	*Number of selections*
2	3	$4C_2 \times 4C_3$
3	2	$4C_3 \times 4C_2$
4	1	$4C_4 \times 4C_1$

Therefore, the required number of ways of answering is

$$= 4C_2 \times 4C_3 + 4C_3 \times 4C_2 + 4C_4 \times 4C_1$$

$$= 4C_2 \times 4C_3 (1+1) + 1 \times 4$$

$$= 4C_2 \times 4C_1 (2) + 4$$

$$= \frac{4!}{2!\,(4-2)!} \times 4 \times 2 + 4$$

$$= \frac{\cancel{4}^{\,2} \times 3 \times \cancel{2!}}{\cancel{2} \times \cancel{2!}} \times 8 + 4 = 48 + 4 = 52$$

Example 4.57: Out of 7 consonants and 4 vowels, how many strings of 3 consonants and 2 vowels can be formed?

Number of ways of selecting

(3 *consonants out of* 7) *and* (2 *vowels out of* 4) *is* $7C_3 \times 4C_2$

Each string contains 5 *letters.*

Number of ways of arranging 5 *letters among themselves is*

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Hence required number of ways is,

$$7C_3 \times 4C_2 \times 5! = \frac{7!}{3!\,(7-3)!} \times \frac{4!}{2!\,(4-2)!} \times 5!$$

$$= \frac{7!}{3!\,4!} \times \frac{4!}{2!\,2!} \times 120 = \frac{7 \times \cancel{6}^{\,3} \times 5 \times \cancel{4!}}{\cancel{3} \times \cancel{2} \times 1 \times \cancel{4!}} \times \frac{\cancel{4}^{\,2} \times 3 \times \cancel{2!}}{\cancel{2} \times 1 \times \cancel{2!}} \times 120$$

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$= 35 \times 6 \times 120 = 25200$

Example 4.58: Find the number of strings of 5 letters that can be formed with the letters of the word PROPOSITION.

There are 11 *letters in the word,*

with respect to number of repetitions of letters

there are 4 *distinct letters* (R, S, T, N), 2 *sets of two alike letters* (PP, II),

1 *set of three alike letters* (OOO).

The following table will illustrate the combination of these sets and the number of words

SI.No	Letter options	Selections	Arrangements
1	5 distinct (R, S, T, N, P, I, O)	$7C_5$	$7C_5 \times 5! = 2520$
2	1 set of 3 alike (OOO), 1 set of 2 alike (PP, II)	$1C_1 \times 2C_1$	$1C_1 \times 2C_1 \times \frac{5!}{3! \times 2!} = 20$
3	1 set of 3 alike (OOO), 2 distinct (R, S, T, N, P, I)	$1C_1 \times 6C_2$	$1C_1 \times 6C_1 \times \frac{5!}{3!} = 300$
4	2 set of 2 alike (PP, II, OO), 1 distinct $\left(\begin{array}{c} R, S, T, N \text{ and} \\ \text{remaining one} \\ \text{in 2 alike} \end{array}\right)$	$3C_2 \times 5C_1$	$3C_2 \times 5C_1 \times \frac{5!}{2! \times 2!} = 450$
5	1 set of 2 alike (PP, II, OO), 3 distinct $\left(\begin{array}{c} R, S, T, N \text{ and} \\ \text{remaining two} \\ \text{in 2 alike} \end{array}\right)$	$3c_1 \times 6C_3$	$3C_1 \times 6C_3 \times \frac{3!}{2!} = 3600$

Hence, the total number of strigs are $2520 + 20 + 300 + 450 + 3600$
$= 6890.$

Example 4.59: If a set of m parallel line interesect anotherr set of n parallel lines (not parallel to the lines in the first set), then find the number of parallelograms formed in this lattic structure.

Whenever we select 2 *lines from the first set of m lines*

and 2 *lines from the second set of n lines, one parallelogram is formed.*

Thus the number of parallelograms formed is $mC_2 \times nC_2$.

Example 4.60: How many diagonals are there in a polygon with n sides?

A polygon of n sides has n vertices.

By joining any two vertices of a polygon,

we obtain either a side or a diagonal of the polygon.

Number of line segments obtained by joining the vertices of a n sided polygon taken two at a time is $nC_2 = \frac{n(n-1)}{2}$

Out of these lines, there are n sides of polygon.

Therefore, number of diagonals of the polygon is $\frac{n(n-1)}{2} - n$

$$= \frac{n(n-1)-2n}{2} = \frac{n(n-1-2)}{2} = \frac{n(n-3)}{2}$$

number of diagonals for pentagon $= \frac{5(5-3)}{2} = \frac{5(\cancel{2})}{\cancel{2}} = 5$

number of diagonals for heptagon(Septagon) $= \frac{7(7-3)}{2} = \frac{7(\cancel{4})^{2}}{\cancel{2}} = 14$

1. If $nC_{12} = nC_9$ find $21C_n$.

$\boxed{nC_r = nC_{n-r}}$

$nC_{12} = nC_{n-9}$

$\therefore 12 = n - 9 \Rightarrow n = 12 + 9 \Rightarrow n = 21$

$\therefore 21C_n = 21C_{21} = 1$

2. If $15C_{2r-1} = 15C_{2r+4}$ find r.

$\boxed{nC_r = nC_{n-r}}$

$15C_{2r-1} = 15C_{2r+4}$

$= 15C_{15-2r-4}$

$2r - 1 = 15 - 2r - 4$

$2r + 2r = 15 - 4 + 1$

$4r = 12$

$r = \frac{12}{4} \Rightarrow \boxed{r = 3}$

3. If $nP_4 = 720$, and $nC_r = 120$, find n, r.

$nC_r = 120,\ nP_r = 720$

$\boxed{nP_r = nC_r \times r!}$

$720 = 120 \times r!$

$r! = \frac{\cancel{720}^{6}}{\cancel{120}} \Rightarrow r! = 6 \Rightarrow r = 3$

4. Prove that $15C_3 + 2 \times 15C_4 + 15C_5 = 17C_5$.

$$\text{LHS} = \frac{15 \times 14 \times 13}{1 \times 2 \times 3} + 2 \times \frac{15 \times 14 \times 13 \times 12}{1 \times 2 \times 3 \times 4} + \frac{15 \times 14 \times 13 \times 12 \times 11}{1 \times 2 \times 3 \times 4 \times 5}$$

$$= \frac{15\times 14\times 13}{1\times 2\times 3}\left[1+\frac{24}{4}+\frac{132}{20}\right]$$

$$= \frac{15\times 14\times 13}{1\times 2\times 3}\left[\frac{20+120+132}{20}\right] = \frac{15\times 14\times 13\times 272}{1\times 2\times 3\times 4\times 5}$$

$$= \frac{17\times 16\times 15\times 14\times 13}{1\times 2\times 3\times 4\times 5} = 17C_5 = RHS$$

5. Prove that $35C_5 + \sum_{4=0}^{4} (39-r)C_4 = 40C_5$.

$$LHS = 35C_5 + \sum_{4=0}^{4} (39-r)C_4$$

$$= 35C_5 + (39-0)C_4 + (39-1)C_4 + (39-2)C_4 + (39-3)C_4 + (39-4)C_4$$

$$= 35C_5 + 39C_4 + 38C_4 + 37C_4 + 36C_4 + 35C_4$$

$$= 35C_5 + 35C_4 + 36C_4 + 37C_4 + 38C_4 + 39C_4 \qquad \boxed{nC_r + nC_{r-1} = n+1C_r}$$

$$= 36C_5 + 36C_4 + 37C_4 + 38C_4 + 39C_4$$

$$= 37C_5 + 37C_4 + 38C_4 + 39C_4$$

$$= 38C_5 + 38C_4 + 39C_4 = 39C_5 + 39C_4 = 40C_3 = RHS$$

6. If $(n+1)C_n : (n-3)P_4 = 57:16$, find the value of n.

$$\frac{(n+1)C_8}{(n-3)P_4} = \frac{57}{16} \Rightarrow 16(n+1)C_8 = 57(n-3)P_4$$

$$16\,\frac{(n+1)!}{(n+1-8)!\,8!} = 57\,\frac{(n-3)!}{(n-3-4)!}$$

$$\frac{(n+1)n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-7)!}\,8!} = \frac{57}{16}\,\frac{\cancel{(n-3)!}}{\cancel{(n-7)!}}$$

$$(n+1)n(n-1)(n-2) = \frac{57\times 8!}{16}$$

$$(n+1)n(n-1)(n-2) = \frac{57\times \overset{2}{\cancel{8}}\times 7\times 6\times 5\times \cancel{4}\times 3\times 2\times 1}{\cancel{16}\,\cancel{4}}$$

$$= 19\times 3\times 7\times 6\times 5\times 4\times 3$$

$$(n+1)n(n-1)(n-2) = 21\times 20\times 19\times 18$$

Equating corresponding value $n+1 = 21$

$n = 21 - 1 = 20$

n is not an integer.

7. Prove that $\mathbf{2nC_n = \dfrac{2^n \times 1 \times 3 \times \cdots (2n-1)}{n!}}$

$L.H.S = 2C_n$

$$= \frac{2n!}{(2n-n)!\,n!}$$

$$= \frac{2n(2n-1)(2n-2)\ldots 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \ldots n \times n!}$$

$$= \frac{2n \cdot\ (2n-2)(2n-4)\ldots\ldots..2 \cdot (2n-1)(2n-3)\ldots\ldots 3 \cdot 1}{n!\,n!}$$

$$= \frac{[2(n)2(n-1)2(n-2)\ldots 2(1)][(2n-1)(2n-3)\ldots 3.1]}{n!\,n!}$$

$$= \frac{2^n[n(n-1)\ldots 1][(2n-1)(2n-3)\ldots 3.1]}{n!\,n!} = \frac{2^n \times 1 \times 3 \times ..(2n-1)}{n!}$$

$= \text{RHS}$

8. Prove that if $\mathbf{1 \le r \le n}$ **then** $\mathbf{n \times (n-1)C_{(r-1)} = (n-r+1)nC_{r-1}}$.

$\text{LHS} = n \times n - 1C_{r-1}$

$$= n\,\frac{(n-1)!}{(n-1-r+1)!\,(r-1)!} = \frac{n!}{(n-r)!\ (r-1)!}$$

$$= \frac{n!\,(n-r+1)}{(n-r+1)!\ \ (r-1)!} = (n-r+1) \cdot nC_{r-1} = \text{RHS}$$

9. (i) A Kabaddi coach has 14 players ready to play. How many different teams of 7 players could the coach put on the court?

(ii) There are 15 persons in a part and if each 2 of them shakes hands with each other, how many handshakes happen in the party?

(iii) How many chords can be drawn through 20 points on a circle?

(iv) In a parking lot one hundred, one year old cars, are parked. Out of them five are to be chosenat random for to check its pollution devices. How many different set of five cars can be chosen?

(v) How many ways can a team of 3 boys, 2 girls and 1 transgender be select from 5 boys, 4 girlsand 2 transgenders?

$$(i)\ 14C_7 = \frac{\overset{7}{\cancel{14}} \times 13 \times \overset{4}{\cancel{12}} \times 11 \times \overset{2}{\cancel{10}} \times \overset{3}{\cancel{9}} \times 8}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5} \times \underset{3}{\cancel{6}} \times \cancel{7}} = 3432$$

$$(ii)\ Number\ of\ handshakes = 15C_2 = \frac{15 \times \overset{7}{\cancel{14}}}{\cancel{2}} = 105$$

(iii) To draw a chord use need two points

$$\therefore \text{Number of chord} = 20C_2 = \frac{\overset{10}{\cancel{20}} \times 19}{\cancel{2}} = 190$$

$$\text{(iv) Number of selections} = 100C_5 = \frac{\overset{\cancel{20}\ 5}{\cancel{100}} \times \overset{33}{\cancel{99}} \times \overset{49}{\cancel{98}} \times 97 \times 96}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5}} = 75287520$$

$$\text{(v) Required number of selection} = 5C_3 \times 4C_2 \times 2C_1$$

$$= \frac{5 \times \overset{\cancel{2}}{\cancel{4}} \times \cancel{3}}{1 \times \cancel{2} \times \cancel{3}} \times \frac{4 \times 3}{1 \times \cancel{2}} \times 2 = 120$$

10. Find the total number of subsets of a set with
[Hint: $nC_0 + nC_1 + nC_2 + \cdots nC_n = 2^n$]
(i) 4 elements (ii) 5 elements (iii) n elements.

(i) 4 elements

$$2^4 = 2 \times 2 \times 2 \times 2 = 16$$

(ii) 5 elements

$$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$$

(iii) n elements $= 2^n$

11. A trust has 25 members.

(i) How many ways 3 officers can be selected?

(ii) In how many ways can a President, Vice President and a Secretary to be selected?

$(i)\ 25C_3$

$(ii)\ 25P_3$

12. How many ways a committee of six persons from 10 persons can be chosen along with a chair person and a secretary?

$$Required\ number\ of\ selection = 10C_1 \times 9C_1 \times 8C_6$$

$$= 10 \times 9 \times 8C_{8-6}$$

$$= 90 \times 8C_2$$

$$= 90 \times \frac{\overset{4}{\cancel{8}} \times 7}{1 \times \cancel{2}} = 2520$$

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13. How many different selections of 5 books can be made from 12 different books if,
(i) Two particular books are always selected?
(ii) Two particular books are never select?

There are 12 books. We have to select 5 books.
(*i*) *Two particular books be taken away and included.*
It is enough if we select 3 books out of remaining 10 books.

$$\text{Number of selections} = 10C_3 = \frac{10 \times \cancel{9}^{3} \times \cancel{8}^{4}}{1 \times \cancel{2} \times \cancel{3}} = 120$$

(*ii*) *Two particular books to be taken away and excluded.*
Therefore we have to select 5 books out of remaining 10 books in

$$10C_5 = \frac{\cancel{10}^{2} \times 9 \times \cancel{8}^{\cancel{4}} \times 7 \times \cancel{6}^{2}}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5}} = 252$$

14. There are 5 teachers and 20 students. Out of them a committeee of 2 teachers and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees.
(i) a particular teacher is included? (ii) a particular student is excluded?

The no. of selection of 2 teachers and 3 students from 5 teachers and 20 students is

$$5C_2 \times 20C_3 = \frac{5 \times \cancel{4}^{2}}{1 \times \cancel{2}} \times \frac{\cancel{20}^{10} \times 19 \times \cancel{18}^{6}}{1 \times \cancel{2} \times \cancel{3}} = 11400$$

(i) Let the particular teacher be included:

The number of selections $= 4C_1 \times 20C_3$

$$= 4 \times \frac{\cancel{20}^{10} \times 19 \times \cancel{18}^{6}}{1 \times \cancel{2} \times \cancel{3}} = 4560$$

(ii) Let the particular student be excluded:

$$\text{Number of selection} = 5C_2 \times 19C_3 = \frac{5 \times \cancel{4}^{2}}{1 \times \cancel{2}} \times \frac{19 \times \cancel{18}^{\cancel{9}\,3} \times 17}{1 \times \cancel{2} \times \cancel{3}} = 9690$$

15. In an examination a student has to answer 5 questions, out of 9 questions in which 2 are compulsory. In how many ways a student can answer the questions?

$$\text{Number of selections} = 7C_3 = \frac{7 \times \cancel{6}^{\cancel{3}} \times 5}{1 \times \cancel{2} \times \cancel{3}} = 35$$

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16. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly three aces in each combination.

Number of selections of 5 card combination from 52 cards with 3 aces in each combination is

$$48C_2 \times 4C_3 = \frac{48 \times 47}{\cancel{2}} \times \cancel{4}^{2} = 4512$$

$$\left\langle \begin{array}{c} No.of\ cards\ = 52 \\ No.of\ aces\ = 4 \\ No.of\ non-aces\ cards = 48 \end{array} \right\rangle$$

17. Find the number of ways of forming a committee of 5 members out of 7 Indians and 5 Americans, so that always Indians will be the majority in the committee.

Committee $= 5$

Possibilities	Indians (7)	American (5)	Combinations
(1)	5	−	$7C_5 = 7C_2 = \frac{7 \times \cancel{6}^{3}}{1 \times \cancel{2}} = 21$
(2)	4	1	$7C_4 \times 5C_1 = \frac{7 \times \cancel{6}^{2} \times 5}{1 \times \cancel{2} \times \cancel{3}} \times 5 = 175$
(3)	3	2	$7C_3 \times 5C_2 = \frac{7 \times \cancel{6}^{2} \times 5}{1 \times \cancel{2} \times \cancel{3}} \times \frac{5 \times \cancel{4}^{2}}{1 \times \cancel{2}} = 350$

Total $= 21 + 175 + 350 = 546$

18. A committee of 7 peoples has to be formed from 8 men and 4 women. In how many ways can this be done when the committee consists of (i) exactly 3 women? (ii) at least 3 women? (iii) at most 3 women?

(i) ***Exactly 3 women***: *that means 4 men in the committee*

$$Number\ of\ combinations = 8C_4 \times 4C_3 \qquad \boxed{\because 4C_3 = 4C_1}$$

$$= \frac{8 \times 7 \times \cancel{6}^{2} \times 5}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}} \times \cancel{4} = 280$$

(ii) ***At least 3 women***

means 4 men + 3 women (or) 3 men + 4 women in the committee.

$$Number\ of\ combinations = (8C_4 \times 4C_3) + (8C_3 \times 4C_4)$$

$$= 280 + \frac{8 \times 7 \times \cancel{6}^{2}}{1 \times \cancel{2} \times \cancel{3}} \times 1$$

$$= 280 + 56 = 336$$

(iii) ***At most 3 women***

means $(7\ men + 0\ women)$; $(6\ men + 1\ women)$, $(5\ men + 2\ women)$; $(4\ men + 3\ women)$ *in the committee*

Number of combinations

$$= (8C_7 \times 4C_0) + (8C_6 \times 4C_1) + (8C_5 \times 4C_2) + (8C_4 \times 4C_3)$$

$$= (8C_1 \times 1) + (8C_2 \times 4C_1) + (8C_3 \times 4C_2) + 280$$

$$= (8 \times 1) + \frac{8 \times 7}{1 \times \cancel{2}} \times \overset{2}{\cancel{4}} + \frac{8 \times 7 \times \overset{3}{\cancel{6}}}{1 \times \cancel{2} \times \cancel{3}} \times \frac{\overset{2}{\cancel{4}} \times 3}{1 \times \cancel{2}} + 280$$

$$= 8 + 112 + 336 + 280 = 736$$

19. 7 relatives of a man comprises 4 ladies and 3 gentlemen, his wife also has 7 relaives, 3 of them are ladies and 4 gentlemen. In how many ways can they invit a dinner paty of 3 ladies and 3 gentlemen so that there are 3 of men's relative and 3 of the wife's relative?

Possibilities	Man		Woman		Combinations
	3 Gents	4 Ladies	4 Gents	3 Ladies	
(1)	3	–	–	3	$3C_3 \times 3C_3 = 1$
(2)	2	1	1	2	$3C_2 \times 4C_1 \times 4C_1 \times 3C_2 = 144$
(3)	1	2	2	1	$3C_1 \times 4C_2 \times 4C_2 \times 3C_1 = 324$
(4)	–	3	3	–	$4C_3 \times 4C_3 = 16$
				Total	485

20. ***A box contains two white balls, three black balls and four red balls.*** **In how many ways can three balls be drawn from the box, if at last one black ball is to be included in the draw?**

Possibilities	$2W$	$3B$	$4R$	*Combinations*	
(1)	2	1	–	$2C_2 \times 3C_1 = 1 \times 3$	$= 3$
(2)	1	1	1	$2C_1 \times 3C_1 \times 4C_1 = 2 \times 3 \times 4$	$= 24$
(3)	–	1	2	$3C_1 \times 4C_2 = 3 \times 6$	$= 18$
(4)	1	2	–	$2C_1 \times 3C_2 = 2 \times 3$	$= 6$
(5)	–	2	1	$3C_2 \times 4C_1 = 3 \times 4$	$= 12$
(6)	–	3	–	$3C_3$	$= 1$
				Total	$= 64$

21. *Find the number of strings of 4 letters that can be formed with the letters of the word EXAMINATION.*

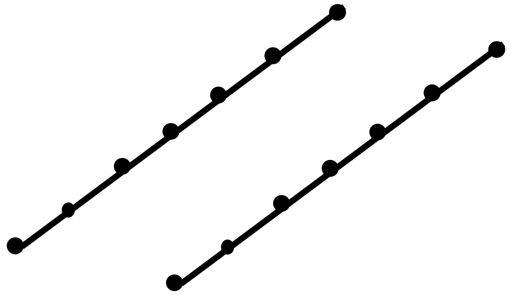
SI.NO.	Letter option	Selection	Arrangements
1	4 distict (E, X, M, T, O, A, I, N)	$8C_4$	$8C_4 \times 4! = 1680$
2	1 set of 2 alike, 2 distinct	$3C_1 \times 7C_2$	$3C_1 \times 7C_2 \times \frac{4!}{2!} = 756$
3	2 set of 2 alike	$3C_2$	$3C_2 \times \frac{4!}{2! \times 2!} = 18$
			Total $= 2454$

22. How many triangles can be formed by joining 15 points on the plane, in which no linejoining any three points?

To get a triangles we need 3 points (non collinear points)

$$\therefore \text{Number of triangles} = 15C_3 = \frac{\overset{5}{\cancel{15}} \times \overset{7}{\cancel{14}} \times 13}{1 \times \cancel{2} \times \cancel{3}} = 445$$

23. How many triangles can be formed by 15 points, in which 7 of them lie on one line and the remaining 8 on another parallel line??



(i) Select 2 points from the group of 7 points which are in first line and select 1 point from the other group of 8 points.

$$\text{Number of triangles} = 7C_2 \times 8C_1 = \frac{7 \times \overset{3}{\cancel{6}}}{1 \times \cancel{2}} \times 8 = 168$$

Mathematical Induction

EXERCISE: 4.4

4.61. *By the principle of Mathematical induction prove that, for all integers* $n \geq 1, 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

Let P(n): $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

Put $n = 1$

$P(n): L.H.S = n = 1$

$P(n): R.H.S = \frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$

$L.H.S = R.H.S$ $\therefore P(1)$ is true.

Let us assume that the statement $P(n)$ is true for $n = k$

$P(k): 1+2+3+\cdots+k = \frac{k(k+1)}{2}$ ....(1) *Assume be true*

We need to show that P(k + 1) is true

$$P(k+1) = \underbrace{1+2+3+\cdots+k}+(k+1)$$

$$= P(k)+k+1$$

$$= \frac{k(k+1)}{2}+(k+1)$$

$$P(k+1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)(k+1+1)}{2}$$

This implies, $P(k+1) =$ *is true*.

The validity of $P(k+1)$ *follows from that of* $P(k)$.

Hence by mathematical induction, for all integers, $n \geq 1$

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

4.62. *Prove that the sum of first n positive odd numbers is* n^2.

Let P(n): $1+3+5+\cdots+(2n-1) = n^2$ $P(n): L.H.S = 2(1)-1 = 1$

$P(n): R.H.S = n^2 = 1^2 = 1$ *is true*

We assume that $P(k) = 1+3+5+\ldots(2k-1)$ is true for $n = k$

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$That\ is\ P(k) = k^2$

We need to show that $P(k+1) = (k+1)^2$

$$P(k+1) = \underbrace{1+3+5+7+\cdots 2k-1}+2k+1 \qquad \boxed{2k-1+2 = 2k+1}$$

$$= P(k)+2k+1 = k^2+2k+1$$

$$= (k+1)^2 \quad This\ implies, P(k+1)\ is\ true.$$

Hence by mathematical induction,
$P(n)$ is true for all natural numbers.

4.63. *By the principle of mathematical induction, prove that, for all integers* $n \geq 1, 1^2+2^2+3^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Let P(n) *denote the statements*

$$Let\ P(n): 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Put $n = 1$

$P(n): L.H.S = n^2 = 1$

$$P(n): R.H.S = \frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2\times 1+1)}{6} = \frac{2\times 3}{6} = 1$$

$L.H.S = R.H.S \quad \therefore P(1)$ is true.

Let us assume that the statement $P(n)$ is true for $n = k$

$$P(k): 1^2+2^2+3^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6} \quad \ldots.(1)\ Assume\ be\ true$$

To prove $P(k+1)$ is true

$$P(k+1) = \underbrace{1^2+2^2+3^2+\cdots+k^2} + (k+1)^2$$

$$= P(k) + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6} = \frac{(k+1)[2k^2+k+6k+6]}{6}$$

$$= \frac{(k+1)[2k^2+7k+6]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$P(k+1) = \frac{(k+1)\big((k+1)+1\big)(2(k+1)+1)}{6}$$

if $P(k)$ is true, then $P(k+1)$ is true

Hence by mathematical induction $P(n)$ is true for all $n \in N$

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4.64. *Using the mathematical induction, show that for any natural numbers n,* $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

$$Let\,P(n):\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Put $n=1$ $P(n): L.H.S=\frac{1}{n(n+1)}=\frac{1}{1\times 2}=\frac{1}{2}$

$$P(n): R.H.S=\frac{n}{n+1}=\frac{1}{1+1}=\frac{1}{2}$$

$L.H.S=R.H.S$ $\therefore P(1)$ is true.

Let us assume that the statement is true for $n=k$

$$Then,\mathrm{P}(k):\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}\quad \ldots.(1)\quad Assume\ be\ true$$

We need to show that P(k + 1) is true

$$P(k+1)=\underbrace{\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{k(k+1)}}+\frac{1}{(k+1)(k+2)}.$$

$$=\quad P(k)\quad+\quad\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{1}{(k+1)}\left(\frac{k}{1}+\frac{1}{k+2}\right)=\frac{1}{(k+1)}\left[\frac{k(k+2)+1}{k+2}\right]$$

$$=\frac{1}{(k+1)}\left(\frac{k^2+2k+1}{k+2}\right)=\frac{1}{(k+1)}\left[\frac{(k+1)^2}{k+2}\right]=\frac{k+1}{(k+1)+1}$$

$$P(k+1)=\frac{k+1}{(k+1)+1}$$

This implies, $P(k+1)$ *is true.*

The validity of $P(k+1)$ *follows from that of* $P(k)$.

Hence by mathematical induction, for any natural numbers n,

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

4.65. ***Prove that for any natural numbers*** $\boldsymbol{n, a^n - b^n}$ ***is divisible by*** $\boldsymbol{a - b}$***, where*** $\boldsymbol{a > b}$**.**

Let P(n) *denote the statements*

P(n): $a^n - b^n$ is divisible by a − b for all $n \in N$

Put $n = 1$

$P(1): a^1 - b^1$

$= a - b$ is divisible by $a - b$ for all $n \in N$

$\therefore P(1)$ is true

Let us assume that the statement $P(n)$ is true for $n = k$

P(k): $a^k - b^k$ is divisible by a − b *Assume be true*

$$\frac{a^k - b^k}{a - b} = c$$

$$a^k - b^k = c(a-b) \implies a^k = b^k + c(a-b)$$

$\mathbf{P(k): a^k - b^k}$ $\boxed{a^k = b^k + c(a-b)}$

To prove P(k + 1) is true

To prove : $P(k+1) = a^{k+1} - b^{k+1}$ is divisible by a − b

$$= a^{k+1} - b^{k+1} = a^k a - b^k b$$

$$= [b^k + c(a-b)]a - b^k b$$

$$= ab^k + ac(a-b) - b^k b$$

$$= ab^k - b^k b + ac(a-b)$$

$$= b^k(a-b) + ac(a-b)$$

$= (a-b)(b^k + ac)$ is divisible by a − b

$\therefore P(k+1)$ is true.

if P(k) is true, then P(k + 1) is true

Hence by mathematical induction P(n) is true for all $n \in N$

4.66. ***Prove that*** $\boldsymbol{3^{2n+2} - 8n - 9}$ ***is divisible by*** $\boldsymbol{8}$ ***for all*** $\boldsymbol{n \geq 1}$**.**

Let P(n): $= 3^{2n+2} - 8n - 9$, *is divisible by* 8

Put $n = 1$

$P(1) = 3^{2+2} - 8(1) - 9$

$= 3^4 - 8 - 9 = 81 - 17 = 64$ *which is divisible by* 8 $\therefore P(1)$ is true.

Let us assume that the statement is true for $n = k$

Let P(k): $3^{2k+2} - 8k - 9$, *is divisible by* 8

$$\frac{3^{2k+2} - 8k - 9}{8} = c \implies 3^{2k+2} - 8k - 9 = 8c$$

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$\therefore 3^{2k+2} = 8c + 8k + 9$

We need to show that $P(k+1) = 3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8

$P(k+1) = \ 3^{2(k+1)+2} - 8(k+1) - 9$

$= 3^2 . 3^{2k+2} - 8k - 8 - 9$ $\quad \boxed{3^{2k+2} = 8c + 8k + 9}$

$= 3^2\ (8c + 8k + 9)\ - 8k - 17$

$= 9(8c + 8k + 9) - \ 8k - 17 = 72c + 72k + 81 - 8k - 17$

$= 72c + 64k + 64$

$= 8(9c + 8k + 8)$ *which is divisible by* 8

This implies, $P(k+1)$ *is true.*

The validity of $P(k+1)$ *follows from that of* $P(k)$.

Hence by mathematical induction, $3^{2n+2} - 8n - 9$ *is divisible by* 8 *for all* $n \geq 1$.

4.67. *Using the mathematical induction, show that for any integer* $\boldsymbol{n \geq 2, 3n^2 > (n+1)^2}$

Let $P(n): 3n^2 > (n+1)^2$ *with* $n \geq 2$

Put $n = 2$

$P(2) = 3 \times 2^2 = 12$ *and* $(2+1)^2 = 9$

As $12 > 9$ $\therefore P(2)$ is true.

Let us assume that the statement is true for $n = k$

Let $P(k): 3k^2 > (k+1)^2$

$P(k+1) = 3(k+1)^2 = 3(k^2 + 2k + 1) = 3k^2 + 6k + 3$

$= P(k) + 6k + 3$

$> (k+1)^2 + 6k + 3$

$> k^2 + 2k + 1 + 6k + 3$

$> k^2 + 8k + 4$

$> k^2 + 4k + 4k + 4 = k^2 + 4k + 4 + 4k$

$= (k+2)^2 + 4k$

$>\ (k+1+1)^2$ *since* $k > 0$

This is the statement $P(k+1)$.

The validity of $P(k+1)$ *follows from that of* $P(k)$.

Hence by mathematical induction, for all $n \geq 2, 3n^2 > (n+1)^2$

$4.68.$ *Using the mathematical induction, show that for any integer $n \geq 2, 3^n > n^2$.*

$Let\, P(n): 3^n > n^2 \ with \ n \geq 2$

Put $n = 2$

$P(2) = 3^2 = 9 \ \ and \ 2^2 = 4$

$As\ 9 > 4 \quad \therefore P(2)$ is true.

Let us assume that the statement is true for $n = k$

$That\ is$ $\mathrm{P(k)} > k^2$

$$P(k+1) = 3^{k+1} = 3 \times 3^k = 3 \times P(k)$$

$$> 3k^2$$

$$> (k+1)^2$$

Hence by mathematical induction, $for\ all\ integer\ n \geq 2, 3^n > n^2$

$4.69.$ *By the principle of mathematical induction, prove that for $n \in N$,*

$$\boldsymbol{cos\alpha + cos(\alpha+\beta) + cos(\alpha+2\beta) + \cdots + cos(\alpha+(n-1)\beta)}$$

$$\boldsymbol{= cos\left(\alpha\frac{(n-1)\beta}{2}\right) \times \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}}$$

Let $\mathrm{P(n)} := cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(n-1)\beta)$

Put $n = 1$

$$P(1) = \quad \cos(\alpha)\frac{\cos(\alpha)\,.\sin\left(\frac{\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \qquad \therefore P(1) \text{ is true.}$$

Let us assume that the statement is true for $n = k$

$$cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(n-1)\beta)$$

$$= cos\left(\alpha\frac{(k-1)\beta}{2}\right) \times \frac{\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

We need to show that $\mathrm{P(k+1)}$ is true.

$$P(k+1) = \ \cos(\alpha) + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(n-1)\beta) + \cos(\alpha+k\beta)$$

$$P(k+1) = P(k) + \cos(\alpha+k\beta)$$

$$= \frac{cos\left(\alpha\frac{(k-1)\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} + \cos(\alpha+k\beta)$$

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$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha+\frac{(k-1)\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)+\cos(\alpha+k\beta)\sin\left(\frac{\beta}{2}\right)\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\left(\alpha+\frac{k\beta}{2}\right)-\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)+\cos(\alpha+k\beta)\sin\left(\frac{\beta}{2}\right)\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\left(\cos\left(\alpha+\frac{k\beta}{2}\right)\cos\frac{\beta}{2}\right)+\sin\left(\alpha+\frac{k\beta}{2}\right)\sin\left(\frac{\beta}{2}\right)\right)\sin\left(\frac{k\beta}{2}\right) + \cos(\alpha+k\beta)\sin\left(\frac{\beta}{2}\right)\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha+\frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)+\ \sin\frac{\beta}{2}\left(\sin\left(\alpha+\frac{k\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + \cos(\alpha+k\beta)\right)\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha+\frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + \frac{\sin\frac{\beta}{2}}{2}\left(2\sin\left(\alpha+\frac{k\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)+2\cos(\alpha+k\beta)\right)\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha+\frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)+ \frac{\sin\frac{\beta}{2}}{2}[(\cos\alpha-\cos(\alpha+k\beta)+2\cos(\alpha+k\beta)]\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha+\frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)+\frac{\sin\frac{\beta}{2}}{2}(\cos\alpha+\cos(\alpha+k\beta))\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha+\frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + \frac{\sin\frac{\beta}{2}}{2}\left(2\cos\left(\alpha+\frac{k\beta}{2}\right)\cos\left(\frac{-k\beta}{2}\right)\right)\right]$$

$$= \frac{\cos\left(\alpha+\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\left[\sin\left(\frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\frac{\beta}{2}\cos\left(\frac{k\beta}{2}\right)\right]$$

$$= \frac{\cos\left(\alpha+\frac{k\beta}{2}\right) sin\left(\frac{(k+1)\beta}{2}\right)}{sin\left(\frac{\beta}{2}\right)}$$

$That\ is\ cos\,\alpha + cos(\alpha+\beta) + cos(\alpha+2\beta) + \cdots + cos(\alpha+(k-1)\beta)$

$+ cos(\alpha+k\beta)$

$= cos\left(\alpha+\frac{k\beta}{2}\right)\frac{sin\left(\frac{(k+1)\beta}{2}\right)}{sin\left(\frac{\beta}{2}\right)}$ $This\ implies, P(k+1) = is\ true.$

$The\ validity\ of\ P(k+1)\ follows\ from\ that\ of\ P(k).$

$Hence\ by\ the\ principle\ of\ mathematical\ induction,$

$cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(n-1)\beta)$

$$= cos\left(\alpha\frac{(n-1)\beta}{2}\right) \times \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

6.70. $\boldsymbol{Using\ the\ mathematical\ induction, show\ that\ for\ any\ natural\ n;}$ $\boldsymbol{with\ the\ assumption\ i^2 = -1, (r(cos\theta + i\ sin\theta))^n = r^n(\cos n\,\theta + i\sin n\theta).}$

$Let\,P(n): [r(cos\theta + i\ sin\theta)]^n = r^n(cos\,n\,\theta + i\,sin\,n\theta)$

Put $n = 1$

$P(1) =\ (r(cos\theta + i\ sin\theta))^1 = r(cos\theta + i\sin\theta)$

$\therefore P(1)$ is true.

Let us assume that the statement is true for $n = k$

$Then, (r(cos\theta + i\ sin\theta))^k = r^k(\cos k\,\theta + i\sin k\,\theta)$

We need to show that P(k + 1) is true.

$P(k+1) = (r(cos\theta + i\ sin\theta))^{k+1}$

$=(r(cos\theta + i\ sin\theta))^k \times r(cos\theta + i\sin\theta)$

$= r^k(\cos k\,\theta + i\sin k\theta) \times r(cos\theta + i\sin\theta)$

$= r^{k+1} \times (cos\,k\,\theta cos\theta + i^2\,sin\,k\theta\,sin\,\theta) + i(\sin k\,\theta cos\theta + cos\,k\theta\sin\theta)$

$= r^{k+1} \times (cos(k+1)\,\theta + i\,sin(k+1)\theta)$

$This\ implies, P(k+1) = is\ true.$

$The\ validity\ of\ P(k+1)\ follows\ from\ that\ of\ P(k).$

$Hence\ by\ the\ principle\ of\ mathematical\ induction,$

$(r(cos\theta + i\ sin\theta))^n = r^n(\cos(n\,\theta) + i\sin(n\theta))$

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1. *By the principle of mathematical induction, prove that, for $n \geq 1$,*

$$\mathbf{1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2}$$

$$Let\, P(n): 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Put $n = 1$

$P(1)$: $L.H.S$: $n^3 = 1$

$$R.H.S := \left[\frac{n(n+1)}{2}\right]^2 = \left[\frac{1(1+1)}{2}\right]^2 = \left[\frac{1(2)}{2}\right]^2 = 1$$

$L.H.S = R.H.S$ $\therefore P(1)$ is true.

Let us assume that the statement $P(n)$ is true for $n = k$

$$\text{P(k)}: 1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2 \ldots.(1) \quad \textit{Assume be true}$$

We need to show that P(k + 1) is true

$$P(k+1) = 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3$$

$$= P(k) + (k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 = (k+1)^2\left[\frac{k^2}{4} + k + 1\right]$$

$$= (k+1)^2\left[\frac{k^2 + 4k + 4}{4}\right] = \frac{(k+1)^2(k+2)^2}{4}$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2 \quad \therefore P(k+1) \text{is true.}$$

Hence by mathematical induction P(n) is true for all values of n.

2. *By the principle of mathematical induction, prove that, for $n \geq 1$,*

$$\mathbf{1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}}$$

$$\text{Let P(n)}: 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Put $n = 1$

$$P(n): L.H.S = (2n-1)^2 = (2 \times 1 - 1)^2 = 1^2 = 1$$

$$P(n): R.H.S = \frac{1(2-1)(2+1)}{3} = \frac{3}{3} = 1$$

$L.H.S = R.H.S \quad \therefore P(1)$ is true.

Let us assume that the statement $P(n)$ is true for $n = k$

$$P(k): 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3} \;\ldots.(1)$$

Assume be true

We need to show that P(k + 1) is true

$\boxed{[2k-1+2]^2 = (2k+1)^2}$

$$P(k+1) = 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2k+1)^2$$

$$= P(k) + (2k+1)^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= (2k+1)\left[\frac{k(2k-1)}{3} + 2k + 1\right]$$

$$= (2k+1)\left[\frac{2k^2 - k + 6k + 3}{3}\right]$$

$$= \frac{(2k+1)(2k^2+5k+3)}{3}$$

$$= \frac{(2k+1)(k+1)(2k+3)}{3}$$

$\therefore P(k+1)$ is true.

This implies, $P(k+1) = $ *is true*.

if P(k) is true, then P(k + 1) is true

Hence by mathematical induction P(n) is true for all values of n

3. *Prove that the sum of first 'n' non − zero even number is $n^2 + n$.*

Let $P(n): 2 + 4 + 6 + \cdots + 2n = n^2 + n$

Put $n = 1$

$P(n): L.H.S = 2n = 2$

$P(n): R.H.S = n^2 + n = 1^2 + 1 = 2$

$L.H.S = R.H.S \quad \therefore P(1)$ is true.

Let us assume that the statement $P(n)$ is true for $n = k$

P$(k): 2 + 4 + 6 + \cdots + 2k = k^2 + k \;\ldots.(1)$ *Assume be true*

We need to show that P(k + 1) is true

P(k + 1): $2 + 4 + 6 + \cdots + 2k + 2k + 2$

$= P(k) + 2(k+1)$

$= k^2 + k + 2k + 2$

$= k^2 + 3k + 2 = (k+2)(k+1)$

$This\ implies, P(k+1) = is\ true.$

if P(k) is true, then P(k + 1) is true

Hence by mathematical induction P(n) is true for all values of n

4. $\boldsymbol{By\ the\ principle\ of\ mathematical\ induction, prove\ that, for\ n \geq 1,}$

$$\mathbf{1.2 + 2.3 + 3.4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.}$$

Let P(n): $1.2 + 2.3 + 3.4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

Put $n = 1$

$P(n): L.H.S = 1(1+1) = 1(2) = 2$

$$P(n): R.H.S = \frac{n(n+1)(n+2)}{3} = \frac{1(1+1)(1+2)}{3} = \frac{2(3)}{3} = 2$$

$L.H.S = R.H.S$ $\therefore P(1)$ is true.

Let us assume that the statement $P(n)$ is true for $n = k$

$$P(k) : 1.2 + 2.3 + 3.4 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3} \quad \ldots.(1)$$

$Assume\ be\ true$

We need to show that P(k + 1) is true

$$P(k+1) = 1.2 + 2.3 + 3.4 + \cdots + k(k+1) + (k+1)(k+2)$$

$$= P(k) + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= (k+1)(k+2)\left[\frac{k}{3} + 1\right]$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$This\ implies, P(k+1) = is\ true.$ if P(k) is true, then P(k + 1) is true

Hence by mathematical induction P(n) is true for all values of n

5. $\boldsymbol{Using\ the\ mathematical\ induction, show\ that\ for\ any\ natural\ number\ n \geq 2,}$

$$\boldsymbol{\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \ldots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}}$$

$$\text{P(n)}: \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \ldots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

Put $n = 2$

$$L.H.S = 1 - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{4-1}{4} = \frac{3}{4}$$

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$$R.H.S = \frac{2+1}{2(2)} = \frac{3}{4}$$

$L.H.S = R.H.S$ $\therefore P(1)$ is true.

Let us assume that the statement $P(n)$ is true for $n = k$

$$P(k) : \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left[1-\frac{1}{k^2}\right] = \frac{k+1}{2k} \quad \ldots.(1)$$

Assume be true

We need to show that $P(k+1)$ is true.

$$P(k+1) = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$= P(k)\left(1-\frac{1}{(k+1)^2}\right) = \left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$= \frac{k+1}{2k}\left[\frac{(k+1)^2-1}{(k+1)^2}\right] = \frac{k+1}{2k}\left(\frac{k^2+2k+1-1}{(k+1)^2}\right)$$

$$= \frac{k+1}{2k}\left[\frac{k^2+2k}{(k+1)^2}\right] = \frac{\cancel{k+1}}{\cancel{2k}} \times \frac{\cancel{k}(k+2)}{\cancel{(k+1)^2}_{k+1}}$$

$$= \frac{k+2}{2(k+1)}$$

$\therefore P(k+1)$ is true.

Hence by mathematical induction $P(n)$ is true for all values of n

6. *Using the mathematical induction, show that for any natural number $n \geq 2$.*

$$\frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots + \frac{1}{1+2+3+\cdots+n} = \frac{n-1}{n+1}$$

$$P(n): \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+3+..+n} = \frac{n-1}{n+1}$$

$$\boxed{\because \sum n = \frac{n(n+1)}{2}}$$

$$P(n): \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{\frac{n(n+1)}{2}} = \frac{n-1}{n+1}$$

$$P(n): \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{2}{n(n+1)} = \frac{n-1}{n+1}$$

Put $n = 2$

$$P(n): L.H.S = \frac{2}{n(n+1)} = \frac{2}{2(2+1)} = \frac{2}{2(3)} = \frac{1}{3}$$

$$P(n): R.H.S = \frac{2-1}{2+1} = \frac{1}{3}$$

$L.H.S = R.H.S$ $\therefore P(1)$ is true.

Let us assume that the statement $P(n)$ is true for $n = k$

$$P(k): \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{2}{k(k+1)} = \frac{k-1}{k+1} \quad \ldots.(1)$$

We need to show that P(k + 1) is true. *Assume be true*

$$P(k+1): \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{2}{k(k+1)} + \frac{2}{(k+1)(k+2)}$$

$$= P(k) + \frac{2}{(k+1)(k+2)} = \frac{k-1}{k+1} + \frac{2}{(k+1)(k+2)}$$

$$= \frac{1}{k+1}\left(k - 1 + \frac{2}{k+2}\right) = \frac{1}{k+1}\left[\frac{(k-1)(k+2)+2}{k+2}\right]$$

$$= \frac{1}{k+1}\left[\frac{k^2+2k-k-2+2}{k+2}\right] = \frac{1}{k+1}\left[\frac{k^2+2k-k\cancel{-2}\cancel{+2}}{k+2}\right]$$

$$= \frac{1}{k+1}\left[\frac{k^2+k}{k+2}\right] = \frac{k}{k+1}\left[\frac{k+1}{k+2}\right] = \frac{k}{k+2}$$

$\therefore P(k+1)$is true.

Hence by mathematical induction P(n) is true for all values of n

7. *Using the mathematical induction, show that for any natural number n,*

$$\mathbf{\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}}$$

$$P(n): \frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

Put $n = 1$

$$P(n): L.H.S = \frac{1}{n(n+1)(n+2)} = \frac{1}{1(1+1)(1+2)} = \frac{1}{1\times2\times3} = \frac{1}{6}$$

$$P(n): R.H.S = \frac{n(n+3)}{4(n+1)(n+2)} = \frac{1(1+3)}{4(1+1)(1+2)} = \frac{4}{4(2)(3)} = \frac{1}{6}$$

$L.H.S = R.H.S$ $\therefore P(1)$ is true.

Let us assume that the statement $P(n)$ is true for $n = k$

$$\because \quad P(k): \frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \cdots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)} \quad \ldots.(1)$$

We need to show that P(k + 1) is true. *Assume be true*

$$P(k+1) = \frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \cdots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= P(k) + \frac{1}{(k+1)(k+2)(k+3)} = \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{(k+2)(k+1)}\left[\frac{k(k+3)}{4} + \frac{1}{(k+3)}\right]$$

$$= \frac{1}{(k+2)(k+1)}\left[\frac{k(k+3)^2+4}{4(k+3)}\right]$$

$$= \frac{1}{4(k+1)(k+2)(k+3)}\,[k(k^2+6k+9)+4]$$

$$= \frac{1}{4(k+1)(k+2)(k+3)}\,[k^3+6k^2+9k+4]$$

$Using\ synthetic\ division\ \ k^3+6k^2+9k+4$

$$\begin{array}{r|rrrr} -1 & 1 & 6 & 9 & 4 \\ & 0 & -1 & -5 & -4 \\ \hline & 1 & 5 & 4 & \boxed{0} \end{array}$$

$one\ of\ the\ factor\ is\ (k+1)$

$\therefore The\ other\ factor\ is\ k^2+5k+4 = (k+1)(k+4)$

$$= \frac{1}{4(k+1)(k+2)(k+3)}[(k+1)(k+1)(k+4)]$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)} \quad This\ implies, P(k+1) is\ true.$$

Hence by mathematical induction P(n) is true for all values of n

8. *Using the mathematical induction, show that for any natural number n,*

$$\boldsymbol{\frac{1}{2\cdot5} + \frac{1}{5\cdot8} + \frac{1}{8\cdot11} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}}$$

P(n): $\frac{1}{2\cdot5} + \frac{1}{5\cdot8} + \frac{1}{8\cdot11} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$

Put $n = 1$

$P(n): L.H.S = \frac{1}{(3n-1)(3n+2)} = \frac{1}{(3-1)(3+2)} = \frac{1}{2\times 5} = \frac{1}{10}$

$P(n): R.H.S = \frac{n}{6n+4} = \frac{1}{6(1)+4} = \frac{1}{10}$ $L.H.S = R.H.S$ $\therefore P(1)$ is true.

Let us assume that the statement $P(n)$ is true for $n = k$

$\because P(k): \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \cdots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4}$ $\quad \ldots.(1)$

Assume be true

We need to show that P(k + 1) is true.

$$P(k+1) = \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \cdots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k-1+3)(3k+2+3)}$$

$$= P(k) + \frac{1}{(3k+2)(3k+5)} = \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)} = \frac{1}{3k+2}\left[\frac{k}{2} + \frac{1}{3k+5}\right]$$

$$= \frac{1}{3k+2}\left[\frac{k(3k+5)+2}{2(3k+5)}\right] = \frac{1}{3k+2}\left[\frac{3k^2+5k+2}{2(3k+5)}\right]$$

$$= \frac{1}{2(3k+2)(3k+5)}(k+1)(3k+2)$$

$$= \frac{k+1}{2(3k+5)} = \frac{k+1}{6k+10}$$ *This implies,* $P(k+1) =$ *is true.*

if P(k) is true, then P(k + 1) is true

Hence by mathematical induction P(n) is true for all values of n

9. *Prove by mathematical induction that*

$\mathbf{1! + (2\times 2!) + (3\times 3!) + .. + (n\times n!) = (n+1)! - 1}$

Let $P(n) := 1! + (2\times 2!) + (3\times 3!) + .. + (n\times n!) = (n+1)! - 1$

Put $n = 1$

$P(n): L.H.S = n\times n! = 1\times 1! = 1$

$P(n): R.H.S = (n+1)! - 1 = (1+1!) - 1$

$= 2 - 1$

$= 1$ $\quad L.H.S = R.H.S$ $\therefore P(1)$ is true.

Let us assume that the statement $P(n)$ is true for $n = k$

p (k): $1! + (2\times 2!) + (3\times 3!) + .. + (k\times k!) = (k+1)! - 1$ $\quad \ldots.(1)$

We need to show that P(k + 1) is true. *Assume be true*

$P(k+1): 1! + (2\times 2!) + (3\times 3!) + .. + (k\times k!) + (k+1)(k+1)!$

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$= P(k) + (k+1)(k+1)!$

$\boxed{\because (n+1)(n!) = (n+1)!}$

$= (k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)! + (k+1)(k+1)! - 1$

$= (k+1)!\,[1 + k + 1] - 1 = (k+1)!\,[k+2] - 1$

$(k+2)! - 1$ $\textit{This implies}, P(k+1) \textit{is true}.$

Hence by mathematical induction P(n) is true for all values of n

10. *Using mathematical induction show that for any natural numbers n, $x^{2n} - y^{2n}$ is divisible by $x + y$.*

Let P(n)$:= x^{2n} - y^{2n}$ *is divisible by* $x + y$.

Put $n = 1$

P(1)$: x^2 - y^2 = (x+y)(x-y)$ *which is divisible by* $x + y$ $\therefore P(1)$ is true.

Let us assume that the statement is true for $n = k$

Then, P(k) $= x^{2k} - y^{2k}$ *is divisible by* $x + y$

$$\frac{x^{2k} - y^{2k}}{x + y} = m \Longrightarrow x^{2k} - y^{2k} = m(x+y)$$

$$x^{2k} = m(x+y) + y^{2k}$$

We need to show that P(k + 1)is true

i.e $x^{2(k+1)} - y^{2(k+1)}$ is divisible by $(x+y)$

$P(k+1) = x^{2(k+1)} - y^{2(k+1)}$

$= x^{2(k+1)} - y^{2(k+1)}$

$= x^{2k+2} - y^{2k+2} = x^2 \cdot x^{2k} - y^{2k+2}$

$= x^2[m(x+y) + y^{2k}] - y^{2k+2}$ $\boxed{x^{2k} = m(x+y) + y^{2k}}$

$= m(x+y)(x^2) + x^2y^{2k} - y^{2k} \cdot y^2$

$= m(x+y)(x^2) + y^{2k}(x^2 - y^2)$

$= m(x+y)x^2 + y^{2k}(x+y)(x-y)$

$= (x+y)[mx^2 + (x-y)y^{2k}]$ *which is divisible by* $(x+y)$

$\therefore x^{2(k+1)} - y^{2(k+2)}$ *is divisible by* $(x+y)$

$\textit{This implies}, P(k+1)$ *is true*.

Hence by mathematical induction P(n) is true for all values of n

12. *Use induction to prove that $n^3 - 7n + 3$ is divisible by 3 for all natural number n.*

Let $P(n): n^3 - 7n + 3$ is divisible by 3

Put $n = 1$ $P(1): 1^3 - 7(1) + 3 = 1 - 7 + 3 = -3$ is divisible by 3

$\therefore P(1)$ is true.

Let us assume that $P(n)$ is true for $n = k$

P(k): $k^3 - 7k + 3$ is divisible by 3 *Assume be true*

$$\frac{k^3 - 7k + 3}{3} = m \Longrightarrow k^3 - 7k + 3 = 3m$$

To prove P(k + 1) is true $\boxed{\because\ (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3]}$

To prove : $P(k+1) = (k+1)^3 - 7(k+1) + 3$ is divisible by 3

$$P(k+1) = (k+1)^3 - 7(k+1) + 3$$

$$= k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3$$

$$= (k^3 - 7k + 3) + (3k^2 + 3k - 6)$$

$$= 3m + 3(k^2 + k - 2)$$

$$= 3(m + k^2 + k - 2)$$ which is divisible by 3

$P(k+1)$ is true.

Hence by mathematical induction P(n) is true for all values of n

13. *Use induction to prove that $5^{n+1} + 4 \times 6^n$ when divided by 20 leaves a remainder 9 for all natural numbers n.*

Let P(n): $= 5^{n+1} + 4 \times 6^n$ *is divisible by* 20 *leaves remainder* 9

Put $n = 1$

$$P(1) = 5^2 + 4 \times 6^1$$

$$= 25 + 24 = 49$$

$$= 2(20) + 9$$ *hence leaves a remainder* 9 *when divisible by* 9

$\therefore P(1)$ is true.

Let us assume that the statement is true for $n = k$

Let P(k): $= 5^{k+1} + 4 \times 6^k$ *when divided by* 20 *leaves a remainder* 9

$$5^{k+1} + 4 \times 6^k = 20m + 9$$

$$5^{k+1} = 20m + 9 - 4(6^k)$$

We need to show that P(k + 1) is true.

P(k + 1): $5^{k+2} + 4 \times 6^{k+1}$

To prove P$(k+1)$ *is divided by* 20 *leaves a remainder* 9

$$= 5^{k+2} + 4 \times 6^{k+1}$$

$$= 5^{k+1} \cdot 5^1 + 4 \cdot 6^k \cdot 6^1$$ $\boxed{5^{k+1} = 20m + 9 - 4(6^k)}$

$$= 5(20m + 9 - 4.6^k) + 24 \cdot 6^k$$

$$= 100m + 45 - 20 \cdot 6^k + 24 \cdot 6^k$$

$$= 100m + 45 - 20 \cdot 6^k + 20 \cdot 6^k + 4 \cdot 6^k$$

$$= 100m + 45 + 4 + 4 \cdot 6^k - 4$$

$= 100m + 49 + 4 \cdot 6^k - 4 \quad = 100m + 40 + 9 + 4(6^k - 1)$

$= 100m + 40 + 4(6^k - 1) + 9$

$which\ is\ divisible\ by\ 20\ and\ leaves\ a\ remainder\ 9$

$\therefore P(k+1)$ is true.

Hence by mathematical induction P(n) is true for all values of n

14. *Use induction to prove that* $\mathbf{10^n + 3 \times 4^{n+2} + 5}$, *is divisible by* 9 *for all natural numbers* $\boldsymbol{n}$.

Let P(n): $10^n + 3 \times 4^{n+2} + 5, is\ divisible\ by\ 9$

Put $n = 1$

$P(1) = 10^1 + 3 \times 4^{1+2} + 5 \quad = 10 + 3(4^3) + 5$

$= 15 + 3(64) = 15 + 192$

$= 207\ \ which\ is\ divisible\ by\ 9 \qquad \therefore P(1)$ is true.

Let us assume that the statement is true for $n = k$

Let P(k): $10^k + 3 \times 4^{k+2} + 5, is\ divisible\ by\ 9$

$$\frac{10^k + 3 \times 4^{k+2} + 5}{9} = m$$

$10^k + 3 \times 4^{k+2} + 5 = 9m$

$10^k = 9m - 5 - 3 \times 4^{k+2}$ We need to show that P(k + 1) is true.

$P(k+1): 10^{k+1} + 3 \times 4^{k+1+2} + 5, is\ divisible\ by\ 9$

$\boxed{10^k = 9m - 5 - 3 \times 4^{k+2}}$

$= \ 10^{k+1} + 3 \times 4^{k+3} + 5$

$= \ 10^k \cdot 10 + 3 \cdot 4^{k+2} \cdot 4^1 + 5$

$= 10[9m - 5 - 3 \times 4^{k+2}] + 12 \cdot 4^{k+2} + 5$

$= 90m - 50 - 30 \times 4^{k+2} + 12 \cdot 4^{k+2} + 5$

$= 90m - 45 - 30 \times 4^{k+2} + 12 \cdot 4^{k+2}$

$= 90m - 45 - 18 \cdot 4^{k+2}$

$= 9[10m - 5 - 2(4^{k+2})] which\ is\ divisible\ by\ 9$

$P(k+1)$ is true.

Hence by mathematical induction P(n) is true for all values of n